Automorphic Vector Bundles on Shimura Varieties

Kush Singhal

2023

Contents

1 Shimura Varieties **1**
- 1.1 Shimura Data . . . 1
- 1.2 Hecke Correspondences . . . 3
- 1.3 The Geometry of $Sh_K(G, X)$ . . . 3
- 1.4 Canonical Models . . . 5

2 Local Systems and Vector Bundles on Shimura Varieties **6**
- 2.1 Review of Local Systems and the Riemann-Hilbert Correspondence . . . 6
- 2.2 Automorphic Vector Bundles . . . 7
- 2.3 Computing Automorphic Vector Bundles . . . 8
- 2.4 Automorphic Local Systems . . . 9
- 2.5 A Note on Automorphy Factors . . . 9
- 2.6 Hecke Action on Cohomology of $\tilde{V}_K$ . . . 10

3 Toroidal Compactifications and Canonical Extensions of Bundles **12**
- 3.1 Boundary Components . . . 12
- 3.2 Baily-Borel compactification and the Satake topology . . . 13
- 3.3 Combinatorial Data . . . 14
- 3.4 Toroidal Compactifications - Classical . . . 16
- 3.5 Constructing Toroidal Compactifications Adelically . . . 17
- 3.6 Canonical Extensions of Automorphic Vector Bundles . . . 20
- 3.7 Hecke Action on Canonical Extensions . . . 21

4 Cohomology of Automorphic Vector Bundles **22**
- 4.1 Relative Lie Algebra Cohomology . . . 22
- 4.2 Cohomology of Automorphic Local Systems as $(\mathfrak{g}, K_o)$-cohomology . . . 22
- 4.3 The Matsushima-Murukami Formula . . . 24
- 4.4 Cohomology with Compact Supports, and Poincaré duality . . . 24
- 4.5 Cohomology of Canonical Extensions . . . 26

A Miscellaneous Algebraic Geometry **28**
- A.1 Connections . . . 28
- A.2 Chern Classes and Intersection Theory . . . 28

1 Shimura Varieties

A good reference with examples is [Mil05], though the structure of exposition follows [Har13a].

1.1 Shimura Data

We fix some notation. Let $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ denote the Deligne torus. Then we have

$$\mathbb{S}(R) = \left(R[j]/\left\langle j^2+1\right\rangle\right)^\times \cong (R \oplus Rj)^\times,$$

and $\mathbb{S}_\mathbb{C} \cong \mathbb{G}_{m,\mathbb{C}}^2$ under the isomorphism

$$(R \oplus Rj)^\times = \mathbb{S}_\mathbb{C}(R) \xrightarrow{\sim} \mathbb{G}_{m,\mathbb{C}}^2 = (R^\times)^2, \quad a + bj \mapsto (a + bi, a - bi)$$

where $i \in \mathbb{C} \subset R$ is a square root of unity. Let $X^*(-) := \mathrm{Hom}(-, \mathbb{G}_m)$ and $X_*(-) := \mathrm{Hom}(\mathbb{G}_m, -)$ be the character and co-character groups. Then $X^*(\mathbb{S})_\mathbb{C} \cong \mathbb{Z}^2$ with basis $X^*(\mathbb{S})_\mathbb{C} \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ where $e_1(a+bj) = a+bi$ and $e_2(a+bj) = a-bi$. Complex conjugation acts on $X^*(\mathbb{S})_\mathbb{C}$ by swapping e_1 and e_2.

We have the weight homomorphism $w : \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$ given by $w(a) = a + 0j$. This induces the map

$$w^* : \mathbb{Z}^2 \cong X^*(\mathbb{S}_\mathbb{C}) \to X^*(\mathbb{G}_{m,\mathbb{R}}) \cong \mathbb{Z}, \quad (p,q) \mapsto p+q.$$

Let G be a connected reductive group over $\mathbb{Q}$. Let its Lie algebra be $\mathfrak{g}$. We have the adjoint rep $\mathrm{Ad} : G \to GL(\mathfrak{g})$ with image $G^{ad} \cong G/Z_G$ where Z_G is the center of G. We write $G(\mathbb{R})^+$ for the connected component of $G(\mathbb{R})$ in the real topology, and we set $G(\mathbb{Q})^+ = G(\mathbb{R})^+ \cap G(\mathbb{Q})$. We let $G(\mathbb{R})_+$ denote the preimage of $G^{ad}(\mathbb{R})^+$ under the map $G(\mathbb{R}) \to G^{ad}(\mathbb{R})$, and we set $G(\mathbb{Q})_+ = G(\mathbb{R})_+ \cap G(\mathbb{Q})$. Let A_G denote the connected component of $Z_G(\mathbb{R})$ in the real topology.

Definition. A *Shimura datum* is a pair (G, X) where G is a reductive group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_\mathbb{R}$ satisfying the following axioms.

SD1 For any $h \in X$, $\mathrm{Ad}\big(h(j)\big)$ is a *Cartan involution* of G^{ad}, i.e. the group

$$\{g \in G^{ad}(\mathbb{C}) : h(j) \cdot \overline{g} \cdot h(j)^{-1} = g\}$$

is a compact Lie group, where $\overline{}$ denotes complex conjugation.

SD2 For $h \in X$, only the characters $(-1,1), (0,0), (1,-1) \in X^*(\mathbb{S}_\mathbb{C})$ occur in the adjoint representation $ad \circ g : \mathbb{S} \to GL(\mathfrak{g}_\mathbb{C})$, i.e. we can write

$$\mathfrak{g}_\mathbb{C} \cong \mathfrak{p}_h^- \oplus \mathfrak{p}_h^+ \oplus \mathfrak{m}_h$$

where $z \in \mathfrak{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $\mathfrak{p}_h^-$ as multiplication by $\overline{z}/z$, on $\mathfrak{p}_h^+$ as multiplication by $z/\overline{z}$, and acts trivially on $\mathfrak{m}_h$.

SD3 (*optional*) G^{ad} has no non-trivial $\mathbb{Q}$-rational factor G_0 such that the projection of h on G_0 is trivial.

Remark 1.1. Brian Conrad's notes, especially Section 5, are an excellent place to gain intuition for why the axioms are the way they are.

Notice that $h \circ w : \mathbb{G}_{m,\mathbb{R}} \to G_\mathbb{R}$ acts trivially on $\mathfrak{g}$ by SD2, and so $h \circ w$ has image in $Z_{G,\mathbb{R}}$. Since we are taking $G(\mathbb{R})$-conjugacy classes of morphisms $h \in X$, we see that $h \circ w = h' \circ w$ for any other $h' \in X$. We thus have a well-defined *weight morphism* $w_X : \mathbb{G}_{m,\mathbb{R}} \to Z_{G_\mathbb{R}}$ attached to our Shimura datum (G, X).

Remark 1.2. Milne defines the weight morphism differently. In his notation, the weight morphism is the *inverse* of our weight morphism w_X.

Now suppose K is an open compact subgroup $K \subset G(\mathbb{A}_\mathbb{Q}^\infty)$, we have

$$Sh_K(G, X) := G(\mathbb{Q})\backslash X \times \big(G(\mathbb{A}_\mathbb{Q}^\infty)/K\big).$$

In fact, equipping $G(\mathbb{A}_\mathbb{Q}^\infty)$ with the adelic topology, we have a homeomorphism

$$Sh_K(G, X) \cong \bigsqcup_i \Gamma_i \backslash X^+$$

where $\Gamma_i := gKg^{-1} \cap G(\mathbb{Q})_+$, X^+ is a connected component of X, and where the disjoint union runs over a set of coset representatives of the double quotient space

$$G(\mathbb{Q})_+ \backslash G(\mathbb{A}_\mathbb{Q}^\infty)/K$$

This double coset space is known to be finite.

Remark 1.3. One checks that $G(\mathbb{R})_+$ is the subgroup of $G(\mathbb{R})$ that stabilizes X^+, so that in fact $X^+ \cong G(\mathbb{R})_+/M_h(\mathbb{R})$ where $h \in X$ is fixed, $M_h(\mathbb{R})$ is the stabilizer of h in $G(\mathbb{R})$, and X^+ is the connected component of X containing h.

1.2 Hecke Correspondences

We consider the family $Sh_K(G,X)$ as K varies through the cofiltered poset of compact opens K in $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. It is known that there is a cofinal sub-poset consisting of those K such that $Sh_K(G,X)$ are smooth manifolds (we can take for instance the *neat* subgroups; see [GH22]). For $K' \subset K$ compact open neat subgroups of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$, it is known that the natural map $Sh_{K'}(G,X) \to Sh_K(G,X)$ is smooth.

Now suppose K is a compact open subset of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$, and $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. Then gKg^{-1} is another compact open subset of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$, and we get an isomorphism

$$Sh_{gKg^{-1}}(G,X) \xrightarrow{\sim} Sh_K(G,X), \quad [x, a \cdot gKg^{-1}] \mapsto [x, ag \cdot K].$$

Thus, if $K' \subset gKg^{-1}$ is another compact open subgroup, then we can compose the above two maps to get the family of maps

$$T_g : Sh_{K'}(G,X) \to Sh_K(G,X)$$

which on points is given by

$$G(\mathbb{Q}) \cdot (x, hK') \mapsto G(\mathbb{Q}) \cdot (x, hgK).$$

These maps are finite étale if K and K' are neat (see [GH22, Section 15.2]). This family of maps T_g gives a right action of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ on the inverse system $(Sh_K(G,X))_K$, called the *Hecke action.*

Definition. The *Shimura variety* attached to the Shimura datum (G,X) is the inverse system of varieties $(Sh_K(G,X))_K$ equipped with the given Hecke action. We set

$$Sh(G,X) := \varprojlim_K Sh_K(G,X),$$

this is a scheme over $\mathbb{C}$.

Proposition 1.4. *For any Shimura datum (G,X), we have*

$$Sh(G,X) = \big(G(\mathbb{Q})/Z_G(\mathbb{Q})\big)\Big\backslash X \times \big(G(\mathbb{A}_{\mathbb{Q}}^{\infty})/\overline{Z_G(\mathbb{Q})}\big)$$

where $\overline{Z_G(\mathbb{Q})}$ is the closure of $Z_G(\mathbb{Q})$ in the adelic topology of $Z_g(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

If $Z_G(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_{\mathbb{Q}}^{\infty})$, then we simply have

$$Sh(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty}).$$

Until now, we have just considered these spaces as topological manifolds. However more is true, as we see in the next subsection.

1.3 The Geometry of $Sh_K(G,X)$

Now fix $h \in X$. Suppose M_h is the stabilizer in $G_{\mathbb{R}}$ of the image of $h : \mathbb{S} \to G_{\mathbb{R}}$; then $M_h(\mathbb{R}) = A_G K_h$ where K_h is a maximal connected compact subgroup of $G(\mathbb{R})$ (follows from SD2), and we have $X \cong G(\mathbb{R})/A_G K_h$. Then one immediately sees that $\mathrm{Lie}(M_h) = \mathfrak{m}_h$. Also, $\mathrm{ad}h(i)$ is an involution and so acts by either ± 1 on each element of $\mathfrak{g}$; we see immediately from SD2 that $\mathrm{ad}h(i)$ acts on $\mathfrak{g}_{\mathbb{C}}$ as -1 on $\mathfrak{p}_h^+ \oplus \mathfrak{p}_h^-$ and as 1 on $\mathfrak{m}_h$. Thus, the -1-eigenspace $\mathfrak{g}^{ad=-1}$ of $\mathrm{ad}h(i)$ in $\mathfrak{g}$ satisfies $\mathfrak{g}_{\mathbb{C}}^{ad=-1} = \mathfrak{p}_h^+ \oplus \mathfrak{p}_h^-$, and we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{g}_h \oplus \mathfrak{g}^{ad=-1}$$

corresponding to the Cartan involution $\mathrm{Ad}h(i)$. One checks from a Hodge decomposition argument that $\mathfrak{p}_h^-$ is a commutative Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

Now, we have an obvious bijection $G(\mathbb{R})/A_G K_h \to X$; this endows X with the structure of a smooth manifold. Moreover, we see that $\mathfrak{p}_h$ is the tangent space of X at h. We can endow X with a complex structure by setting the action of i on the tangent space $\mathfrak{p}_h$ to be given by $\mathrm{ad}(h(\zeta_4))$, where ζ_4 is a square root of i (that this actually yields a complex structure is non-trivial). Hence, X is a complex manifold. Since $\Gamma_i := gKg^{-1} \cap G(\mathbb{Q})_+$ acts discretely on X for K small enough, we see that $\Gamma_i\backslash X^+$ are complex manifolds for all i, and hence $Sh_K(G,X)$ is a complex manifold for all K small enough.

We can in fact do something better.

Proposition 1.5. *For K sufficiently small, $Sh_K(G,X)$ is a quasi-projective complex algebraic variety.*

Remark 1.6. This is shown by taking the Baily-Borel compactification of $Sh_K(G,X)$, which can be described easily if G^{ad} has no factors of dimension 3. In this case, if Ω^1 is the sheaf of differentials on $Sh_K(G,X)$ and we set $\omega := \bigwedge^d \Omega^1$ where $d = \dim X$, then we have the graded ring

$$A := \bigoplus_{n \geq 0} \Gamma\big(Sh_K(G,X), \omega^{\otimes n}\big),$$

and we have a canonical inclusion $Sh_K(G,X) \hookrightarrow \mathrm{Proj} A$. The closure of the image of this map is the Baily-Borel compactification of $Sh_K(G,X)$. If however G^{ad} has factors of dimension 3, then we must replace $\Gamma\big(Sh_K(G,X), \omega^{\otimes n}\big)$ with the group of sections having at worst logarithmic singularities along the boundary of some smooth compactification of $Sh_K(G,X)$.

See [Lip] for slightly more detail.

Proposition 1.7 (Borel)**.** *Let G be a reductive group over $\mathbb{Q}$. Then, the space Sh_K is compact for some $K \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ (equivalently, for all such compact open K) if and only if G^{der} is $\mathbb{Q}$-anisotropic.*

Now, let P_h be the subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{p}_h := \mathfrak{p}_h^- \oplus \mathfrak{m}_h$ (this is the 0th step in the Hodge filtration of $\mathfrak{g}_{\mathbb{C}}$ induced by μ_h), and let $U_h \subset P_h$ be the subgroup with Lie algebra $\mathfrak{p}_h^-$. We have $G(\mathbb{R}) \cap P_h(\mathbb{C}) = M_h(\mathbb{R})$

Proposition 1.8 ([Mil90, Chap 3, Prop 1.1])**.** *The subgroup P_h is a parabolic subgroup of $G_{\mathbb{C}}$ with Levi component $M_{h,\mathbb{C}}$ and with unipotent radical U_h. We thus have a smooth open embedding*

$$\beta = \beta_h : X \hookrightarrow G(\mathbb{C})/P_h(\mathbb{C}),$$

called the Borel embedding*, of X into the projective complex algebraic variety $\check{X} := (G/P_h)(\mathbb{C})$. This embedding is equivariant with respect to the $G(\mathbb{R})$-action on X and the $G(\mathbb{C})$-action on $\check{X}$.*

Definition. We call $\check{X}$ the *compact dual of X*.

Note that $(G/P_h)(\mathbb{C})$ is indeed a generalized flag variety. We can view this embedding on points as a certain embedding of X into a flag variety directly as follows.

For $h \in X$, consider the cocharacter μ_h defined by

$$\mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}, \quad \mu_h(z) = h(z,1)$$

where we identify $\mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m,\mathbb{C}}^2$ as above. We describe how this cocharacter defines a filtration functor, following [Mil90].

In general, if μ is a cocharacter of an algebraic group G over a field k of characteristic 0, then for any k-representation $\rho : G \to GL(V)$ we can attach a filtration

$$\cdots \supset F^pV \supset F^{p+1}V \supset \cdots, \quad F^pV := \oplus_{q \geq p} V^q$$

on V where

$$V = \bigoplus_{q \in \mathbb{Z}} V^q$$

where $\rho \circ \mu : \mathbb{G}_{m,k} \to GL(V)$ acts on V^q by the character $z \mapsto z^q$. One checks that this defines a symmetric monoidal functor

$$\mathrm{Filt}(\mu) : \mathrm{Rep}_k(G) \to \mathrm{FiltVect}_k$$

such that the diagram of symmetric monoidal functors

$$\begin{array}{ccc} \mathrm{Rep}_k(G) & \xrightarrow{\mathrm{Filt}(\mu)} & \mathrm{FiltVect}_k \\ & \searrow \quad \swarrow & \\ & \mathrm{Vect}_k & \end{array}$$

where the vertical functors are the forgetful functors.

