Coleman Isomorphisms in Syntomic Cohomology and THH

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Abstract

This paper proves a generalisation of Coleman's isomorphism (between norm compatible cyclotomic units and a group of invertible power series) to various cohomology theories evaluated on proper regular p-adic formal schemes, for future applications to Iwasawa theory. This generalisation is an immediate consequence of a description, as a cyclotomic synthetic spectrum, of the limit over transfer maps as one goes up the cyclotomic tower of motivically filtered THH. This crucially uses a calculation of $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}])$ due to Devalapurkar-Raksit as well as a calculation of the free loop transfer due to Schlichtkrull. Along the way, we construct transfer maps for various cohomology theories along finite flat regular maps, using some elements of $\mathbb{P}^1$-stable motivic homotopy theory.

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1 Introduction

Throughout, fix an odd prime p.

1.1 Statement of Main Results

Consider the power series ring in 1-variable $\mathbb{Z}_p[\![q-1]\!] \simeq \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n]$. This is equipped with a ring endomorphism φ, the Frobenius lift, uniquely determined by $\varphi(q) = q^p$. The ring $\mathbb{Z}_p[\![q-1]\!]$ is also equipped with a $\mathbb{Z}_p^\times$-action via ring maps induced by multiplication on $\mathbb{Z}/p^n$. This article generalises and gives a new more conceptual proof of the following computation of Coleman [Col79].

Theorem 1.1 (Coleman). *Let $N_\varphi : \mathbb{Z}_p[\![q-1]\!] \to \mathbb{Z}_p[\![q-1]\!]$ denote the unique map determined by*

$$(\varphi \circ N_\varphi)(f(q)) = \prod_{\zeta \in \mu_p} f(\zeta q).$$

Then, there is a $\mathbb{Z}_p^\times$-equivariant isomorphism of abelian groups

$$\varprojlim_n \mathbb{Z}_p[\zeta_{p^n}]^\times \cong \left\{ f \in \mathbb{Z}_p[\![q-1]\!]^\times : N_\varphi(f) = f \right\}$$

where the left hand side is the inverse limit over norm maps.

This result (and its generalisation to Galois cohomology of de Rham representations by Cherbonnier–Colmez) is one of the key local inputs in Iwasawa theory, in particular the theory of explicit reciprocity laws[1]. These explicit reciprocity laws are, for example, used in the construction of p-adic L-functions, which are the image of certain motivic classes under the Perrin-Riou p-adic regulator map. This regulator map is constructed using Coleman's isomorphism as well as some (rational) p-adic Hodge theory. Moreover, the construction of these motivic classes, as well as computations of their images under regulator maps, require chasing around certain classes in various cohomology theories along comparison maps between them, as well as under pushforward maps in cohomology (cf. [LZ21]). While the starting motivic input is global, the cohomology arguments most often occur locally at p.

In integral p-adic Hodge theory, comparison results between many p-adic cohomology theories are mediated by the prismatic cohomology of [BMS19; BS22; BL22a]. Moreover, prismatic cohomology can be used to construct an integral form of syntomic cohomology $\mathbb{Z}_p(r)^{\mathrm{syn}}$ [BMS19; BL22a], which identifies with étale sheafified p-adic motivic cohomology. In fact, one should think of Coleman's isomorphism as a computation of $\varprojlim_n \mathbb{Z}_p(1)^{\mathrm{syn}}(\mathbb{Z}_p[\zeta_{p^n}])$ since

$$\mathbb{Z}_p(1)^{\mathrm{syn}}(-) \simeq \mathrm{R}\Gamma_{\text{ét}}(-, \mathbb{G}_m)_p^\wedge[-1].$$

With a view towards studying (in future work) an integral version of Perrin-Riou's regulator map and its role in constructing p-adic L-functions using modern integral p-adic Hodge theory, we generalise Coleman's isomorphism to higher dimensional algebraic (formal) varieties, other cohomology theories, and higher motivic weights. We obtain such generalisations in one fell swoop by using homotopy-theoretic techniques as in [BMS19]. By [BMS19], topological Hochschild homology THH and its motivic filtration $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}$ package together (Nygaard filtered) prismatic and syntomic cohomology. The following Theorem A is the main result of this paper, with immediate consequences to (Nygaard-filtered) prismatic, syntomic, and (Hodge-filtered) de Rham cohomology listed in the following corollaries B-D. To keep track of motivic filtrations on THH, we need to use the theory of 'cyclotomic synthetic spectra' of [AR24], a brief overview of which is in Section 2.3.