Remark 1.9. Though we will not need this fact, every such functor $\mathrm{Rep}_k(G) \to \mathrm{FiltVect}_k$ arises (non-uniquely) from a cocharacter μ in such a way.

Now, suppose we fix a faithful representation $\rho : G \to GL(V)$. Then to each point $h \in X$ we have an associated $G(\mathbb{C})$filtration $\mathrm{Filt}(\mu_h)(V)$ on V, and so this gives a map from X to a Grassmann variety. One checks that P_h is the stabilizer of the filtration $\mathrm{Filt}(\mu_h)(V)$, and so in fact the $G(\mathbb{C})$ conjugacy class of filtrations of V containing $\mathrm{Filt}(\mu_h(V))$ for all $h \in X$ is precisely $G(\mathbb{C})/P_h(\mathbb{C}) \cong \check{X}$. Hence, the embedding $X \hookrightarrow \check{X}$ can be viewed as the map $h \mapsto \mathrm{Filt}(\mu_h)(V)$.

1.4 Canonical Models

A lot can be said about canonical models of Shimura varieties. We summarize things here.

Consider the map $X \to X_*(G)_{\mathbb{C}}$, given by sending h to μ_h. For $h, h' \in X$, we see that μ_h and $\mu_{h'}$ are $G(\mathbb{C})$-conjugate in $X_*(G)_{\mathbb{C}}$. Thus the Shimura datum in particular picks out a $G(\mathbb{C})$-conjugacy class M_X of cocharacters in $X_*(G)_{\mathbb{C}}$. However, G is defined over $\mathbb{Q}$. It thus follows that there exists a minimal number field $E = E(G, X)$ such that M_X is the base-change to $\mathbb{C}$ of a $G(E)$-conjugacy class of cocharacters in $X_*(G)_E$.

Definition. Such a minimal number field E is the *reflex* field of the Shimura datum (G, X).

It is a hard fact of Deligne's that every Shimura variety admits a *canonical model* over E, i.e. we can find a unique inverse system $M(G, X) = (M_K(G, X))_K$ of varieties over K with a $G(\mathbb{A}_{\mathbb{Q}}^\infty)$ action such that there is a $G(\mathbb{A}_{\mathbb{Q}}^\infty)$-equivariant isomorphism of $\mathbb{C}$-varieties $Sh_K(G, X) \cong M_K(G, X) \times_E \mathbb{C}$ and $Sh(G, X) \cong M(G, X) \times_E \mathbb{C}$. There are other requiremens for a model over E to be a canonical model, but we ignore them for now.

It is also known that the projective variety $\check{X}$ is in fact also defined over the reflex field E.

2 Local Systems and Vector Bundles on Shimura Varieties

2.1 Review of Local Systems and the Riemann-Hilbert Correspondence

Suppose for now S is a complex manifold.

Definition. A *k-local system of vector spaces* on S is a locally constant sheaf of k-vector spaces on S.

Given a k-local system $\mathcal{V}$ of vector spaces we get a so-called *monodromy representation*, as follows. Notice first that if two points lie in the same (path-)connected component, the fibres over the two points will be (non-canonically) isomorphic via the point. Moreover, this isomorphism between the fibres is uniquely determined by (and uniquely determines) the path we chose between the two points. Thus, if S is connected, specifying a local system $\mathcal{V}$ is the same as (upon picking a base point $x \in S$) specifying a vector space $V \cong \mathcal{V}_x$ and isomorphisms $\sigma_\gamma : V \cong V$ along every loop γ such that if two paths are homotopic, then the isomorphism must be the identity. Therefore, a k-local system $\mathcal{V}$ on a connected space S is the same as a k-representation $\sigma : \pi_1(S, x) \to GL(V)$ where $V := \mathcal{V}_x$.

Remark 2.1. Similar definitions can be made for a scheme S. In this case, we also obtain a monodromy representation from a local system, though this time the representation is of the étale fundamental group.

We now consider connections and flat bundles on schemes. We follow [Fon], though Brian Conrad's notes also seem to be pretty good. Suppose X is a scheme over S with structure map $\pi : X \to S$, and let $\mathbb{V}$ be a quasi-coherent $\mathcal{O}_X$-module (for instance, $\mathbb{V}$ could be a locally free sheaf, i.e. a vector bundle on X).

Definition. A connection on $\mathbb{V}$ is a morphism of $\pi^{-1}\mathcal{O}_S$-modules

$$\nabla : \mathbb{V} \to \mathbb{V} \otimes_{\mathcal{O}_X} \Omega^1_{X/S},$$

satisfying the Leibniz rule

$$\nabla(f \otimes v) = v \otimes df + f \cdot \nabla(v)$$

for $f \in \mathcal{O}_X(U)$ and $v \in \Gamma(U, \mathbb{V})$, where U is any open S-subscheme of X.

Let $T_{X/S} := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X)$ be the relative tangent sheaf (sections of this sheaf are vector fields). A connection induces (and is uniquely determined by) a morphism

$$T_{X/S} \to \underline{\mathrm{End}}_{\mathcal{O}_S}(\mathbb{V}), \quad \text{written } v \mapsto \nabla_v$$

where $\nabla_v(e) = \langle v, \nabla e \rangle$ is the *covariant derivative of e along v*. Notice that

$$\nabla_v(fe) - v(f)e + f\nabla_v e$$

where $v(f) = \langle v, df \rangle$ is the canonical pairing of vector fields with 1-forms.

Example 2.2. $\mathcal{O}_X$ can be equipped with the connection $d : \mathcal{O}_X \to \Omega^1_{X/S} = \mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$.

Definition. A connection is *flat* if for any local sections v, w of $T_{X/S}$ we have

$$\nabla_{[v,w]} = \nabla_v \circ \nabla_w - \nabla_w \circ \nabla_v.$$

A bundle is *flat* if it can be equipped with a flat connection.

Of course, all of these definitions carry over to complex manifolds as well (i.e. we take $S = \mathbb{C}$). In fact, these definitions are compatible with Serre's GAGA.

Theorem 2.3 (Riemann-Hilbert Correspondence)**.** *Suppose X is a complex manifold. Then, the category of flat vector bundles on M is equivalent to the category of $\mathbb{C}$-local systems of finite rank.*

Let us write down this categorical equivalence. Suppose $(\mathbb{V}, \nabla)$ is a flat vector bundle. We have a $\mathbb{C}$-local system $\mathcal{V} = \mathbb{V}^\nabla$ given by

$$\mathcal{V}(U) = \{e \in \mathbb{V}(U) : \nabla e = 0\}.$$

On the other hand, suppose $\mathcal{V}$ is a $\mathbb{C}$-local system of finite rank. Set $\mathbb{V} := \mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_X$. Then, the map

$$\mathrm{id} \otimes d : \mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{V} \otimes_{\mathbb{C}} \Omega^1_X = (\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega^1_X$$

is in fact a flat connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X.$$

2.2 Automorphic Vector Bundles

Our exposition will (mostly) be pulling from [Su18], [Har85], and [Mil90]. The following is a key definition.

Definition. Let S be an algebraic variety over a field k equipped with an action of an algebraic group G. A *G-vector bundle* on S is a vector bundle $\mathbb{V}$ on S together with an action of G on the total space of $\mathbb{V}$ (as an algebraic variety), such that

- $p(g \cdot v) = g \cdot p(v)$ for all $g \in G$ and $v \in \mathbb{V}$, where $p : \mathbb{V} \to S$ is the projection map from the total space of the vector bundle onto S; and
- the maps $g : \mathbb{V}_s \to \mathbb{V}_{gs}$ are linear for all $s \in S$.

We have the obvious analog for vector bundles on manifolds with a Lie group action.

The idea behind automorphic vector bundles is simply this: we want to take a $G_{\mathbb{C}}$ vector bundle $\mathcal{V}$ on $\check{X}$. We want to descend this vector bundle to get a vector bundle $\mathcal{V}_K$ on $Sh_K(G, X)$ for each level K. This bundle turns out to be algebraic. Manifestly, the global sections of $\mathcal{V}_K$ are functions on X that are invariant with respect to discrete subgroups of the form $K \cap G(\mathbb{Q})$, which is precisely what we want for automorphic forms. However, there are technical complications that can arise when constructing $\mathcal{V}_K$ from $\mathcal{V}$, since it is unclear whether after quotienting by K the vector space structure on the stalks still survives.

A stupid obstruction to constructing $\mathcal{V}_K$ is if the groups $gKg^{-1} \cap G(\mathbb{Q})_+$ fail to act discretely with no fixed points, for then even $Sh_K(G, X)$ fails to be a manifold. Thus, for instance, we would need to assume that our level K is neat. Further technical obstructions are in fact closely related to this one; we need the groups $gKg^{-1} \cap G(\mathbb{Q})_+$ to act discretely without fixed points on the vector bundle $\mathcal{V}$ as well! Different authors impose slightly (at least superficially) different conditions:

1. Let Z_G^s denote the largest subtorus of Z_G that splits over $\mathbb{R}$ such that no subtorus of Z_G^s splits over $\mathbb{Q}$. Then, we require that $Z_g^s(\mathbb{C})$ acts trivially on $\mathcal{V}$ (this is how it is phrased in [Mil90]).
2. The maximal $\mathbb{Q}$-split torus in Z_G is also the maximal $\mathbb{R}$-split torus in Z_G (this is how it is phrased in [Su18]).

From now on, we will suppose that one of the two conditions hold, and that also G is connected. Fix a point $o \in X$, and let P_o be the corresponding parabolic subgroup of $G_{\mathbb{C}}$ so that $\check{X} \cong G(\mathbb{C})/P_o(\mathbb{C})$. Then the Levi factor of P_o is $M_{o,\mathbb{C}}$, where M_o is the stabilizer in $G_{\mathbb{R}}$ of the image of $h : \mathbb{S} \to G_{\mathbb{R}}$. Recall that $M_o(\mathbb{R}) = A_G K_o$ where K_o is a maximal compact subgroup of $G(\mathbb{R})$.

Suppose that $\mathcal{V}$ is a $G_{\mathbb{C}}$-vector bundle on $\check{X}$. The Borel embedding $\beta_o : X \hookrightarrow \check{X}$ is an open embedding, and so the sheaf $\beta_o^* \mathcal{V}$ is still a vector bundle on X. We then obtain a $G(\mathbb{A}_{\mathbb{Q}}^\infty)$-homogeneous holomorphic vector bundle $\beta^* \mathcal{V} \times \big(G(\mathbb{A}_{\mathbb{Q}}^\infty)/K\big)$ over $X \times \big(G(\mathbb{A}_{\mathbb{Q}}^\infty)/K\big)$. Under the above assumptions, for a neat level K the group $G(\mathbb{Q})$ acts freely on $X \times \big(G(\mathbb{A}_{\mathbb{Q}}^\infty)/K\big)$, and so we obtain a vector bundle

$$\mathcal{V}_K := G(\mathbb{Q}) \backslash \beta_o^* \mathcal{V} \times G(\mathbb{A}_{\mathbb{Q}}^\infty)/K$$

on $Sh_K(G, X)$.

Definition. An *automorphic vector bundle* is a bundle $\mathcal{V}_K$ on $Sh_K(G, X)$ obtained by the above construction from a $G_{\mathbb{C}}$-vector bundle $\mathcal{V}_K$ on $\check{X}$.

For each $g \in G(\mathbb{A}_{\mathbb{Q}}^\infty)$ and pair of neat levels K and K' such that $K' \subset gKg^{-1}$, we get a morphism

$$\mathcal{V}_K \to \mathcal{V}_{K'}, \quad [x, a] \mapsto [x, ag]$$

just as we had for $Sh_K(G, X)$. It is clear that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{V}_{K'} & \longrightarrow & \mathcal{V}_K \\ \big\downarrow & & \big\downarrow \\ Sh_{K'}(G, X) & \xrightarrow{T_g} & Sh_K(G, X) \end{array} \tag{1}$$

Proposition 2.4. *The vector bundles $\mathcal{V}_K$ and the maps $\mathcal{V}_{K'} \to \mathcal{V}_K$ are algebraic.*

Proposition 2.5. *If G has no factors of dimension 3, then every holomorphic section of $\mathcal{V}_K$ is algebraic, and the space of such sections is finite-dimensional over $\mathbb{C}$.*

Remark 2.6. The requirement that G has no factors of dimension 3 is due to the same issues that show up in Baily-Borel compactification. In this case, X^+ has a factor isomorphic to the unit disk (or what is essentially the same thing, the upper half-plane). Thus, one has to take care of logarithmic singularities occurring at the cusps, and so extra conditions are required.

Of course, we can take the limit over all K to then get a vector bundle $[\mathcal{V}]$ on Sh. As usual, we get an action of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ on $[\mathcal{V}]$, and it turns out that this makes $[\mathcal{V}]$ a $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-homogeneous vector bundle over Sh. The main theorem of [Har85] is the following.

Theorem 2.7. *The functor $\mathcal{V} \to [\mathcal{V}]$ from $G_{\mathbb{C}}$-homogeneous vector bundle on $\check{X}$ to $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-homogeneous vector bundles on Sh is rational over the reflex field E of the Shimura data (G, X).*

Concretely, if $\mathcal{V}$ is defined over E, then so is $[\mathcal{V}]$.

2.3 Computing Automorphic Vector Bundles

We can construct $G(\mathbb{C})$-vector bundles as follows. If $\rho : P_o \to GL(V)$ is an algebraic representation with V a finite dimensional $\mathbb{C}$-vector space, then we set

$$\tilde{V} := G(\mathbb{C}) \times^{P_o(\mathbb{C})} V = \big(G(\mathbb{C}) \times V\big)/P_o(\mathbb{C})$$

where $p \in P_o(\mathbb{C})$ acts on $(g, v) \in G(\mathbb{C}) \times V$ by $p \cdot (g, v) = (gp^{-1}, \rho(p)v)$. There is an obvious projection map $\tilde{V} \to \check{X}$, and it is easy to see that $\tilde{V}$ is a vector bundle over $\check{X}$. The $G(\mathbb{C})$-action given by multiplying on the left on the first coordinate makes $\tilde{V}$ a $G(\mathbb{C})$-homogeneous bundle. We thus have a vector bundle $\tilde{V}_K$ on $Sh_K(G, X)$.

Remark 2.8. The above construction actually defines an equivalence of (symmetric monoidal) categories between the category of finite-dimensional representations of P_o, and the category of $G(\mathbb{C})$-vector bundles on $\check{X} = G(\mathbb{C})/P_o(\mathbb{C})$ [Har90]. If $\mathcal{V}$ is a $G(\mathbb{C})$-vector bundle on $\check{X}$, then the fibre $\mathcal{V}_o$ at the base point $o \in X \subset \check{X}$ has a natural action of $P_o(\mathbb{C})$.

Now, if V is a $M_{o\mathbb{C}}$-representation, then under the Levi projection $P_o \twoheadrightarrow M_o$ we view V as a P_o-representation, and so we have a $G(\mathbb{C})$-vector bundle on $\check{X}$. It turns out that this vector bundle is semi-simple, and in fact, the above equivalence restricts to an equivalence between the category of finite-dimensional representations of $M_{o\mathbb{C}}$ and *semi-simple* $G(\mathbb{C})$-vector bundles on $\check{X}$ [EH17].

Remark 2.9. I think $\beta_o^* \tilde{V} \cong G(\mathbb{R}) \times^{K_o(\mathbb{R})} V$ as $G(\mathbb{R})$-homogeneous vector bundles on X, where the action of $K_o(\mathbb{R})$ on V is obtained by restriction.

Example 2.10. If we take $V := \bigwedge^p(\mathfrak{g}_{\mathbb{C}}/\mathrm{Lie}(P_o))$, then $\tilde{V} = \Omega^p_{\check{X}}$ is the bundle of smooth p-forms on $\check{X}$. It then follows that

$$\tilde{V}_K = \Omega^p_{Sh_K(G,X)}$$

is the bundle of smooth p-forms on $Sh_K(G, X)$.

Let $\mathcal{O}$ denote the sheaf of holomorphic functions on the complex manifold $Sh_K(G, X)$. As usual with algebraic vector bundles, we can view $\tilde{V}_K$ as a locally free sheaf of $\mathcal{O}$ modules as follows. Consider the quotient map

$$\pi_K : G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/A_G K = G(\mathbb{Q})\backslash\big(G(\mathbb{R})/A_G \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K\big) \twoheadrightarrow Sh_K(G, X).$$

Let $G(\mathbb{A}_{\mathbb{Q}}) = A_G G^1(\mathbb{A}_{\mathbb{Q}})$ where G^1 is the intersection of the kernels of all $\mathbb{Q}$-characters of G. Write $\mathfrak{g}^1 := Lie(G^1)$, and write $\mathfrak{p}_o^1 = \mathfrak{p}_o \cap \mathfrak{g}^1$; then writing $\mathfrak{a}_G = Lie(A_G)$ we have $\mathfrak{p}_h = \mathfrak{p}_o \oplus \mathfrak{a}_G$. There is a natural $(\mathfrak{p}_o^1, K_o)$-module structure on the space of smooth functions

$$C^\infty(\pi_K^{-1}(U)) \otimes V,$$

for $U \subset Sh_K(G, X)$ open, where V is a $P_o(\mathbb{C})$-module and so is a $(\mathfrak{p}_o^1, K_o)$-module, whereas $\mathfrak{p}_o^1$ acts on $C^\infty(\pi_K^{-1}(U))$ by right differentiation and K_o acts on $C^\infty(\pi_K^{-1}(U))$ by the right regular representation.

Proposition 2.11. *$\tilde{V}_K$, viewed as a sheaf of $\mathcal{O}$-modules, is the sheaf*

$$U \mapsto \big(C^\infty(\pi_K^{-1}(U)) \otimes V\big)^{(\mathfrak{p}_o^1, K_o)},$$

where for concreteness

$$\big(C^\infty(\pi_K^{-1}(U)) \otimes V\big)^{(\mathfrak{p}_o^1, K_o)} = \big\{f : \pi_K^{-1}(U) \to V : f \text{ smooth}, \ v(f) = 0 \ \forall v \in \mathfrak{p}_o^1, \text{ and } f(gk) = f(g) \ \forall k \in K_o\big\}$$

with V endowed with the usual analytic topology induced by the $\mathbb{C}$-vector space structure on V.

Now suppose K, K' are compact open subgroups of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. The commuting square (1) yields a natural isomorphism $T_g^*\tilde{V}_K \cong \tilde{V}_{K'}$. By the pull-back–push-forward adjunction, we thus have a natural map

$$\tilde{V}_K \to (T_g)_*\tilde{V}_{K'}.$$

It is straightforward to check that as sheaf maps, this is simply the map

$$\left(C^\infty(\pi_K^{-1}(U)) \otimes V\right)^{(\mathfrak{p}_o^1, K_o)} = \tilde{V}_K(U) \to (T_g)_*\tilde{V}_{K'}(U) = \left(C^\infty\left(\pi_{K'}^{-1}(T_g^{-1}(U))\right) \otimes V\right)^{(\mathfrak{p}_o^1, K_o)}$$
$$f \mapsto gf(\cdot\, g)$$

for every open $U \subset Sh_K(G,X)$. Here, we can define an action of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ action on V as follows: for any $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$, write $a_g \in A_G$ to be the unique element such that $g \in a_g G^1(\mathbb{A}_{\mathbb{Q}}^{\infty})$. Then g acts on V by multiplication by a_g^{-1}.

2.4 Automorphic Local Systems

Suppose now V is a representation of $G_{\mathbb{C}}$. Then we have a local system

$$\underline{V}_K := G(\mathbb{Q})\backslash(V \times X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K)$$

on $Sh_K(G,X)$, for every compact open neat subgroup $K \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. Here, $G(\mathbb{Q})$ acts on V via the inclusion $G(\mathbb{Q}) \hookrightarrow G(\mathbb{C})$.

Example 2.12. For V the trivial representation of $G_{\mathbb{C}}$, we have $\underline{V}_K = \underline{\mathbb{C}}$, the sheafification of the constant sheaf $\mathbb{C}$.

Of course, any representation $\rho : G_{\mathbb{C}} \to GL(V)$ of $G_{\mathbb{C}}$ restricts to a representation of P_o, and so defines the automorphic vector bundle $\mathcal{V}_K$. It turns out that under the Riemann-Hilbert correspondence, the local system $\underline{V}_K$ corresponds to the vector bundle $\mathcal{V}_K$ equipped with a certain flat connection. We can construct this flat connection from ρ directly (we follow the construction in Section 5 of [GP02]). In fact, we define a $G(\mathbb{R})$-invariant connection

$$\nabla : \tilde{V}_K \to \tilde{V}_K \otimes_{\mathcal{O}} \Omega^1_{Sh_K(G,X)}$$

by defining a morphism $T_{Sh_K} \to \underline{\mathrm{End}}(\tilde{V}_K)$. First, notice that $\rho : G_{\mathbb{C}} \to GL(V)$ induces a map $d\rho : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(V)$. Now, consider any local vector field v on Sh_K and local (holomorphic) section f of $\tilde{V}_K$. We view f as a smooth map $f : \pi_K^{-1}(U) \to V$ such that $d\rho|_{\mathfrak{p}_o^1} \cdot f \equiv 0$ and $f(gk) = \rho(k^{-1})f(g)$, where

$$\pi_K : G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/A_G K \to Sh_K.$$

Then, we set $\nabla_v f$ to be the section

$$(\nabla_v f)(g) := \tilde{v}(f)(g) + \rho(g)d\rho(g^{-1} \cdot \tilde{v}) \cdot f(g),$$

where $\tilde{v}$ is any lift of v under the map π_K, and we consider $\rho(g)d\rho(g^{-1} \cdot \tilde{v}_g) \in \mathrm{End}(V)$ acting on $f(g) \in V$ (where g^{-1} pushes $v_g \in T_g G(\mathbb{C})$ to $T_0 G(\mathbb{C}) = \mathfrak{g}$).

Remark 2.13. In fact, the functor taking a representation $\rho : G \to GL(V)$ to $\mathcal{V}_K$ equipped with the above flat connection defines an equivalence of (symmetric monoidal) categories between the category of finite-dimensional representations of $G_{\mathbb{C}}$ and the category of flat automorphic vector bundles on Sh_K.

2.5 A Note on Automorphy Factors

Classically, automorphic forms were defined using so-called automorphy factors. That is, an automorphic form for Γ of type J, for J an automorphy factor, was a function $f : X^+ \to V$ such that $f(\gamma x) = J(\gamma, x)f(x)$ and satisfying other nice conditions (holomorphicity, growth conditions at ∞, etc). For instance, the automorphy factor for a discrete subgroup Γ of $SL_2(\mathbb{Z})$ acting on the upper half plane with values in $\mathbb{C}$ given by

$$J\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) = (cz+d)^k$$

is classically used to define weight k-modular forms.

Suppose $\rho : P_o \to GL(V)$ is a representation with corresponding automorphic vector bundle $\tilde{V}_K$. Fix a base point $x_0 \in X$ (say $x_0 = [1] \in G(\mathbb{R})/K_o(\mathbb{R})$).

Definition. A (holomorphic) *automorphy factor for* $\tilde{V}_K$ is a smooth map $J : G(\mathbb{R}) \times X \to GL(V)$ such that

- $J(g,-) : X \to GL(V)$ is holomorphic for all $g \in G(\mathbb{R})$,
- $J(gg', x) = J(g, g'x)J(g', x)$ for all $g, g' \in G(\mathbb{R})$ and $x \in X$, and
- $J(k, x_0) = \rho(k)$ for all $k \in P_o(\mathbb{C}) \cap G(\mathbb{R}) = K_o(\mathbb{R})$.

An automorphy factor J determines a holomorphic trivialization

$$\Phi_J : \beta_o^* \tilde{V} \cong G(\mathbb{R}) \times^{K_o(\mathbb{R})} V \to X \times V, \quad [h, v] \mapsto (hx_0, J(h, x_0)v)$$

where the action of $G(\mathbb{R})$ is $g \cdot (x, v) = (gx, J(g, x)v)$.

2.6 Hecke Action on Cohomology of $\tilde{V}_K$

We now construct an action of $G(\mathbb{A}_{\mathbb{Q}}^\infty)$ on the cohomology of bundles on Shimura varieties. Throughout, we fix a representation (ρ, V) of K_o. We follow the construction in [Nic20]. A slightly less general construction is carried out in [GH22], where some concrete discussion also takes place.

Let K and K' be arbitrary compact open neat subgroups of $G(\mathbb{A}_{\mathbb{Q}}^\infty)$, and let $g \in G(\mathbb{A}_{\mathbb{Q}}^\infty)$ be arbitrary. We have the finite étale map

$$\pi_{K \cap g^{-1}K'g, K} : Sh_{K \cap g^{-1}K'g} \twoheadrightarrow Sh_K,$$

which induces a map on cohomology

$$\pi^*_{K \cap g^{-1}K'g, K} : H^\bullet(Sh_K, \tilde{V}_K) \to H^\bullet\left(Sh_{K \cap g^{-1}K'g}, \tilde{V}_{K \cap g^{-1}K'g}\right).$$

Now, we have the right multiplication isomorphism

$$[g^{-1}] : Sh_{gKg^{-1} \cap K'} \xrightarrow{\cong} Sh_{K \cap g^{-1}K'g},$$

which yields an isomorphism

$$[g^{-1}]^* : H^\bullet(Sh_{K \cap g^{-1}K'g}, \tilde{V}_{K \cap g^{-1}K'g}) \to H^\bullet\left(Sh_{gKg^{-1} \cap K'}, \tilde{V}_{gKg^{-1} \cap K'}\right).$$

We finally construct a *trace* map

$$Tr_{gKg^{-1} \cap K', K'} : H^\bullet\left(Sh_{K'}, \tilde{\pi}_{gKg^{-1} \cap K', K' *} V_{gKg^{-1} \cap K'}\right) \to H^\bullet(Sh_{K'}, \tilde{V}_{K'}).$$

Write $K'' := gKg^{-1} \cap K'$ for notational simplicity. Since $\pi_{K'', K'}$ is finite étale, the functors $(\pi_{K'', K'})_!$ and $(\pi_{K'', K'})_*$ coincide. Under the shriek-pushforward–pullback adjunction, the identity map on $\pi^*_{K'', K'} \tilde{V}_{K'}$ induces a map

$$\pi_{K'', K' *} \pi^*_{K'', K'} \tilde{V}_{K'} \to \tilde{V}_{K'},$$

which on fibres is simply the map

$$\oplus_{x' \in \pi^{-1}_{K'', K'}(x)} V \cong \left(\pi_{K'', K'} {}_* \pi^*_{K'', K'} \tilde{V}_{K'}\right)_x \to V, \quad (v_{x'})_{x' \in \pi^{-1}_{K'', K'}(x)} \mapsto \sum_{x' \in \pi^{-1}_{K'', K'}(x)} v_{x'};$$

here, we have identified $(\tilde{V}_{K'})_x \cong V$. Since $\pi^*_{K'', K'} \tilde{V}_{K'} \cong \tilde{V}_{K''}$, we thus have a map

$$\pi_{K'', K' *} \tilde{V}_{K''} \to \tilde{V}_{K'},$$

and hence a trace map on cohomology

$$Tr_{K'', K'} : H^\bullet\left(Sh_{K'}, \pi_{K'', K' *} \tilde{V}_{K''}\right) \to H^\bullet(Sh_{K'}, \tilde{V}_{K'}).$$

Of course, the push-forward also induces the map

$$\pi_{K'', K' *} : H^\bullet\left(Sh_{K''}, \tilde{V}_{K''}\right) \to H^\bullet\left(Sh_{K'}, \pi_{K'', K' *} \tilde{V}_{K''}\right).$$

Remark 2.14. This is a general construction, the *Méthode de la trace*.

Composing these four maps together, we get the *Hecke operator* on cohomology

$$T_{KgK'} = \pi_{K'',K'*} \circ Tr_{K'',K'} \circ [g^{-1}]^* \circ \pi^*_{g^{-1}K''g,K} : H^\bullet(Sh_K, \tilde{V}_K) \to H^\bullet(Sh_{K'}, \tilde{V}_{K'}).$$

In particular, taking $K' = K$, we get Hecke operators

$$T_g : H^\bullet(Sh_K, \tilde{V}_K) \to H^\bullet(Sh_K, \tilde{V}_K).$$

It thus follows that we have an action of $G(\mathbb{A}_\mathbb{Q}^\infty)$ on the system

$$H^\bullet(Sh(G,X), \tilde{V}) := \varinjlim_{K \leq G(\mathbb{A}_\mathbb{Q}^\infty)} H^\bullet(Sh_K, \tilde{V}_K).$$

For a fixed k, since $H^k(Sh_K, \tilde{V}_K)$ is finite dimensional, and since $H^k(Sh(G,X), \tilde{V})^K = H^k(Sh_K, \tilde{V}_K)$ (resulting from the Hochschild-Sere spectral sequence), it follows that the $G(\mathbb{A}_\mathbb{Q}^\infty)$ on the profinite space $H^k(Sh(G,X), \tilde{V})$ is an admissible representation.

3 Toroidal Compactifications and Canonical Extensions of Bundles

We want to study the automorphic vector bundles constructed on the Shimura varieties Sh_K. We know that Sh_K is a quasi-projective variety, and is a Zariski open subset of a projective variety, the *Baily-Borel compactification* $\overline{Sh_K}^{BB}$ of Sh_K. However, while the Baily-Borel compactification is projective, it has a lot of singularities. Specifically, the 'boundary components' $\overline{Sh_K}^{BB} \backslash Sh_K$ are usually singular. In particular, extending automorphic vector bundles to $\overline{Sh_K}^{BB}$ may be bad, or at the very least, difficult. This is where *toroidal compactifications* are useful: they are essentially a family of resolutions of singularities of $\overline{Sh_K}^{BB}$ indexed by some combinatorial data. It turns out that automorphic vector bundles do admit extensions to toridal compactifications, and the cohomology of these extended bundles are interesting. Moreover, there are also nice compatibility-type results between the cohomology of these extended bundles over different toroidal compactifications.

Compactifications are usually constructed by taking the Hermitian symmetric space and adjoining 'boundary components' and then endowing this union with the *Satake topology.* The boundary components are usually indexed by various parabolic subgroups. Then, one tries to extend the action of the discrete group to this compactification so as to get the quotient.

Fix $o \in X$, and let X^+ be its connected component of X. We continue to let $H(\mathbb{R})^+$ denote the identity component in the real topology for an algebraic group H over $\mathbb{R}$. Let G^{der} be the derived subgroup of G and G^{ad} the adjoint group of G. Recall that $G(\mathbb{R})_+$ denotes the stabilizer of X^+ under the action of $G(\mathbb{R})$ on $\pi_0(X)$; equivalently, it the preimage of $G^{ad}(\mathbb{R})^+$ under the map $G(\mathbb{R}) \to G^{ad}(\mathbb{R})$.

Let $\overline{K}_o$ be the image of K_o under the map $G(\mathbb{R}) \to G^{ad}(\mathbb{R})$. Since the stabiliser in $G^{ad}(\mathbb{R})$ of a component is $G^{ad}(\mathbb{R})^+$ and K_o fixes X pointwise, it follows that $\overline{K}_o \subset G^{ad}(\mathbb{R})^+$. Thus

$$X^+ = G^{ad}(\mathbb{R})^+/\overline{K}_o.$$

Since X is a disjoint union of such X^+ (for varying $o \in X$), to compactify X it suffices to compactify X^+. Moreover, as the map $G(\mathbb{R}) \to G^{ad}(\mathbb{R})$ is a finite map and the map $G(\mathbb{R})^+ \to G^{ad}(\mathbb{R})^+$ is a finite covering map (i.e. is surjective as well), it then follows that $\overline{K}_o$ is a maximal compact subgroup of $G^{ad}(\mathbb{R})^+$.

Good expositional reference with examples is [Gor05]. Computations of toroidal compactifications of the modular curve and of Hilbert modular varieties are given in [Ash+10, Section I.4] and [Ash+10, Section I.5] respectively. The reference [Ash+10] is the canonical reference for toroidal compactifications; indeed, this is where toroidal compactifications were first written about. However, [Har89] translated toroidal compactifications to the adelic setting. We thus follow [Har89] for the sections on toroidal compactifications.

Throughout we assume that G^{ad} is connected.