Theorem A (Theorem 4.41 and Corollary 4.45). *Suppose X is a proper regular p-adic formal scheme over $\mathbb{Z}_p[\zeta_p]$ and write $X_n := X \times_{\mathrm{Spf}\,\mathbb{Z}_p[\zeta_p]} \mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^n}]$. There is a functorial-in-X $\mathbb{Z}_p^\times$-equivariant equivalence of $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(X)$-modules of cyclotomic synthetic spectra*

$$\varprojlim_n \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(X_n)_p^\wedge \simeq \left(\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(X/\mathbb{S}_p[\![q-1]\!])_p^\wedge\right) \widehat{\otimes}_{\mathbb{S}_p[\![q-1]\!]^{\mathrm{Tate}}} \mathscr{M}_{\mathrm{Iw}}(1)[1] \tag{1}$$

where:

- *the limit on the left hand side is over transfer maps (see Section 1.2 below),*
- *$\mathbb{S}_p[\![q-1]\!]^{\mathrm{Tate}}$ is the cyclotomic spectrum whose underlying spectrum is the E_∞-ring*

  $$\mathbb{S}_p[\![q-1]\!] := (\mathbb{S}[q])^\wedge_{(p,q-1)} \simeq \varprojlim_n \mathbb{S}_p[\mathbb{Z}/p^n]$$

  with trivial circle action and with cyclotomic Frobenius given by the Tate diagonal [NS18, Section III.1], and endowed with a $\mathbb{Z}_p^\times$-action coming from the usual multiplication action on $\mathbb{Z}/p^n$ as $n \to \infty$;
- *$\mathbb{Z}_p[\zeta_p]$ is a $\mathbb{S}_p[\![q-1]\!]$-algebra via $q \mapsto \zeta_p$;*
- *$\mathrm{THH}(X/\mathbb{S}_p[\![q-1]\!])_p^\wedge$ is equipped with a $\mathbb{Z}_p^\times$-action coming from $\mathbb{S}_p[\![q-1]\!]$;*

[1]For a comprehensive survey of explicit reciprocity laws and the role that Coleman's isomorphism and its generalisations play in this story, see the survey of Venjakob [Ven25] and the references therein.

- $\mathscr{M}_{\mathrm{Iw}}$ *is the cyclotomic spectrum with underlying spectrum with S^1-action is*

$$\mathscr{M}_{\mathrm{Iw}} := \varprojlim_{\tau} \mathbb{S}_p[\![q-1]\!]^{\mathrm{triv}}$$

equipped with the trivial circle action where

$$\tau : \mathbb{S}_p[\![q-1]\!] \to \mathbb{S}_p[\![q-1]\!], \quad q^i \mapsto \begin{cases} 0 & p \nmid i, \\ q^{i/p} & p | i, \end{cases}$$

and with the cyclotomic Frobenius given by the shift map

$$\varprojlim_{\tau} \mathbb{S}_p[\![q-1]\!]^{\mathrm{triv}} \to \left(\varprojlim_{\tau} \mathbb{S}_p[\![q-1]\!]^{\mathrm{triv}}\right)^{tC_p} \simeq \varprojlim_{\tau} \mathbb{S}_p[\![q-1]\!]^{\mathrm{triv}}, \quad (f_n) \mapsto (f_{n+1});$$

- (1) *denotes a Tate twist (for a more precise formulation see Section 4.1); and*
- *the tensor product on the right is the p-complete tensor product of filtered spectra*[2].

In additional to the usual contravariant functoriality in X, this equivalence is also covariantly functorial in transfer maps for regular finite flat maps, as well as Gysin maps for regular closed immersions.

We explain how this recovers Coleman's isomorphism in Remark 1.5.

Remark 1.2. Readers familiar with Iwasawa theory may recognise the map

$$\mathbb{Z}_p[\![q-1]\!] \simeq \pi_0\mathbb{S}_p[\![q-1]\!] \xrightarrow{\pi_0\tau} \pi_0\mathbb{S}_p[\![q-1]\!] \simeq \mathbb{Z}_p[\![q-1]\!]$$

as being the normalised trace of Frobenius operator usually denoted by ψ in the number theory literature. However, the symbol ψ is also used to denote the Adams operations on KU_p which will be used frequently in Section 4, and so we will use the symbol τ for the map defined above. We will abuse notation by also writing τ for the map $\mathbb{Z}_p[\![q-1]\!] \to \mathbb{Z}_p[\![q-1]\!]$ on π_0.