3.1 Boundary Components

Consider the Borel embedding $X \hookrightarrow \check{X} = G(\mathbb{C})/P(\mathbb{C})$, where recall $\check{X}$ is a projective (generalized) flag variety. Let $\overline{X}^+$ be the closure of X^+ in $\check{X}$.

Remark 3.1. Some places define $\overline{X}^+$ as the closure of X^+ in Euclidean space $\mathbb{C}^m$ for some m, via the *Harish-Chandra embedding.* This is actually the same thing, since the Harish-Chandra embedding is an open embedding

$$X^+ \hookrightarrow \mathfrak{p}_o^+/\mathfrak{p}_o^+ \cap \mathrm{Lie}(Z_G),$$

and by [Ash+10, Theorem III.2.1], we have an open embedding

$$\mathfrak{p}_o^+/\mathfrak{p}_o^+ \cap \mathrm{Lie}(Z_G) \hookrightarrow \check{X}.$$

The image of X^+ in $\mathfrak{p}_o^+/\mathfrak{p}_o^+ \cap \mathrm{Lie}(Z_G)$ is explicitly described in [Ash+10, Theorem III.2.9].

Definition. A *boundary component* of the symmetric space X^+ is an analytic submanifold F that is also a single holomorphic path component, i.e. F is an equivalence class in $\overline{X}^+$ under the equivalence relation generated by $x\tilde{y}$ iff there exists a holomorphic map $\lambda : \Delta \to \overline{X}^+$ such that $x, y \in \lambda(\Delta)$, where $\Delta = \{|z| < 1\}$ is the open unit disk in $\mathbb{C}$.

Note that X^+ is also a boundary component by this definition. We say that a boundary component F is *proper* if $F \neq X^+$, i.e. $F \subset \partial X^+ = \overline{X}^+ \setminus X^+$.

A boundary component F is thus a maximal analytic submanifold of $\overline{X}^+$. We include X^+ as a 'boundary component', even though it is manifestly not the boundary of X^+ in $\overline{X}^+$, to make the statements of various results a lot neater.

Remark 3.2. In [Ash+10, Section III.3], one can precisely write down what boundary components F look like in terms of roots in $\mathfrak{g}$.

Here is a collection of all the important facts about boundary components. All of these results are in [Ash+10, Section III.3].

- $\overline{X}^+$ is the disjoint union of boundary components, and K_o acts on the set of boundary components via permutations.
- If $G^{ad}_{\mathbb{R}}$ decomposes over $\mathbb{R}$ into simple factors $G^{ad} = G_1 \times G_2 \times \cdots \times G_k$, then X^+ decomposes as
$$X^+ = X_1^+ \times X_2^+ \times \cdots \times X_k^+$$
into hermitian symmetric domains. The boundary components of X^+ are then precisely the products of boundary components of each of the simple factors X_i^+.
- Each boundary component is itself a bounded symmetric domain. Boundary components of a boundary component F of X^+ are also boundary components of X^+.
- For F a boundary component, the normaliser
$$\{g \in G(\mathbb{R}) : gF = F\}$$
form the real points of a parabolic subgroup $P_F \subset G^{ad}_{\mathbb{R}}$. Moreover, two boundary components are equal if and only if their normalizers coincide. Obviously, $P_{X^+} = G^{ad}$.
- If G^{ad} is $\mathbb{Q}$-simple, then for any proper boundary component F, P_F is a maximal parabolic of G^{ad}. Moreover, the association $F \leftrightarrow P_F$ is a 1-1 correspondence between maximal proper parabolics of $G^{ad}_{\mathbb{R}}$ and proper boundary components.

Definition. A boundary component F of X^+ is *rational* if the corresponding parabolic P_F is defined over $\mathbb{Q}$, i.e. is the base-change to $\mathbb{R}$ of a parabolic subgroup of G^{ad} over $\mathbb{Q}$. Obviously, X^+ is a rational boundary component.

There is another way to think about proper rational boundary components (c.f. [Lip]). Suppose P is a proper maximal parabolic of G^{ad}. Let N_P and M_P denote its unipotent radical and its Levi factor respectively. Let $K_P = \overline{K}_o \cap M_P(\mathbb{R})$. Finally, let A_P denote the identity component of $\mathbb{R}$-points of the maximal $\mathbb{Q}$-split torus in M_P. Consider the *reductive Borel-Serre component*

$$e_{RBS}(P) = P(\mathbb{R})/N_P(\mathbb{R})A_P K_P.$$

Then, it turns out that $e_{RBS}(P)$ contains a unique factor F that is a Hermitian symmetric domain. It turns out that F is the unique boundary component of X^+ satisfying $P_F = P$.

3.2 Baily-Borel compactification and the Satake topology

Let $\overline{X^+}^{BB}$, as a set, be the disjoint union of X^+ with its rational boundary components. Obviously, this is a subset of $\overline{X}^+ \subset \check{X}$. However, the topology we will endow on $\overline{X^+}^{BB}$ is going to be different from the subspace topology induced from $\check{X}$.

The reason for this is simple. Given a discrete subgroup $\Gamma < \mathrm{Aut}(X^+)$ (say $\Gamma = G(\mathbb{Q}) \cap K$ for $K \subset G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ a compact open), we can extend the action of Γ to (the set) $\overline{X^+}^{BB}$. For example, if $X^+ = \mathfrak{h}$ is the upper half plane, then $\overline{X^+}^{BB}$ is given by $\overline{\mathfrak{h}} = \mathfrak{h} \cup \mathbb{Q} \cup \{\infty\}$ as expected. Following the example of the modular curve, we would like to define the Baily-Borel compactification of $\Gamma \backslash X^+$ to be $\Gamma \backslash \overline{X^+}^{BB}$. However, this latter quotient is not Hausdorff. The issue is that Γ fails to act discretely on $\overline{X^+}^{BB} \backslash X^+$, and this is simply because the subspace topology from the Borel embedding does not provide enough open sets. We thus need to retopologise. We in fact see this issue crop up in the example of the upper half plane. Endowing $\overline{\mathfrak{h}}$ with the subspace topology from $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, the boundary is very pathological, and the quotient $SL_2(\mathbb{Z}) \backslash \overline{\mathfrak{h}}$ fails to be Hausdorff at the cusp ∞.

We retopologise with the *Satake topology*. For a point $p \in X^+ \subset \overline{X^+}^{BB}$, this is just the usual topology on X^+. On the other hand, suppose $p \in \overline{X^+}^{BB} \setminus X^+$. Fix some choice of minimal $\mathbb{Q}$-parabolic of G^{ad}. Fix any arithmetic subgroup $\Gamma \subset G^{ad}(\mathbb{Q})$ and let $\Omega_\Gamma = C \cdot \mathfrak{S} \subset X^+$ be some fundamental set for Γ, where $C \subset G^{ad}(\mathbb{Q})$ is some finite subset and where $\mathfrak{S}$ is a Siegel set with respect to the minimal $\mathbb{Q}$-parabolic (see [GH22, Section 2.7] for more on Siegel sets). Then, by definition, a fundamental system of neighbourhoods of p in the Satake topology is given by those subsets $U \subset \overline{X^+}^{BB}$ such that

- $\Gamma_p \cdot U = U$, where $\Gamma_p = \{\gamma \in \Gamma : \gamma \cdot p = p\}$; and
- $\gamma \cdot U \cap \overline{\Omega}$ is a neighbourhood for $\gamma \cdot x$ in $\overline{\Omega}$, where $\overline{\Omega}$ is the closure of Ω in $\check{X}$.

For example, for $X^+ = \mathfrak{h}^+$ the upper half-plane, the choice of minimal $\mathbb{Q}$-parabolic corresponding to the cusp ∞, and for the choice of $\Gamma := SL_2(\mathbb{Z})$, the open sets U satisfying the above two conditions are precisely given by

$$U = \{\infty\} \cup \{z \in \mathfrak{h} : \mathrm{Im}(z) > C\}$$

for $C > 0$.

The Satake topology is characterised as follows.

Theorem 3.3 ([Ash+10, Theorem III.6.1]). *The Satake topology on $\overline{X^+}^{BB}$ is the unique topology satisfying the following properties:*

1. *it induces the natural topology on X^+ and on the closures $\overline{\mathfrak{S}}$ of any Siegel set $\mathfrak{S}$, where the closure is taken in $\check{X}$;*
2. *the group $G(\mathbb{Q})$ acts continuously on $\overline{X^+}^{BB}$;*
3. *for any Γ an arithmetic subgroup of $G^{ad}(\mathbb{Q})$, if $p, p' \in \overline{X^+}^{BB}$ are not in the same Γ-orbit, then there exist neighbourhoods U of p and U' of p' such that $(\Gamma \cdot U) \cap U' = \emptyset$;*
4. *for any Γ an arithmetic subgroup of $G^{ad}(\mathbb{Q})$ and for any $p \in X^+$, there exists a fundamental set of neighbourhoods $\mathcal{U}$ of p such that, for all $U \in \mathcal{U}$, we have $\gamma \cdot U = U$ if $\gamma \cdot p = p$ and $\gamma \cdot U \cap U = \emptyset$ if $\gamma \cdot x \neq x$.*

Remark 3.4. Essentially, conditions 2, 3 and 4 guarantee that the quotient $\Gamma\backslash\overline{X^+}^{BB}$ is a well-defined topological manifold, for any arithmetic subgroup $\Gamma \subset G^{ad}(\mathbb{Q})$. Condition 1 can be viewed as some sort of minimality condition, i.e. we want to make sure that the topology on $\overline{X^+}^{BB}$ is as close to the subspace topology induced from $\check{X}$.

Definition. For Γ an arithmetic subgroup of $G^{ad}(\mathbb{Q})$, The *Baily-Borel* compactification of $\Gamma\backslash X^+$ is $\Gamma\backslash\overline{X^+}^{BB}$.

The Baily-Borel compactification enjoys the following properties:

- $\Gamma\backslash\overline{X^+}^{BB}$ is a compact Hausdorff space containing $\Gamma\backslash X^+$ as an open dense subset. It is a *finite* union of subspaces of the form $\Gamma_F\backslash F$ for F a rational boundary component, where $\Gamma_F := \Gamma \cap P_F(\mathbb{Q})$.
- The closure of $\Gamma_F\backslash F$ in $\Gamma\backslash\overline{X^+}^{BB}$ is the union of $\Gamma_F\backslash F$ and of subspaces $\Gamma_{F'}\backslash F'$, of strictly smaller dimension.
- The image of F under the quotient map $\overline{X^+}^{BB} \to \Gamma\backslash\overline{X^+}^{BB}$ is precisely $\Gamma_F\backslash F$.
- For any rational boundary component $F \subset \overline{X^+}^{BB}$, the closure $\overline{F}$ in $\overline{X^+}^{BB}$ is precisely the Baily-Borel compactification $\overline{F}^{BB}$ of F.
- $\Gamma\backslash\overline{X^+}^{BB}$ is a complex projective algebraic variety for all arithmetic subgroups Γ (using embedding via enough automorphic forms).

Remark 3.5. [Gor05, Section 2.5] gives an explicit computation for the Baily-Borel compactification $\overline{X^+}^{BB}$ for the group $G^{ad} = Sp_{2n}$ and for X^+ the Siegel upper half space. This is fairly illuminating.

3.3 Combinatorial Data

Let V denote a $\mathbb{Q}$-vector space.

Definition. A *rational polyhedral cone (rpc)* $\pi \subset V_{\mathbb{R}}$ is a closed subset of $V_{\mathbb{R}}$ of the form

$$\{x \in V_{\mathbb{R}} : \ell_i(x) \geq 0 \ \forall 1 \leq i \leq k\}$$

for linear functionals $\ell_i \in V^*$, or equivalently, of the form

$$\left\{ \sum_{i=1}^{k} \lambda_i y_i \in V : \lambda_i \geq 0 \right\}$$

for some finite set $y_i \in V$.

An rpc is *pointed* if it does not contain any non-zero linear subspace of $V_{\mathbb{R}}$.

If an rpc π spans $V_{\mathbb{R}}$ and so has non-empty interior, the interior of π is called an *open rpc*.

The *dual* $\check{\pi}$ of an rpc π is

$$\check{\pi} := \{\lambda \in V_{\mathbb{R}}^* : \lambda(v) \geq 0 \; \forall v \in \pi\}$$

The dual of an rpc in $V_{\mathbb{R}}$ is an rpc in $V_{\mathbb{R}}^*$.

Suppose the dual rpc $\check{\pi}$ is generated by $\lambda_1, ..., \lambda_r$, Then for any $I \subset [1, b]$ the subset

$$\pi_I := \{v \in \pi : \lambda_i(v) = 0 \; \forall i \in I\}$$

is a *face* of π.

If an rpc π has non-empty interior and $|I| = r - 1$, then $\pi_I = \pi \cap H_I$ for some affine hyperplane. Such an H_I is called a *supporting hyperplane.*

Suppose T is a split torus over some field k. Let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_{m,k})$ and

$$X_*(T) = \mathrm{Hom}(\mathbb{G}_{m,k}, T) = \mathrm{Hom}(X^*(T), \mathbb{Z}).$$

These are free $\mathbb{Z}$-modules of rank equal to the rank of T.

Definition. An *equivariant (affine) embedding* if T is an open embedding $T \hookrightarrow X$ into a (affine) k-variety X such that the group action on T extends to an action of T on X.

Lemma 3.6 ([Ash+10, Section I.1])**.** *Given a split torus T, there is a 1-1 correspondence between pointed rpcs $\sigma \subset X_*(T)_{\mathbb{R}}$, and normal equivariant affine embeddings T_σ of T. This correspondence is given by*

$$\sigma \mapsto T_\sigma = \operatorname{Spec} k[M \cap \check{\sigma}].$$

If $\sigma' \subset \sigma$ is a face, then $k[M \cap \check{\sigma}']$ is a quotient of $k[M \cap \check{\sigma}]$, and so $T_{\sigma'}$ is an open subvariety of T_σ.

Definition. A *rational partial polyhedral decomposition (rppd)* of $X_*(T)_{\mathbb{R}}$ is a collection Σ of pointed rpcs such that

- for every $\sigma \in \Sigma$, every face of σ lies in Σ_F;
- for $\sigma, \sigma' \in \Sigma$, the cone $\sigma \cap \sigma'$ is a common face of both σ and σ'.

A *refinement* of an rppd Σ is an rppd Σ' such that

- every $\sigma' \in \Sigma'$ is contained in some $\sigma \in \Sigma$, and
- every $\sigma \in \Sigma$ can be written as a *finite* union

$$\sigma = \bigcup_{\sigma' \in \Sigma', \sigma' \subset \sigma} \sigma'.$$

Suppose Σ is an rppd. Then, for every $\sigma, \sigma' \in \Sigma$, the affine varieties T_σ and $T_{\sigma'}$ have the common open subvariety $T_{\sigma \cap \sigma'}$, along which they may be glued. By glueing all the T_σ, we get a scheme T_Σ with an action of T. We thus have a T-equivariant embedding $T \hookrightarrow T_\Sigma$ into a separated normal irreducible k-scheme locally of finite type. For Σ' a refinement of Σ, we have commutative triangles

$$\begin{array}{ccc} & T & \\ \swarrow & & \searrow \\ T_{\Sigma'} & \xrightarrow{\pi_{\Sigma',\Sigma}} & T_\Sigma \end{array}$$

where the map $\pi_{\Sigma',\Sigma}$ is proper.

Definition. Fix an open convex cone $C \subset V_{\mathbb{R}}$, and suppose there is a discrete group $\Gamma \subset GL(V_{\mathbb{R}})$ acting transitively on C. A Γ-*admissible rppd* of C is an rppd Σ in $V_{\mathbb{R}}$ such that

- for all $\sigma \in \Sigma$, we have $\sigma \subset \overline{C}$;
- for every $\sigma \in \Sigma$ and $\gamma \in \Gamma$, we have $\gamma \cdot \sigma \in \Sigma$ (so that Γ acts on the set Σ);
- there are only finitely many Γ_F-orbits in Σ; and
- $C = \bigcup_{\sigma \in \Sigma} \sigma \cap C.$

3.4 Toroidal Compactifications - Classical

Remark 3.7. While the next section makes this one redundant, in order to construct canonical extensions of automorphic vector bundles, it is easier to work over connected components than it is to work adelically. Thus, this section has been included as well.

Fix a neat arithmetic group $\Gamma \subset G(\mathbb{Q})_+$. We will now suppose throughout that F denotes a rational boundary component. All the statements made here are proven in [Ash+10, Section III].

Recall there is a parabolic subgroup $P_F \subset G$ attached to a boundary component F of X^+. For F rational, this group is defined over $\mathbb{Q}$. Let W_F denote the unipotent radical of P_F, and U_F the centre of W_F. Let M_F be a Levi factor of P_F. Now, there exists a maximal $\mathbb{Q}$-rational connected reductive subgroup $G_{\ell,F} \subset M_F$ over $\mathbb{Q}$ such that $G_{\ell,F}$ acts trivially on F and (modulo a finite subgroup) acts faithfully by conjugation on U_F.

Since W_F is unipotent and U_F is its centre, the exponential map yields a canonical identification (as algebraic groups) of the $\mathbb{Q}$-vector space $\mathfrak{u}_F$ with U_F. On this $\mathbb{Q}$-vector space U_F, there is a natural $\mathbb{Q}$-rational inner product on U_F. Inside $U_F(\mathbb{R})$ there is an open convex cone C_F that is self-adjoint with respect to the inner product, on which $G_{\ell,F}(\mathbb{R})^+$ acts transitively, such that for a certain distinguished point $\Omega_F \in U_F(\mathbb{R})$, the stabiliser of Ω_F under the action of $G_{\ell,F}(\mathbb{R})^+$ is $G_{\ell,F}(\mathbb{R})^+ \cap K_o$. In other words, there is a (non-canonical) isomorphism

$$C_F \cong \frac{G_{\ell,F}(\mathbb{R})^+}{G_{\ell,F}(\mathbb{R})^+ \cap K_o}.$$

Let $\Gamma_{\ell,F} := \Gamma \cap G_{\ell,F}(\mathbb{R})^+$; then it is known that $\Gamma_{\ell,F}$ acts freely on C_F.

Lemma 3.8. *Suppose F, F' are boundary components with $F \subset \overline{F'}$.*

- *We have the following compatibilities:*

$$W_F \supset W_{F'}, \qquad G_{\ell,F} \supset G_{\ell,F'}, \qquad \overline{C_{F'}} = \overline{C_F} \cap U_{F'};$$

- *$C_{F'}$ is a boundary component (as a cone) of C_F. Fixing F, the map $F' \mapsto C_{F'}$ is an order-reversing bijection between the set of boundary components F' of X^+ with $F \subset \overline{F'}$, and the set of boundary components of C_F (as a cone). Moreover, under this bijection, rational boundary components F' of X^+ with $F \subset \overline{F'}$ correspond precisely to rational boundary components of C_F.*

Definition. A Γ-*admissible collection of polyhedra* is the data of a $\Gamma_{\ell,F}$-admissible rppd Σ_F of C_F (in the ambient vector space $U_F(\mathbb{R})$) for each rational boundary component F, satisfying the following compatibility conditions:

- if $F' = \gamma F$ with $\gamma \in \Gamma$, then
$$\Sigma_{F'} = \{\gamma\sigma : \sigma \in \Sigma_F\},$$
via the natural isomorphism $\gamma : C_F \to C_{F'}$; and
- if $F \subset \overline{F'}$ then
$$\Sigma_{F'} = \{\sigma \cap \overline{C_{F'}} : \sigma \in \Sigma_F\}.$$

These Γ-admissible collection of polyhedra are precisely the combinatorial data required to construct a toroidal compactification. Suppose we fix a Γ-admissible collection of polyhedra Σ. Rather than constructing the compactification $(\Gamma\backslash X^+)^{tor}_{\Sigma}$ (a construction of which is found in [Har89, Section 2]), let us describe the toroidal compactification via its properties. This requires a little more notation.

Let P'_F be the centralizer of U_F in P_F, and let $\Gamma'_F = \Gamma \cap P'(\mathbb{Q})$. By an abuse of notation, we write $U_F(\mathbb{Z}) = U_F(\mathbb{Q}) \cap \Gamma$; this is a lattice in the vector space $U_F(\mathbb{R})$. Since $U_F \subset P'$, we have $U_F(\mathbb{Z}) \subset \Gamma'_F$. Now, there is a certain open set $X^+_F \subset \check{X}$ of X^+ satisfying:

1. $X^+_F \cong U_F(\mathbb{C}) \times \mathbb{C}^g \times F$, for some $g \geq 0$;
2. X^+_F is equipped with an action of Γ'_F extending the action of Γ'_F on X^+; and
3. there is a certain fibration $\Gamma'_F\backslash X^+_F \xrightarrow{\pi_2} A_F \xrightarrow{\pi_1} M_F$ such that
    - $M_F \cong \Gamma_F\backslash F$ for some arithmetic group Γ_F (in particular, M_F is a quasi-projective variety),
    - A_F is an abelian scheme over M_F with structure map π_1,
    - $\pi_2 : \Gamma'_F\backslash X^+_F \to A_F$ is a principal homogeneous space with fibres the torus $T_F := U_F(\mathbb{Z})\backslash U_F(\mathbb{C})$, and
    - $\Gamma'_F\backslash X^+_F$ has the structure of an algebraic variety, with respect to which π_2 is a morphism of varieties.

Now, suppose given a boundary component F and a pointed rpc $\sigma \subset U_F(\mathbb{R})$. One easily checks that $X_*(T_F) \cong U_F(\mathbb{Z})$ and $X_*(T_F)_{\mathbb{R}} \cong U_F(\mathbb{R})$, and so we have the equivariant embedding $T_F \hookrightarrow T_{F,\sigma}$ of the torus T_F corresponding to the pointed rpc σ. Consider the contracted product

$$(\Gamma'_F \backslash X_F^+)_\sigma := (\Gamma'_F \backslash X_F^+) \times^{T_F} T_{F,\sigma},$$

and let $X_{F,\sigma}^+$ be the interior of the closure in $(\Gamma'_F \backslash X_F^+)_\sigma$ of $\Gamma'_F \backslash X^+$. Similarly, let ${X^+}_{F,\sigma}^1$ be the interior of the closure of $U_F(\mathbb{Z})\backslash X^+(F)$ in the contracted product

$$(U_F(\mathbb{Z})\backslash X_F^+) \times^{T_F} T_{F,\sigma}.$$

Since $U_F(\mathbb{Z}) \subset \Gamma'_F$, we have a quotient map ${X^+}_{F,\sigma}^1 \to {X^+}_{F,\sigma}$.

Theorem 3.9. *We assume the notation and setup as above.*

1. *For any Γ-admissible collection of polyhedra Σ, there exists a unique Hausdorff analytic variety $\overline{\Gamma\backslash X^+}_\Sigma$ containing $\Gamma\backslash X^+$ as an open dense subset such that:*

   - *for every rational boundary component F of X^+, there are open analytic morphisms $(\rho_{F,\sigma})_{\sigma\in\Sigma_F}$ making*

   $$\begin{array}{ccc} U_F(\mathbb{Z})\backslash X^+ & \hookrightarrow & {X^+}_{F,\sigma}^1 \\ \downarrow & & \downarrow \rho_{F,\sigma} \\ \Gamma\backslash X^+ & \hookrightarrow & \overline{\Gamma\backslash X^+}_\Sigma \end{array}$$

   commute;

   - *every point of $\overline{\Gamma\backslash X^+}_\Sigma$ is in the image of one of the maps ρ_F.*

   This space $\overline{\Gamma\backslash X^+}_\Sigma$ is a compact algebraic space.

2. *The natural map $\Gamma'_F\backslash X^+ \to \Gamma\backslash X^+$ extends to a local analytic isomorphism $\phi_{F,\sigma} : X_{F,\sigma}^+ \to \overline{\Gamma\backslash X^+}_\Sigma$. Moreover, an open cover of $\overline{\Gamma\backslash X^+}_\Sigma^{tor}$ is given by*

   $$\{X_{F,\sigma}^+ : \sigma \in \Sigma_F, F \text{ a rational boundary component of } X^+\}.$$

3. *There exists a natural morphism $\overline{\Gamma\backslash X^+}_\Sigma \to \overline{\Gamma\backslash X^+}^{BB}$ inducing the identity morphism on the dense open subset $\Gamma\backslash X^+$.*

4. *Suppose Γ is neat. Then, for every Γ-admissible collection Σ of polyhedra, there exists a subdivision Σ' such that the morphism $\overline{\Gamma\backslash X^+}_{\Sigma'} \to \overline{\Gamma\backslash X^+}_\Sigma$, induced by the identity morphism on $\Gamma\backslash X^+$, is projective and $\overline{\Gamma\backslash X^+}_{\Sigma'}$ is smooth (in fact, is a blow-up).*

Definition. The toroidal compactification $\overline{\Gamma\backslash X^+}_\Sigma$ is said to be *SNC* if it is smooth and $\overline{\Gamma\backslash X^+}_\Sigma \setminus (\Gamma\backslash X^+)$ is a divisor with normal crossings.

Lemma 3.10. *The compactification $\overline{\Gamma\backslash X^+}_\Sigma$ is SNC if and only if for all rational boundary components F and all $\sigma \in \Sigma_F$, the semigroup $\sigma \cap U_F(\mathbb{Z})$ is generated by a subset of a basis for the free abelian group $U_F(\mathbb{Z})$.*

It is a fact that every Γ-admissible collection of polyhedra admits a refinement such that the corresponding toroidal compactification is SNC. It is also a fact that every Γ-admissible collection of polyhedra Σ admits a refinement Σ' such that the map $\overline{\Gamma\backslash X^+}_{\Sigma'} \to \overline{\Gamma\backslash X^+}_\Sigma$ is projective. There also exists a Σ such that $\overline{\Gamma\backslash X^+}_\Sigma$ is a projective SNC toroidal compactification.

3.5 Constructing Toroidal Compactifications Adelically

This is Sections 2.5-2.7 of [Har89].

Choose a $\mathbb{Q}$-rational minimal parabolic subgroup B of G. Let P_i $(1 \le i \le r)$ be the set of maximal $\mathbb{Q}$-parabolic subgroups of G containing B. Let U_i be the centre of the unipotent radical of P_i, and let $C_i \subset U_i(\mathbb{R})$ be the corresponding $P_i(\mathbb{R})^+$-stable self-adjoint convex cone. Let K be a neat open compact subgroup of $G(\mathbb{A}_\mathbb{Q}^\infty)$.

For each i, choose a collection of rational polyhedral cones $\sigma_{\alpha,g}^{(i)} \subset \overline{C}_i$ for $\alpha \in \Xi_i$ and $g \in G(\mathbb{A}_\mathbb{Q}^\infty)$, indexed by some set Ξ_i and $G(\mathbb{A}_\mathbb{Q}^\infty)$. We require the following compatibility properties:

1. for each $1 \leq i \leq r$ and each $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$, $\{\sigma_{\alpha,g}^{(i)}\}_{\alpha \in \Xi_i}$ is a rppd;
2. for any $1 \leq i \leq r$, any $\alpha \in \Xi_i$ and any $\gamma \in P_i(\mathbb{R})_+$, there exists $\beta \in \Xi_i$ such that $\gamma \cdot \sigma_{\alpha,g}^{(i)} = \sigma_{\beta,\gamma g}^{(i)}$;
3. $\{\sigma_{\alpha,gk}^{(i)}\}_{\alpha,g} = \{\sigma_{\alpha,g}^{(i)}\}_{\alpha,g}$ for all i and all $k \in K$;
4. for each i, the double orbit space $P_i(\mathbb{R})_+ \backslash \{\sigma_{\alpha,g}^{(i)}\}/K$ is finite;
5. $C_i = \bigcup_{\alpha} \sigma_{\alpha,g}^{(i)} \cap C_i$ for any $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ and any $1 \leq i \leq r$;
6. if F_i is a boundary component of F_j, then for all $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$,
$$\{\sigma_{\alpha,g}^{(j)} : \alpha \in \Xi_j\} = \{\sigma_{\alpha,g}^{(i)} \cap \overline{C}_j : \alpha \in \Xi_i\},$$
with respect to a certain natural identification of $\overline{C}_j$ with $\overline{C}_i \cap U_j(\mathbb{R})$.

For each i, set
$$\Delta_i = \left\{\sigma_{\alpha,g}^{(i)} \times \{g\} : \alpha \in \Xi_i, g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})\right\} \subset \overline{C}_i \times G(\mathbb{A}_{\mathbb{Q}}^{\infty}),$$
and set
$$\Sigma_i := \mathrm{Ind}_{P_i(\mathbb{R})_+}^{G(\mathbb{Q})_+} \Delta_i.$$
By definition, this is the set
$$\mathrm{Ind}_{P_i(\mathbb{R})_+}^{G(\mathbb{Q})_+} \Delta_i := (G(\mathbb{Q})_+ \times \Delta_i)/P_i(\mathbb{R})_+$$
equipped with the obvious $G(\mathbb{Q})_+$ action, where with $P_i(\mathbb{R})_+$ acting on $G(\mathbb{Q})_+ \times \Delta_i$ by the formula
$$p(\gamma, \sigma_{\alpha,g}^{(i)}, g) = \big(\gamma p^{-1}, p(\sigma_{\alpha,g}^{(i)}), pg\big).$$
Let $\Sigma = \bigsqcup_{i=1}^{r} \Sigma_i$.

Definition. We say that a Σ constructed as above is a *K-admissible collection of polyhedra.*

All of this combinatorial data is required to construct a toroidal compactification $\overline{Sh_K}^{\Sigma}$ of Sh_K.

Fix a connected component X^+ of X, so that $G(\mathbb{R})_+$ can be thought of as the stabiliser of X^+ under the action of $G(\mathbb{R})$ on $\pi_0(X)$. Let F_i be the boundary component of X^+ corresponding to the parabolic P_i, and let $X_i^+ = X_{F_i}^+$ be as defined in the previous section. Set
$$\begin{aligned} A_i &:= U_i(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K, \\ B_i &= U_i(\mathbb{Q}) \backslash X_i^+ \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K, \\ \mathcal{T}_i &:= U_i(\mathbb{Q}) \backslash U_i(\mathbb{C}) \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K. \end{aligned}$$
It is easy to see that $\mathcal{T}_i$ is a group variety over A_i all of whose fibres are tori. Let
$$\begin{array}{ccc} \mathcal{T}_i & \hookrightarrow & \mathcal{T}_{i,\Delta_i} \\ & \searrow \quad \nearrow & \\ & A_i & \end{array}$$
be the torus embedding, where for any $\overline{g} \in A_i$ (with $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$), the fibre of the map $\mathcal{T}_i \hookrightarrow \mathcal{T}_{i,\Delta_i}$ is precisely the usual torus embedding
$$(\mathcal{T}_i)_{\overline{g}} \hookrightarrow \big((\mathcal{T}_i)_{\overline{g}}\big)_{\{\sigma_{\alpha,g}^{(i)}\}}.$$
As before, we take B_{i,Δ_i} to be the interior of the closure of $U_i(\mathbb{Q}) \backslash X^+ \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K$ in the contracted product
$$B_i \times^{\mathcal{T}_i} \mathcal{T}_{i,\Delta_i}.$$
If F_i is a boundary component of F_j, then there is a canonical étale map $\pi_{ji} : B_{j,\Delta_j} \to B_{i,\Delta_i}$. There is also an action of $P_i(\mathbb{R})_+$ on B_{i,Δ_i}, and so we can define
$$B_{i,\Sigma_i} = \mathrm{Ind}_{P_i(\mathbb{R})_+}^{G(\mathbb{Q})_+} B_{i,\Delta_i}.$$

Finally, set

$$\tilde{Sh}_K^{\Sigma} = \bigsqcup_{i=1}^{r} B_{i,\Sigma_i}.$$

We define an equivalence relation on $\tilde{Sh}_K^{\Sigma}$: for $x_i \in B_{i,\Sigma_i}$ and $x_j \in B_{j,\Sigma_j}$, we say that $x_i \sim x_j$ if and only if there exists $k \in \{1, ..., r\}$ and $x \in B_{k,\Delta_k}$ such that F_i and F_j are both boundary components of F_k, and there exist $\gamma_i, \gamma_j \in G(\mathbb{Q})_+$ such that

$$\gamma_i x_i = \pi_{ki}(x) \quad \text{and} \quad \gamma_j x_j = \pi_{kj}(x).$$

The quotient of $\tilde{Sh}_K^{\Sigma}$ under the above equivalence relation will be denoted by $\overline{Sh_K}^{\Sigma}$.