Remark 1.3. The reader may have observed the shift in the filtration as one goes from the left to the right. In fact, the isomorphism in Theorem A has an implicit factor of $\log_{\Delta}(q^p) \in \mathrm{Fil}^1_{\mathrm{Nyg}}\mathbb{Z}_p[\![q-1]\!]^{(1)}\{1\}$ (see [BL22a, Section 2] for details on this element) on the right; indeed, $\log_{\Delta}(q^p)$ can be viewed as a generator of the Tate twist present in the above formula. Since Frobenius preserves this element [BL22a, Proposition 2.5.18], we safely suppress this factor.

For more on this see Remark 4.42.

Theorem A has a lot of immediate consequences for other cohomology theories. For the first corollary, we recall some basic facts about prismatic cohomology of a p-adic formal scheme X:

1. The r'th Breuil-Kisin twisted absolute prismatic cohomology of X, denoted by $\Delta_X\{r\}$, is an object of the derived category $\mathcal{D}(\mathbb{Z}_p)^\wedge_p$ of p-complete $\mathbb{Z}_p$-modules. It is further endowed with a 'Nygaard' filtration $(\mathrm{Fil}^{\geq *}_{\mathrm{Nyg}}\Delta_X)\{r\}$ and a 'Frobenius' map $\varphi : \Delta_X \to \Delta_X$ that induces a 'divided Frobenius' map

$$\varphi\{r\} : (\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\Delta_X)\{r\} \to \Delta_X\{r\}.$$

   If X lives over a perfectoid R, then the above complexes are $A_{\mathrm{inf}}(R)$-modules, and $\varphi\{r\}$ is given by $d^{-r}\varphi$ for d a choice of generator of $\ker(A_{\mathrm{inf}}(R) \to R)$. This choice of d also trivialises the Breuil-Kisin twists, so that $\varphi\{r\}$ is actually independent of the choice of d; a similar phenomenon happens for relative prismatic cohomology.

2. Relative prismatic cohomology $\Delta_{X/\mathbb{Z}_p[\![q-1]\!]}$ of X relative to the q-de Rham prism $\mathbb{Z}_p[\![q-1]\!]$ is a $(p, q-1)$-adically complete $\mathbb{Z}_p[\![q-1]\!]$ module. This object is equipped with a $\varphi_{\mathbb{Z}_p[\![q-1]\!]}$ semi-linear Frobenius map

$$\varphi_X : \Delta_{X/\mathbb{Z}_p[\![q-1]\!]} \to \Delta_{X/\mathbb{Z}_p[\![q-1]\!]}.$$

   We write

$$\Delta_{X/\mathbb{Z}_p[\![q-1]\!]}\{r\} := \mathrm{qdR}_X \otimes_{\mathbb{Z}_p[\![q-1]\!]} \mathbb{Z}_p[\![q-1]\!]\{r\}$$

[2] The synthetic structure on $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(X/\mathbb{S}_p[\![q-1]\!])$ then induces a synthetic structure on the full tensor product.

for the r'th Breuil-Kisin twist, where $\mathbb{Z}_p[\![q-1]\!]\{r\} := \mathbb{Z}_p[\![q-1]\!]\{1\}^{\otimes r}$ and $\mathbb{Z}_p[\![q-1]\!]\{1\}$ is the invertible $\mathbb{Z}_p[\![q-1]\!]$-submodule defined in [BL22a, Section 2].

The Frobenius twist $\Delta^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]} := \Delta_{X/\mathbb{Z}_p[\![q-1]\!]} \otimes_{\mathbb{Z}_p[\![q-1]\!],\varphi} \mathbb{Z}_p[\![q-1]\!]$ is further endowed with a Nygaard filtration $\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\Delta^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}$, such that the Frobenius map upgrades to a filtered map

$$\varphi_X : \mathrm{Fil}^{\geq \bullet}_{\mathrm{Nyg}}\Delta^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]} \to [p]_q^{\bullet}\Delta_{X/\mathbb{Z}_p[\![q-1]\!]}$$

where $[p]_q := \sum_{i=0}^{p-1} q^i$. The divided Frobenius map

$$\varphi_X\{r\} : (\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\Delta^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]})\{r\} \to \Delta_{X/\mathbb{Z}_p[\![q-1]\!]}\{r\}$$

is then given by $[p]_q^{-r}\varphi_X$.

Theorem A has the following concrete application to prismatic cohomology.