Proposition 3.11. *$\overline{Sh_K}^{\Sigma}$ is the disjoint union of the toroidal compactifications of each of the connected components $\Gamma_i \backslash X^+$, where $\Gamma_i = g_i K g_i^{-1} \cap G(\mathbb{Q})_+$ and $\{g_i\}$ is a set of coset representatives of $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K$.*

Thus, we can define $\overline{Sh_K}^{\Sigma}$ to be the toroidal compactification of the Shimura variety Sh_K.

Remark 3.12. Of course, the whole construction above is moot if we cannot find such combinatorial data Σ. Harris constructs one such Σ in [Har89, 2.5.7(a)]. It turns out that this special Σ makes $\overline{Sh_K}^{\Sigma}$ a projective variety.

Now, suppose Σ' be a refinement of Σ (i.e. we take refinements of the rppds $\{\sigma_{\alpha,g}^{(i)}\}$ and continue with the above setup). Then, the induced map

$$\pi_{\Sigma',\Sigma} : \overline{Sh_K}^{\Sigma'} \to \overline{Sh_K}^{\Sigma}$$

is a proper birational morphism.

Lemma 3.13. *Suppose K' is an open subgroup of K, and Σ is a K-admissible collection of polyhedra. Then Σ is automatically K'-admissible as well, and the canonical projection $Sh_{K'} \to Sh_K$ extends to a natural morphism*

$$\overline{Sh_{K'}}^{\Sigma} \to \overline{Sh_K}^{\Sigma}.$$

If K' is moreover a normal subgroup of K, then K acts naturally on $\overline{Sh_{K'}}^{\Sigma}$ and there is a natural isomorphism

$$\overline{Sh_{K'}}^{\Sigma}/K \cong \overline{Sh_K}^{\Sigma}.$$

Let us now briefly discuss models of the toroidal compactifications over the reflex field. For each standard maximal $\mathbb{Q}$-parabolic $P \subset G$, let P'' be the maximal subgroup with the property that the homomorphism

$$Ad|_{P''} : P'' \to GL(\mathrm{Lie}(U_i))$$

factors through a $\mathbb{Q}$-morphism $\nu : P'' \to \mathbb{G}_{m,\mathbb{Q}}$, with $\mathbb{G}_{m,\mathbb{Q}}$ acting as scalars in GL. In particular, one checks that the centraliser P' of U in P satisfies

$$1 \to P' \to P'' \to \mathbb{G}_{m,\mathbb{Q}} \to 1.$$

Set $L_i(g) := U_i(\mathbb{Q}) \cap gKg^{-1}$.

Definition. Let Σ be a K-admissible collection of polyhedra. Say Σ is *projective* (in the sense of Tai) if for each i there exists a continuous function

$$\phi_i : \Delta_i \to \mathbb{R}$$

such that

- ϕ_i is piecewise linear in the variable $\overline{C}_i$;
- $\phi_i(x, g) > 0$ for all $x \neq 0$;
- ϕ_i is linear on the image of $\{\sigma^{(i)_{\alpha,g}}\}$ for all α;
- the $\{\sigma^{(i)_{\alpha,g}}\}$ are the maximal polyhedral cones in Δ_i on which ϕ_i is linear;
- $\phi_i(L_i(g) \cap \overline{C}_i, g) \subset \mathbb{Z}$ for all $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

Definition. A projective K-admissible collection of polyhedra Σ is *equivariant* if

$$\phi_i(x, pg) = \|\nu(p)\|^{-1} \phi_i(x, g)$$

for all $p \in P_i''(\mathbb{A}_{\mathbb{Q}}^{\infty})$ and all i, g, x. Here, $\|.\|$ is the adele norm on $\mathbb{A}_{\mathbb{Q}}^{\infty}$.

Proposition 3.14 ([Har89, Proposition 2.8])**.** *If Σ is projective and equivariant, then the complex variety $\overline{Sh_K}^{\Sigma}$ and the divisor $\overline{Sh_K}^{\Sigma} \setminus Sh_K$ are both defined over the reflex field of the Shimura datum (G, X).*

Remark 3.15. This result is proven by first noting that $\overline{Sh_K}^{\Sigma}$ is the normalization of a blow up along certain ideal sheaves of the Baily-Borel compactification. These ideal sheaves are *a priori* defined over $\mathbb{C}$. Harris shows that these ideal sheaves are actually defined over the reflex field. Since the Baily-Borel compactification is defined over the reflex field, it then follows that the toroidal compactification is as well.

3.6 Canonical Extensions of Automorphic Vector Bundles

It is easiest to define canonical extensions of automorphic vector bundles over the connected components $\Gamma\backslash X^+$ of Sh_K (where $\Gamma = G(\mathbb{Q})_+ \cap gKg^{-1}$ for some $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$). We use the notation defined in Section 3.4.

Let Σ be a Γ-admissible collection of polyhedra. Suppose ρ is a $G_{\mathbb{C}}$-representation with corresponding flat vector bundle $\mathcal{V}$ on Sh_K (and by restriction, a flat bundle on $\Gamma\backslash X^+$). Let $\check{\mathcal{V}}$ be the corresponding $G_{\mathbb{C}}$ representation on the compact dual space $\check{X}$. Let

$$j_{\Sigma} : \Gamma\backslash X^+ \hookrightarrow \overline{\Gamma\backslash X^+}_{\Sigma}$$

be the canonical open embedding. Recall the open subset $X_F^+ := U_F(\mathbb{C}) \cdot X^+ \subset \check{X}$. Recall also the fibration

$$\Gamma'_F\backslash X_F^+ \xrightarrow{\pi_2} A_F \xrightarrow{\pi_1} M_F$$

attached to each cone $\sigma \in \Sigma_F$.

It is a fact that for every rational boundary component F, the bundle $\check{\mathcal{V}}|_{X_F^+}$ has a basis of $U_F(\mathbb{C})$-invariant holomorphic sections. This implies that the vector bundle

$$\mathcal{V}_F := \Gamma'_F\backslash\check{\mathcal{V}}|_{X_F^+}$$

over $\Gamma'_F\backslash X_F^+$ satisfies

$$\mathcal{V}_F \cong \pi_2^*(\mathcal{V}_F^A)$$

for some vector bundle $\mathcal{V}_F^A$ on A_F. Let $\pi_{2,\sigma} : X_{F,\sigma}^+ \to A_F$ be the canonical map induced by π_2 whose fibres are isomorphic to $T_{F\sigma}$, and set

$$\mathcal{V}_{F,\sigma} = \pi_{2,\sigma}^*(\mathcal{V}_F^A).$$

Recall the open morphisms $\phi_{F,\sigma} : X_{F,\sigma}^+ \to \overline{\Gamma\backslash X^+}_{\Sigma}$ given in Theorem 3.9(2).

Definition. A *canonical extension of* $\mathcal{V}|_{\Gamma\backslash X^+}$ to $\overline{\Gamma\backslash X^+}_{\Sigma}$ is a subsheaf $\mathcal{V}_{\Sigma}$ of $j_{\Sigma,*}(\mathcal{V})$ over $\overline{\Gamma\backslash X^+}_{\Sigma}$ such that there exist isomorphisms

$$f_{\sigma} : \phi_{F,\sigma}^*(\mathcal{V}_{\Sigma}) \xrightarrow{\sim} \mathcal{V}_{F,\sigma}$$

for all $\sigma \in \Sigma$, satisfying obvious compatibility conditions between $f_{\sigma}, f_{\sigma'}$ for $\sigma, \sigma' \in \Sigma_F$.

Definition. A *canonical extension* of the vector bundle $\mathcal{V}$ on Sh_K is a vector bundle $\mathcal{V}_{\Sigma}$ on $\overline{Sh_K}^{\Sigma}$ whose restriction to every connected component $\overline{\Gamma\backslash X^+}_{\Sigma}$ of $\overline{Sh_K}^{\Sigma}$ is a canonical extension of $\mathcal{V}|_{\Gamma\backslash X^+}$.

Obviously, a canonical extension if it exists is determined uniquely up to unique isomorphism. It is also a fact that the canonical extension $\mathcal{V}_{\Sigma}$, if it exists, is an algebraic bundle.

Theorem 3.16 ([Har89, Theorem 4.2])**.** *Assume that $K \subset G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is neat, and let Σ be a K-admissible collection of polyhedra with corresponding toroidal compactification $\overline{Sh_K}^{\Sigma}$.*

1. *Any automorphic vector bundle $\mathcal{V}$ over Sh_K has a canonical extension $\mathcal{V}_{\Sigma}$ to $\overline{Sh_K}^{\Sigma}$.*
2. *The functor $\mathcal{V} \mapsto \mathcal{V}_{\Sigma}$ is exact and commutes with tensor products and the Hom functors.*
3. *If Σ is projective, then the functor $\mathcal{V} \mapsto \mathcal{V}_{\Sigma}$ preserves fields of definitions.*

Example 3.17. Consider the structure sheaf $\mathcal{O}_{\check{X}}$ on $\check{X}$. This is $G_{\mathbb{C}}$-equivariant, and so we get the automorphic bundles $\mathcal{O}_{Sh_K}$. Then, Mumford showed that

$$(\mathcal{O}_{Sh_K})_{\Sigma} \cong \mathcal{O}_{\overline{Sh_K}^{\Sigma}}.$$

Example 3.18. Suppose $\overline{Sh_K}^{\Sigma}$ is SNC. Then, for any $r > 0$, we have

$$(\Omega^r_{Sh_K})_\Sigma \cong \Omega^r_{\overline{Sh_K}^\Sigma}(\log Z),$$

where $Z := \overline{Sh_K}^{\Sigma} \setminus Sh_K$ and $\Omega^\bullet_{\overline{Sh_K}^\Sigma}(\log Z)$ is the *logarithmic de Rham complex* of Deligne.

The existence of canonical extensions is proven as follows. Of course, it suffices to prove existence for the connected component $\Gamma\backslash X^+$. Suppose that $h \in X^+$ and P_h the corresponding parabolic. Every automorphic vector bundle corresponds to a finite dimensional representation of P_h. It is a fact that this representation is a subquotient of the restriction to P_h of a finite dimensional representation of G. It thus suffices to work with flat automorphic bundles. Recall also that every Γ-admissible collection of polyhedra Σ admits a finite refinement such that the corresponding toroidal compactification is SNC. Harris shows that the canonical extension on a SNC compactification $\overline{Sh_K}^{\Sigma'}$ pushes forward to the canonical extension on $\overline{Sh_K}^{\Sigma}$.

Hence, we may suppose that the compactification $\overline{\Gamma\backslash X^+}_\Sigma$ is SNC and that $\check{\mathcal{V}} \cong V \times \check{X}$ where $\rho : G \to GL(V)$ is a representation and G acts diagonally on $\check{\mathcal{V}}$. As usual, the corresponding bundle $\mathcal{V}$ on $\Gamma\backslash X^+$ is flat. Since K is neat, we have $\pi_1(\Gamma\backslash X^+) = \Gamma$, and under the Riemann-Hilbert correspondence this bundle $\mathcal{V}$ corresponds to the monodromy representation $\rho|_\Gamma : \pi_1(\Gamma\backslash X^+) \to GL(V)$. Now, the monodromy representation of $\mathcal{V}$ along the divisor $Z = \overline{\Gamma\backslash X^+}_\Sigma \setminus \Gamma\backslash X^+$ is unipotent. A theorem of Deligne then provides a unique extension of $\mathcal{V}$ to a flat bundle $\mathcal{V}_{can}$ on $\overline{\Gamma\backslash X^+}_\Sigma$ with regular singularities along Z. By studying this bundle $\mathcal{V}_{can}$ on each of the open sets $X^+_{F,\sigma}$, Harris shows that this bundle satisfies the definition of canonical extension given above.

Proposition 3.19. *Suppose Σ is a K-admissible collection of polyhedra and that $\mathcal{V}$ is an automorphic vector bundle on Sh_K. If Σ' is any refinement of Σ, then*

$$\pi_{\Sigma',\Sigma,*}\mathcal{V}_{\Sigma'} \cong \mathcal{V}_\Sigma \quad \text{and} \quad \pi^*_{\Sigma',\Sigma}\mathcal{V}_\Sigma \cong \mathcal{V}_{\Sigma'}.$$

Moreover, these isomorphisms are adjoint to each other. We also have natural isomorphisms

$$H^*(\overline{Sh_K}^{\Sigma}, \mathcal{V}_\Sigma) \cong H^*(\overline{Sh_K}^{\Sigma'}, \mathcal{V}_{\Sigma'}).$$

As one would expect, all of these isomorphisms are defined over a field E if the original automorphic vector bundle on Sh_K was defined over E and if Σ and Σ' are projective and equivariant.

3.7 Hecke Action on Canonical Extensions

Suppose K' is an open neat subgroup of the neat compact open $K \subset G(\mathbb{A}^\infty_\mathbb{Q})$. By Lemma 3.13, we have a map

$$t_{K',K} : \overline{Sh_{K'}}^{\Sigma'} \to \overline{Sh_K}^{\Sigma}$$

for any K-admissible collection of polyhedra Σ (with Σ' the corresponding K'-admissible collection of polyhedra. Suppose the G-homogeneous bundle $\check{\mathcal{V}}$ on $\check{X}$ induces the automorphic vector bundles $\mathcal{V}_K$ and $\mathcal{V}_{K'}$ on Sh_K and $Sh_{K'}$ respectively.

Proposition 3.20. *In the setup above, we have a canonical isomorphism*

$$t^*_{K',K}\mathcal{V}_{K,\Sigma} \cong \mathcal{V}_{K',\Sigma'}$$

Now suppose $h \in G(\mathbb{A}^\infty_\mathbb{Q})$; then $K^h := h^{-1}Kh$ is also neat. Let Δ_i, Σ_i, and Σ be as in Section 3.5. If we replace Δ_i by

$$\Delta_i^h := (\sigma^{(i)}_{\alpha,gh}, gh)$$

for all i. We then get the K^g-admissible collection of polyhedra Σ^g, and so we have the toroidal compactification $\overline{Sh_{K^h}}^{\Sigma^h}$ of Sh_{K^h}. Suppose also that the G-homogeneous bundle $\check{\mathcal{V}}$ yields the automorphic bundles $\mathcal{V}_K$ on Sh_K and $\mathcal{V}_{K^h}$ on Sh_{K^h}.

Proposition 3.21. *The natural isomorphism $Sh_{K^h} \cong Sh_K$ extends to an isomorphism*

$$t_h : \overline{Sh_{K^h}}^{\Sigma^h} \cong \overline{Sh_K}^{\Sigma}.$$

Moreover, we have a canonical isomorphism

$$t^*_h\mathcal{V}_{K,\Sigma} \cong \mathcal{V}_{K^h,\Sigma^h}.$$

As usual, in both of these propositions, if everything in sight is defined over a subfield E of $\mathbb{C}$ and if the admissible collections of polyhedra are projective and equivariant, then the above natural isomorphisms are also defined over E.

Combining the above two propositions, we thus get Hecke actions on cohomology. We will revisit this when we look at coherent cohomology.

4 Cohomology of Automorphic Vector Bundles

We recall the notation previously used. Let (G, X) be a Shimura datum. Write $\mathfrak{g} = Lie(G)$. Fix a point $o \in X$; then the stabiliser of o in $G(\mathbb{R})$ is $A_G K_o$ where K_o is a maximal compact subgroup. Let $\mathfrak{k} = Lie(K_o)$; then we have the Hodge decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_o^+ \oplus \mathfrak{p}_o^- \oplus (\mathfrak{k}_{o,\mathbb{C}} \oplus \mathfrak{a}_G).$$

Let P_o be the parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{p}_o := \mathfrak{k}_{o,\mathbb{C}} \oplus \mathfrak{a}_G \oplus \mathfrak{p}_o^-$, and set $\check{X}$ to be the (projective) flag variety of parabolic subgroups of G conjugate to P_o. The Borel embedding is thus $X \hookrightarrow \check{X}(\mathbb{C})$. Let M_o denote the Levi factor of P_o; then M_o is the stabiliser of $o \in X$ under the G action, and $M_o(\mathbb{R}) = A_G K_o$.

Throughout, for $\mathcal{V}$ a flat bundle we denote by $\mathcal{V}^{\nabla}$ the corresponding local system under the Riemann-Hilbert correspondence.

The survey [Har90] is an excellent reference.

4.1 Relative Lie Algebra Cohomology

In order to prevent confusion with previous notation, consider a Lie group H with Lie algebra $\mathfrak{h}$ and maximal compact C. Let $\mathfrak{c}$ be the Lie subalgebra of $\mathfrak{h}$ corresponding to C. Suppose W is a $(\mathfrak{h}, C)$-module. For a given $q \geq 0$, set $C^q(\mathfrak{h}, C; W)$ to be the space of linear functions $f : \bigwedge^q(\mathfrak{h}/\mathfrak{c}) \to W$ satisfying

$$\sum_{i=1}^{q} f(x_1, ..., x_{i-1}, [x, x_i], x_{i+1}, ..., x_q) = x \cdot f(x_1, ..., x_q)$$

for all $x \in \mathfrak{c}$ and $x_1, ..., x_q \in \mathfrak{h}$, and satisfying

$$\sum_{i=1}^{q} f(x_1, ..., x_{i-1}, Ad_h(x_i), x_{i+1}, ..., x_q) = h \cdot f(x_1, ..., x_q)$$

for all $h \in C$ and all $x_1, ..., x_q \in \mathfrak{h}$. Strictly speaking, the first condition is a special case of the second condition via the exponential map, but we write both conditions separately for emphasis. We have natural differential maps

$$d : C^q(\mathfrak{h}, C; W) \to C^{q+1}(\mathfrak{h}, C; W)$$

given by

$$(df)(x_0, ..., x_q) := \sum_{i=1}^{q} (-1)^i x_i \cdot f(x_0, ..., \hat{x}_i, ..., x_q) + \sum_{1 \leq i < j \leq q} (-1)^{i+j} f([x_i, x_j], x_0, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_q).$$

The cohomology of this complex $C^{\bullet}(\mathfrak{h}, C; W)$ is denoted by $H^*(\mathfrak{h}, C; W)$.

4.2 Cohomology of Automorphic Local Systems as $(\mathfrak{g}, K_o)$-cohomology

We follow the exposition of [Su18]. The survey article [Har90] is also a good reference.

Remark 4.1. The computation carried out in [Su18] actually computes the cohomology for the local system (under the Riemann-Hilbert correspondence), not the flat bundle! Thus, even though we follow the computation of [Su18], we correct this mistake.

Fix a level K, and let V be a complex representation of $G(\mathbb{C})$, so that we have the flat bundle $\tilde{V}_K$ on Sh_K. Consider the Dolbeault complex $\mathcal{A}^{0,\bullet}$ whose differential maps are $\overline{\partial}$. It is a general fact that the cohomology of local systems is computed using a de Rham complex, i.e. $\tilde{V}_K^{\nabla} \otimes_{\mathbb{C}} \mathcal{A}^{0,\bullet}$ can be used to compute the sheaf cohomology $H^*(Sh_K, \tilde{V}_K)$. We thus try to compute the cohomology of the complex $\Gamma(Sh_K, \tilde{V}_K^{\nabla} \otimes_{\mathbb{C}} \mathcal{A}^{0,\bullet})$. Let Ω^q denote the sheaf of smooth q forms, so that

$$\Omega^q = \bigoplus_{i+j=q, i,j \geq 0} \mathcal{A}^{i,j}.$$

First, we know that $\tilde{V}_K^{\nabla} \otimes_{\mathbb{C}} \Omega^0$ (where recall $\Omega^0 = \mathcal{C}^{\infty}$) is the sheaf

$$U \mapsto \left(C^{\infty}(\pi_K^{-1}(U)) \otimes V\right)^{K_o},$$

where π_K is the projection

$$\pi_K : G(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathbb{Q}})/A_G K = G(\mathbb{Q}) \backslash \big(G(\mathbb{R})/A_G \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K\big) \twoheadrightarrow Sh_K(G, X).$$

Suppose $U \subset Sh_K$ is open. Let us define a map

$$\Phi_U : \tilde{V}_K^{\nabla} \otimes_{\mathbb{C}} \Omega^q(U) \to \mathrm{Hom}\left(\bigwedge^q(\mathfrak{g}_{\mathbb{C}}/\mathfrak{a}_G), C^{\infty}(\pi_K^{-1}(U)) \otimes V\right)$$

as follows. Suppose $\omega \in \tilde{V}_K^{\nabla} \otimes_{\mathbb{C}} \Omega^q(U)$. Suppose $v_1, ..., v_q \in \mathfrak{g}_{\mathbb{C}}/\mathfrak{a}_G$; then via left-translation ℓ_p we can view $v_1, ..., v_q$ as elements of $T_p\pi_K^{-1}(U)$ for all $p \in \pi_K^{-1}(U)$. We then get the element

$$\omega_{\pi_K p}\left(\pi_{K*}\ell_{p,*}v_1, ..., \pi_{K*}\ell_{p,*}v_q\right) \in V.$$

Thus, we have a smooth map

$$\pi_K^{-1}(U) \to V, \quad p \mapsto \omega_{\pi_K p}\left(\pi_{K*}\ell_{p,*}v_1, ..., \pi_{K*}\ell_{p,*}v_q\right),$$

and so $v_1, ..., v_q$ have defined an element of $C^{\infty}(\pi_K^{-1}(U)) \otimes V$. This map is obviously linear, and so to ω we have assigned an element $\Phi_U\omega \in \mathrm{Hom}\left(\bigwedge^q(\mathfrak{g}_{\mathbb{C}}/\mathfrak{a}_G), C^{\infty}(\pi_K^{-1}(U)) \otimes V\right)$. We have thus defined the map Φ_U. If some $v_j \in \mathfrak{k}_o = \mathfrak{m}_o/\mathfrak{a}_G$, then $\pi_{K*}v_j = 0$ since π_K is the quotient by K_o map. Thus, $\Phi_U\omega(v_1, ..., v_q) = 0$ if any one of the $v_i \in \mathfrak{m}_o/\mathfrak{a}_G$. Hence, Φ_U actually defines a map

$$\Phi_U : \tilde{V}_K^{\nabla} \otimes_{\mathbb{C}} \Omega^q(U) \to \mathrm{Hom}\left(\bigwedge^q(\mathfrak{g}_{\mathbb{C}}/\mathfrak{m}_o), C^{\infty}(\pi_K^{-1}(U)) \otimes V\right).$$

Suppose $k \in K_o$ and $p \in \pi_K^{-1}(U)$. Since π_K is the quotient by K_o map, it follows that for any $v \in \mathfrak{g}/\mathfrak{a}_G$ we have

$$\pi_{K*}\ell_{p,*}v = \pi_{K*}\ell_{pk^{-1},*}Ad(k) \cdot v,$$

and so in fact the map Φ is K_o-equivariant. Thus, we have a map

$$\Phi_U : \tilde{V}_K^{\nabla} \otimes_{\mathbb{C}} \Omega^q(U) \to C^q(\mathfrak{g}_{\mathbb{C}}/\mathfrak{a}_G, K_o; C^{\infty}(\pi_K^{-1}(U)) \otimes V).$$

Since π_K is a principal K_o-bundle, it follows that Φ_U actually induces an isomorphism. Thus, if $C^{\infty}(Sh_K, -)$ denotes the functor of global smooth sections, we get the following result, where for simplicity, set

$$[G] := G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/A_G.$$

Proposition 4.2. *With notation as above, we have*

$$R^qC^{\infty}(Sh_K, \tilde{V}_K^{\nabla}) \cong H^q\left(\mathfrak{g}_{\mathbb{C}}/\mathfrak{a}_G, K_o; C^{\infty}\left([G]/K\right)_{K_o-fin} \otimes V\right).$$

Let us now consider the functor $\Gamma(Sh_K, -)$ of *holomorphic* global sections. Since, by definition, the complex structure on Sh_K is induced by the decomposition $\mathfrak{g}_{\mathbb{C}}/\mathfrak{m}_o \cong \mathfrak{p}_o^+ \oplus \mathfrak{p}_o^-$, and since $\mathfrak{p}_o^- = \mathfrak{p}_o^1/\mathfrak{k}_o$, it follows that Φ_U induces an isomorphism

$$\tilde{V}^{\nabla} \otimes_{\mathbb{C}} \mathcal{A}^{0,q}(U) \cong C^q\left(\mathfrak{p}_o^1, K_o; C^{\infty}(\pi^{-1}(U)) \otimes V\right).$$

We thus have the following result.

Proposition 4.3. *With notation as above, we have*

$$H^q(Sh_K, \tilde{V}_K^{\nabla}) \cong H^q\left(\mathfrak{p}_o^1, K_o; C^{\infty}\left([G]/K\right)_{K_o-fin} \otimes V\right).$$

Taking limits, we then have the following.

Proposition 4.4. *There are $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-equivariant isomorphism*

$$R^qC^{\infty}(Sh, \tilde{V}^{\nabla}) \cong H^q\left(\mathfrak{g}_{\mathbb{C}}/\mathfrak{a}_G, K_o; C^{\infty}\left([G]\right)_{K_o-fin} \otimes V\right),$$
$$H^q(Sh, \tilde{V}^{\nabla}) \cong H^q\left(\mathfrak{p}_o^1, K_o; C^{\infty}\left([G]\right)_{K_o-fin} \otimes V\right),$$

where the right hand side has the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ action induced by its diagonal action on $C^{\infty}([G])_{K_o-fin} \otimes V$.

As a reminder, $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ acts on V via projection to A_G.

4.3 The Matsushima-Murukami Formula

What is the point of computing the cohomology of (the associated local system of) automorphic vector bundles? It turns out that the cohomology of automorphic vector bundles is intimitely connected with automorphic forms. This is exemplified in the situation where G^{der} is anisotropic (i.e. has $\mathbb{Q}$-rank 0), so that Sh_K are already compact. In this case, the $G(\mathfrak{A}_{\mathbb{Q}})$-module $C^\infty([G])$ is unitarizable. We compute $R^q C^\infty(Sh, \tilde{V})$ differently. This is precisely the remarks made by Harris in the beginning of Section 1.3 of [Har90]. The reference [Nic20] is also good.

Let us decompose $C^\infty([G])$ as a $(\mathfrak{g}, K_o) \times G(\mathbb{A}_{\mathbb{Q}}^\infty)$-module. Suppose

$$C^\infty([G])_{K_o-fin} \cong \bigoplus_{\pi_\infty,\pi^\infty} m(\pi_\infty \otimes \pi^\infty) V_{\pi_\infty} \otimes V_{\pi^\infty},$$

where the direct sum runs over all irreducible unitarizable $(\mathfrak{g}, K_o)$-modules $(\pi_\infty, V_{\pi_\infty})$ and all irreducible $G(\mathbb{A}_{\mathbb{Q}}^\infty)$-modules $(\pi^\infty, V_{\pi^\infty})$. The $m(\pi_\infty \otimes \pi^\infty)$ are non-negative integers denoting the multiplicity of each representation. Since $C^\infty([G])$ is an admissible representation, these multiplicities are finite. This decomposition should then yield a decomposition of $R^q C^\infty(Sh, \tilde{V})$ indexed by automorphic representations.

This is the content of the Matsushima-Murukami theorem.

Theorem 4.5. *Let (G, X) be as above, so that the corresponding Shimura variety Sh is compact. Let ρ be any irreducible finite-dimensional representation of $G(\mathbb{C})$, with corresponding flat automorphic vector bundle $\mathcal{V}_\rho$. Then, we have*

$$R^q C^\infty(Sh, \mathcal{V}_\rho^\nabla) \cong \bigoplus_{\pi_\infty,\pi^\infty} m(\pi_\infty \otimes \pi^\infty)\pi^\infty \otimes H^q(\mathfrak{g}, K_o, \pi_\infty \otimes \rho)$$

as admissible $G(\mathbb{A}_{\mathbb{Q}}^\infty)$-representations, where the direct sum runs over all irreducible unitarizable $(\mathfrak{g}, K_o)$-modules $(\pi_\infty, V_{\pi_\infty})$ and all irreducible $G(\mathbb{A}_{\mathbb{Q}}^\infty)$-modules $(\pi^\infty, V_{\pi^\infty})$. The integers $m(\pi_\infty \otimes \pi^\infty)$ is the multiplicity of $\pi_\infty \otimes \pi^\infty$ occurring in the $(\mathfrak{g}, K_o) \times G(\mathbb{A}_{\mathbb{Q}}^\infty)$-module

$$L^2([G])[\omega_\infty^{-1}],$$

where $\omega_\infty : Z_G(\mathbb{C}) \to \mathbb{C}^\times$ is the central character of ρ.

Thus, computing $R^q C^\infty(Sh, \mathcal{V}_\rho)$ gives information about automorphic representations of a certain type.

For Sh_K not compact, the elements of the cohomology groups of $\mathcal{V}_\rho$ need not be automorphic forms. The reason is that the moderate growth conditions required in the definition of automorphic forms. This was rectified via Borel's theory on de Rham cohomology with growth conditions.

4.4 Cohomology with Compact Supports, and Poincaré duality

We require the general theory of de Rham cohomology with growth conditions. Let $G_0 := G^{der}(\mathbb{R})^+$, and let $g \mapsto \tilde{g}$ by the Cartan involution $\mathrm{Ad}(o(i))$ on G_0 with respect to the K_o. Set

$$\|g\|_{G_0} := \mathrm{Trace}\left(Ad(\tilde{g}^{-1}g)\right)$$

for each $g \in G_0$.

Definition. Suppose V is a normed $\mathbb{C}$-vector space.

- A function $f \in C^\infty(A_G \backslash G(\mathbb{A}_{\mathbb{Q}}), V)$ is *slowly increasing* if f is a finite sum of eigenfunctions for $Z_G(\mathbb{A}_{\mathbb{Q}})$ and if there exist $m \geq 0$ and $C > 0$ such that

  $$\|f(g\gamma)\|_V < C\|g\|_{G_0}^m$$

  for all $g \in G_0$ and all $\gamma \in G(\mathbb{A}_{\mathbb{Q}})$.

- A function $f \in C^\infty(A_G \backslash G(\mathbb{A}_{\mathbb{Q}}), V)$ is *rapidly decreasing* if f is a finite sum of eigenfunctions for $Z_G(\mathbb{A}_{\mathbb{Q}})$ and if for all $m \geq 0$ and all $\gamma \in G(\mathbb{A}_{\mathbb{Q}})$ there exists $C_{\gamma,m} > 0$ such that

  $$\|f(g\gamma)\|_V < C_{\gamma,m}\|g\|_{G_0}^m$$

  for all $g \in G_0$.

For every $K \subset G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ compact open, let $C_{si,K}(G)$ (resp. $C_{rd,K}(G)$) denote the space of all C^{∞} functions on $[G]/K$ which, together with all their right $U(\mathfrak{g})$-derivatives, are slowly increasing (resp. rapidly decreasing). Set

$$C_{si}(G) = \varinjlim_{K} C_{si,K}(G) \quad \text{and} \quad C_{rd}(G) = \varinjlim_{K} C_{rd,K}(G)$$

Proposition 4.6. *Let $\mathcal{V}_{\rho}$ be a flat automorphic vector bundle corresponding to the $G(\mathbb{C})$-representation ρ. There is a natural isomorphism*

$$R^q C^{\infty}(Sh, \mathcal{V}_{\rho}^{\nabla}) \cong H^q(\mathfrak{g}/\mathfrak{a}_G, K_o, C_{si}(G) \otimes V)$$

of admissible $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules.

Now, notice that the construction outlined in the previous section also carries over to cohomology with compact supports. We thus get a similar result for H_c^*.

Proposition 4.7. *Let $\mathcal{V}_{\rho}$ be a flat automorphic vector bundle corresponding to the $G(\mathbb{C})$-representation ρ. There is a natural isomorphism*

$$H_c^*(Sh, \mathcal{V}_{\rho}^{\nabla}) \cong H^q(\mathfrak{g}/\mathfrak{a}_G, K_o, C_{rd}(G) \otimes V)$$

of admissible $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules.

Here, for lack of better notation, we assume that $H_c^*(Sh, \mathcal{V}_{\rho})$ consists of smooth forms (with compact support), as opposed to holomorphic forms.

The theory of holomorphic forms with compact support is not as clean. Let $\mathcal{A}(G)$ (resp. $\mathcal{A}_0(G)$) denote the space of all automorphic forms (resp. automorphic cusp forms) on $[G]$. By definition, $\mathcal{A}(G)$ is the $(\mathfrak{g}/\mathfrak{a}_G, K_o)$-submodule of K_o-finite and $Z(\mathfrak{g}_{\mathbb{C}})$-finite vectors in C_{si}. We now define an operator Δ_V on $C^{\bullet}(\mathfrak{g}/\mathfrak{a}_G, K_o, \mathcal{A}_0(G) \otimes V)$.

Now recall that X possesses a $G(\mathbb{R})$-invariant Hermitian metric which descends to a complete Hermitian metric on Sh_K for any compact open $K \subset G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. Moreover, for V a finite-dimensional representation of K_o, since K_o is compact we can endow V with a K_o-invariant Hermitian inner product. We can thus make any automorphic flat bundle $\mathcal{V}_{\rho}$ corresponding to a representation (V, ρ) of $G(\mathbb{C})$ into a Hermitian vector bundle over Sh. With respect to these metrics, the exterior derivative operator $d_V : \Omega^{\bullet}(V) \to \Omega^{\bullet}(V)[1]$ has a formal adjoint $d_V^* : \Omega^{\bullet}(V)[1] \to \Omega^{\bullet}(V)$. Let $\Delta_V = d_V \circ d_V^* + d_V^* \circ d_V$; this is the usual Laplacian from complex geometry. By using the map Φ_{Sh_K} from the previous section, we thus have an induced map Δ_V on $C^{\bullet}(\mathfrak{g}/\mathfrak{a}_G, K_o, C^{\infty}([G]/K) \otimes V)$. Taking colimits over all K, we obtain an operator Δ_V on $C^{\bullet}(\mathfrak{g}/\mathfrak{a}_G, K_o, C^{\infty}([G]) \otimes V)$. Let $\mathcal{H}^p_{cusp,V}$ denote the kernel of

$$\Delta_V|_{C^p(\mathfrak{g}/\mathfrak{a}_G, K_o, \mathcal{A}_0(G) \otimes V)}.$$

These are essentially the harmonic automorphic forms. Since cusp forms are rapidly decreasing we have a natural map

$$\mathcal{H}^*_{cusp,V} \to H_c^*(Sh, \mathcal{V}_{\rho}^{\nabla}).$$

Also, we have the usual canonical map

$$H_c^*(Sh, \mathcal{V}_{\rho}^{\nabla}) \to R^* C^{\infty}(Sh, \mathcal{V}_{\rho}^{\nabla});$$

let the image of this map be denoted by $\overline{H}^*(Sh, \mathcal{V}_{\rho}^{\nabla})$. Composing the above two maps, we then get a map

$$\mathcal{H}^*_{cusp,V} \to \overline{H}^*(Sh, \mathcal{V}_{\rho}^{\nabla}).$$

Theorem 4.8. *The above map*

$$\mathcal{H}^*_{cusp,V} \to \overline{H}^*(Sh, \mathcal{V}_{\rho}^{\nabla})$$

is injective and $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-equivariant.

Let us now discuss duality in the above cohomology. Of course, usual Poincaré duality yields a pairing

$$H^i(Sh_K, \mathcal{V}^{\nabla}) \otimes H_c^{2n-i}(Sh_K, (\mathcal{V}^*)^{\nabla}) \to \mathbb{C}$$

for any automorphic vector bundle $\mathcal{V}$ on Sh_K, where n is the complex dimension of the variety Sh_K. Here, $\mathcal{V}^*$ is the automorphic vector bundle corresponding to the dual of the representation associated to $\mathcal{V}$. By the above comparison results, this yields a pairing on the relative Lie algebra cohomology. We construct this bilinear pairing explicitly.

Since the product of a slowly increasing function with a rapidly decreasing function is again rapidly decreasing, we get a map of chain complexes

$$C^\bullet(\mathfrak{g}/\mathfrak{a}_G, K_o, C_{si,K}(G)\otimes V)\otimes C^\bullet(\mathfrak{g}/\mathfrak{a}_G, K_o, C_{rd,K}(G)\otimes V^*)\to C^\bullet(\mathfrak{g}/\mathfrak{a}_G, K_o, C_{rd,K}(G)).$$

In particular, upon taking cohomology, the above map yields a bilinear pairing

$$(-)\cup(-): H^p(\mathfrak{g}/\mathfrak{a}_G, K_o, C_{si,K}(G)\otimes V)\otimes H^{2n-p}(\mathfrak{g}/\mathfrak{a}_G, K_o, C_{rd,K}(G)\otimes V^*)\to H^{2n}(\mathfrak{g}/\mathfrak{a}_G, K_o, C_{rd,K}(G)).$$

Define the following map:

$$\tilde{Tr}: C^{2n}(\mathfrak{g}/\mathfrak{a}_G, K_o, C_{rd,K})\to\mathbb{C}; \omega\mapsto (2\pi i)^{-n}\int_{G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/A_G K_o K}\omega.$$

Proposition 4.9. *Let notation be as above.*

1. *The map $\tilde{Tr}$ factors through $H^{2n}(\mathfrak{g}/\mathfrak{a}_G, K_o, C_{rd,K})$, and thus defines a surjective homomorphism*

$$Tr: H_c^{2n}(Sh_K,\mathbb{C})\to\mathbb{C}.$$

2. *For each connected component $\Gamma\backslash X^+$, the restricting Tr defines an isomorphism*

$$H_c^{2n}(\Gamma\backslash X^+,\mathbb{C})\xrightarrow[Tr]{\sim}\mathbb{C}.$$

3. *The bilinear pairing*

$$H^q(Sh_K,\mathcal{V}^\nabla)\otimes H_c^{2n-q}(Sh_K,(\mathcal{V}^*)^\nabla)\to\mathbb{C},\quad \omega\otimes\omega'\mapsto Tr(\omega\cup\omega')$$

coincides, up to a non-zero scalar, with Poincaré duality.

4.5 Cohomology of Canonical Extensions

We follow [Har90, Section 2].

Recall that for any neat level K, any toroidal compactification $\overline{Sh_K}^\Sigma$ of Sh_K, and any automorphic vector bundle $\mathcal{V}_K$ on Sh_K, we have the canonical extension of $\mathcal{V}_K$ to $\overline{Sh_K}^\Sigma$. For this section, we denote this extension by $\mathcal{V}_K^{can,\Sigma}$. Let $Z_K^\Sigma := \overline{Sh_K}^\Sigma\setminus Sh_K$ and let $\mathcal{I}(Z_K^\Sigma)\subset\mathcal{O}_{\overline{Sh_K}^\Sigma}$ be the ideal sheaf[1] defining the divisor Z_K^Σ.

Definition. The *subcanonical extension* $\mathcal{V}_K^{sub,\Sigma}$ is the vector bundle

$$\mathcal{V}_K^{sub,\Sigma} := \mathcal{V}_K^{can,\Sigma}\otimes_{\mathcal{O}_{\overline{Sh_K}^\Sigma}}\mathcal{I}(Z_K^\Sigma).$$

Clearly, $\mathcal{V}_K^{sub,\Sigma}$ is a subsheaf of $\mathcal{V}_K^{can,\Sigma}$, and we have an exact sequence

$$0\to\mathcal{V}_K^{sub,\Sigma}\to\mathcal{V}_K^{can,\Sigma}\to\mathcal{V}_K^{can,\Sigma}|_{Z_K^\Sigma}\to 0.$$

Example 4.10. Consider the automorphic vector bundle $\mathcal{V}_K=\Omega^n_{Sh_K}$. Then, it is known that $\mathcal{V}_K^{sub,\Sigma}$ is the dualising sheaf[2] of the projective variety $\overline{Sh_K}^\Sigma$.

Recall that for any Σ' a refinement of Σ, we get maps

$$\pi_{\Sigma',\Sigma}:\overline{Sh_K}^{\Sigma'}\to\overline{Sh_K}^\Sigma$$

inducing the identity on Sh_K, such that

$$\pi^*_{\Sigma',\Sigma}\mathcal{V}_K^{can,\Sigma}\xrightarrow{\sim}\mathcal{V}_K^{can,\Sigma'}.$$

It follows easily that

$$\pi^*_{\Sigma',\Sigma}\mathcal{V}_K^{sub,\Sigma}\xrightarrow{\sim}\mathcal{V}_K^{sub,\Sigma'}.$$

[1] see [Har13b, Section II.6], especially the discussion leading up to Proposition II.6.18

[2] see [Har13b, Section III.7]

Similarly, the natural isomorphism

$$H^*(\overline{Sh_K}^{\Sigma}, \mathcal{V}_K^{can,\Sigma}) \xrightarrow{\sim} H^*(\overline{Sh_K}^{\Sigma'}, \mathcal{V}_K^{can,\Sigma'})$$

yields a natural isomorphism

$$H^*(\overline{Sh_K}^{\Sigma}, \mathcal{V}_K^{sub,\Sigma}) \xrightarrow{\sim} H^*(\overline{Sh_K}^{\Sigma'}, \mathcal{V}_K^{sub,\Sigma'}).$$

It is known that refinements form a cofiltered system. Hence the following definitions make sense

$$H_K^*(\mathcal{V}^{can}) := \mathrm{colim}_{\Sigma} H^*(\overline{Sh_K}^{\Sigma}, \mathcal{V}_K^{can,\Sigma}),$$
$$H_K^*(\mathcal{V}^{sub}) := \mathrm{colim}_{\Sigma} H^*(\overline{Sh_K}^{\Sigma}, \mathcal{V}_K^{sub,\Sigma}).$$

Definition. The *coherent cohomology* of the automorphic vector bundle $\mathcal{V}_K$ on Sh_K is $H_K^*(\mathcal{V}^{can})$.

Now recall the Hecke actions on cohomology from Section 3.7: for any $K' \subset gKg^{-1}$ and any K-admissible collection of polyhedra Σ, we get a K' admissible collection of polyhedra Σ', and the finite étale map

$$t_{g,K',K} : \overline{Sh'_K}^{\Sigma'} \to \overline{Sh_K}^{\Sigma}$$

induces canonical isomorphisms

$$t_{g,K',K}^* \mathcal{V}_K^{can,\Sigma} \xrightarrow{\sim} \mathcal{V}_{K'}^{\Sigma'}.$$

Set

$$\tilde{H}^*(\mathcal{V}^{can}) := \mathrm{colim}_K H_K^*(\mathcal{V}^{can}) \quad \text{and} \quad \tilde{H}^*(\mathcal{V}^{sub}) := \mathrm{colim}_K H_K^*(\mathcal{V}^{sub}).$$

Remark 4.11. The phrase 'coherent cohomology' is quite often used to refer to the system of $H_K^*(\mathcal{V}^{can})$, or what is essentially the same thing, the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-module $\tilde{H}^*(\mathcal{V}^{can})$.

As usual, these have a Hecke action on them. Since $\mathcal{V}_K^{sub,\Sigma}$ is a subsheaf of $\mathcal{V}_K^{can,\Sigma}$, we get an induced map

$$\tilde{H}^*(\mathcal{V}^{sub}) \to \tilde{H}^*(\mathcal{V}^{can}).$$

Proposition 4.12. *The groups $\tilde{H}^*(\mathcal{V}^{can})$ and $\tilde{H}^*(\mathcal{V}^{sub})$ are naturally admissible graded $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules, and the natural map $\tilde{H}^*(\mathcal{V}^{sub}) \to \tilde{H}^*(\mathcal{V}^{can})$ is $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-equivariant.*

Moreover if the bundle $\mathcal{V}$ on Sh is defined over a subfield E of $\mathbb{C}$, then the $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ action preserves the E-structures on the above cohomology groups.

We can connect the cohomology of the canonical and subcanonical extensions to Lie algebra cohomology. Recall the function spaces $C_{si}(G)$ and $C_{rd}(G)$ on $[G]$. The proof of the following theorem is described in [Har90, Section 2.4].

Theorem 4.13. *Suppose (σ, W) is a representation of M_o, with corresponding semisimple automorphic vector bundle $\mathcal{V}_\sigma$. Then, the following is a natural commutative diagram of admissible graded $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-modules*

$$\begin{array}{ccc} H^*(\mathfrak{p}_o, M_o, C_{rd} \otimes W) & \xrightarrow{\sim} & \tilde{H}^*(\mathcal{V}_\sigma^{sub}) \\ \downarrow & & \downarrow \\ H^*(\mathfrak{p}_o, M_o, C_{si} \otimes W) & \xrightarrow{\sim} & \tilde{H}^*(\mathcal{V}_\sigma^{can}) \end{array}$$

where the horizontal arrows are isomorphisms.

Remark 4.14. For purpose of brevity, apart from mentioning the above theorem, we have skipped discussions on the relationship between coherent cohomology of canonical extensions and Lie algebra cohomology with extra growth conditions. The results are parallel to the ones in the previous section, though instead of working with the cohomology of the Riemann-Hilbert local system attached to the automorphic vector bundle, one can work with the cohomology of the automorphic vector bundles themselves. For instance, this is the subject of the dissertation [Su18].

Remark 4.15. Faltings famous *B-G-G spectral sequence* computes the cohomology of certain local systems attached to flat automorphic vector bundles via coherent cohomology of canonical extensions, thus connecting the cohomology discussed in this section with the cohomology discussed in Section 4.2. This spectral sequence involves a study of the root system attached to G. See [Har90, Section 4] for details.

Remark 4.16. Though we will not mention it here, one can also define spaces of harmonic cusp forms as subspaces of cohomology. There is also a notion of Serre duality, where the duality pairing can be explicitly written down in terms of functions in Lie algebra cohomology. This is Sections 2.5-2.7 of [Har90].

A Miscellaneous Algebraic Geometry

A.1 Connections

A.2 Chern Classes and Intersection Theory

References

[Ash+10] Avner Ash, David Mumford, Michael Rapoport, and Yung-sheng Tai. *Smooth Compactifications of Locally Symmetric Varieties.* 2nd ed. Cambridge Mathematical Library. Cambridge University Press, 2010. DOI: 10.1017/CBO9780511674693.

[EH17] Hélène Esnault and Michael Harris. *Chern classes of automorphic vector bundles.* 2017. arXiv: 1701.09073 [math.AG].

[Fon] Tiago J Fonseca. *Calculus on Schemes - Lecture 5.* URL: https://tjfonseca.github.io/teaching/calculus-5.pdf.

[GH22] Jayce R Getz and Heekyoung Hahn. *An Introduction to Automorphic Representations. with a view toward Trace Formulae.* 2022. URL: https://sites.duke.edu/heekyounghahn/files/2022/04/GTM.pdf.

[Gor05] Mark Goresky. "Compactifications and cohomology of modular varieties". In: *Harmonic analysis, the trace formula, and Shimura varieties* 4 (2005), pp. 551–582.

[GP02] Mark Goresky and William Pardon. "Chern classes of automorphic vector bundles". In: *Inventiones mathematicae* 147.3 (2002), p. 561.

[Har13a] Michael Harris. *Lie Groups, Deligne's Axioms, and Shimura Varieties.* 2013. URL: https://webusers.imj-prg.fr/~michael.harris/Cours_2013/2lie.pdf.

[Har13b] Robin Hartshorne. *Algebraic geometry.* Vol. 52. Springer Science & Business Media, 2013.

[Har85] Michael Harris. "Arithmetic vector bundles and automorphic forms on Shimura varieties. I". In: *Inventiones mathematicae* 82.1 (1985), pp. 151–189.

[Har89] Michael Harris. "Functorial properties of toroidal compactifications of locally symmetric varieties". In: *Proceedings of the London Mathematical Society* 3.1 (1989), pp. 1–22.

[Har90] Michael Harris. "Automorphic forms and the cohomology of vector bundles on Shimura varieties". In: *Automorphic forms, Shimura varieties, and L-functions* 2 (1990), pp. 41–91.

[Lip] Michael Lipnowski. *Baily-Borel compactification.* URL: http://virtualmath1.stanford.edu/~conrad/shimsem/2013Notes/BailyBorelcompactification.pdf.

[Mil05] James S Milne. "Introduction to Shimura varieties". In: *Harmonic analysis, the trace formula, and Shimura varieties* 4 (2005), pp. 265–378.

[Mil90] James S Milne. "Canonical models of (mixed) Shimura varieties and automorphic vector bundles". In: *Automorphic forms, Shimura varieties, and L-functions* 1 (1990), pp. 283–414.

[Nic20] Marc-Hubert Nicole. "Unitary Shimura varieties". In: *Shimura Varieties* (2020), pp. 72–95.

[Su18] Jun Su. "Coherent cohomology of Shimura varieties and automorphic forms". In: *arXiv preprint arXiv:1810.12056* (2018).