Corollary B. *Suppose X is a proper regular p-adic formal scheme over $\mathbb{Z}_p$, and write $X_n := X \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^n}]$. Then, there are $\mathbb{Z}_p^\times$-equivariant equivalences*

$$\varprojlim_n \mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\Delta}_{X_n} \simeq \mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\Delta}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}\widehat{\otimes}_{\mathbb{Z}_p[\![q-1]\!]}\mathcal{M}_{\mathrm{Iw}}(1)[-1],$$

$$\varprojlim_n \widehat{\Delta}_{X_n} \simeq \widehat{\Delta}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}\widehat{\otimes}_{\mathbb{Z}_p[\![q-1]\!]}\mathcal{M}_{\mathrm{Iw}}(1)[-1],$$

where the limit on the left hand side is over prismatic trace maps (see Section 1.2 below), the right hand side of both these equivalences are $(p, q-1)$-adically completed, both the relative and absolute prismatic cohomologies are Nygaard completed[3]*, (1) is the Tate twist denoting tensoring by the $\mathbb{Z}_p^\times$-module $T_p\mu_{p^\infty}$, and*

$$\mathcal{M}_{\mathrm{Iw}} := \pi_0\mathscr{M}_{\mathrm{Iw}} \simeq \varprojlim_\tau \mathbb{Z}_p[\![q-1]\!].$$

Moreover, in addition to the usual contravariant functoriality in X, these equivalences are covariantly functorial for Gysin maps associated to regular closed immersions and for transfers along finite flat regular maps.

Remark 1.4. The module $\mathcal{M}_{\mathrm{Iw}} = \varprojlim_\tau \mathbb{Z}_p[\![q-1]\!]$ is a rather large $\mathbb{Z}_p[\![q-1]\!]$-module. It can be rewritten as

$$\mathcal{M}_{\mathrm{Iw}} \cong \mathrm{Hom}_{\mathbb{Z}_p[\![q-1]\!]}(A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cyc}}), \mathbb{Z}_p[\![q-1]\!]),$$

the module of (continuous) $\mathbb{Z}_p[\![q-1]\!]$-linear maps from $A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cyc}}) \to \mathbb{Z}_p[\![q-1]\!]$, where

$$A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cyc}}) \simeq \left(\operatorname*{colim}_n \mathbb{Z}_p[\![q^{p^{-n}}-1]\!]\right)^\wedge_{p,q-1}$$

is Fontaine's period ring for the integral perfectoid ring $\mathbb{Z}_p^{\mathrm{cyc}}$. This presentation as linear functionals on $A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cyc}})$ is perhaps more psychologically useful. Beyond this presentation, the author does not know of a useful interpretation of $\mathcal{M}_{\mathrm{Iw}}$.

The r'th integral syntomic cohomology $\mathbb{Z}_p(r)^{\mathrm{syn}}(X)$ of [BMS19; BL22a] is given by the fibre

$$\mathbb{Z}_p(r)^{\mathrm{syn}}(X) := \mathrm{fib}\left((\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\Delta_X)\{r\} \xrightarrow{\mathrm{can}-\varphi\{r\}} \Delta_X\{r\}\right) \simeq \mathrm{fib}\left((\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\Delta}_X)\{r\} \xrightarrow{\mathrm{can}-\varphi\{r\}} \widehat{\Delta}_X\{r\}\right)$$

where $\mathrm{can} : (\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\Delta_X)\{r\} \to \Delta_X\{r\}$ is the canonical map from a filtration to the underlying object.

Corollary C. *Suppose X is a qcqs proper regular p-adic formal scheme over $\mathbb{Z}_p[\zeta_p]$. Write $X_n := X \times_{\mathrm{Spf}\,\mathbb{Z}_p[\zeta_p]} \mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^n}]$. Then, for any $r \in \mathbb{Z}$, there is a $\mathbb{Z}_p^\times$-equivariant equivalence of objects in $\mathcal{D}(\mathbb{Z}_p)^\wedge_p$*

$$\begin{aligned}&\varprojlim_n \mathbb{Z}_p(r)^{\mathrm{syn}}(X_n)\\ &\quad\simeq \mathrm{fib}\left(\mathrm{Fil}^{r-1}_{\mathrm{Nyg}}\widehat{\Delta}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}\{r-1\}\widehat{\otimes}_{\mathbb{Z}_p[\![q-1]\!]}\mathcal{M}_{\mathrm{Iw}} \xrightarrow{\Phi_X\{r-1\}-\mathrm{can}} \widehat{\Delta}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}\{r-1\}\widehat{\otimes}_{\mathbb{Z}_p[\![q-1]\!]}\mathcal{M}_{\mathrm{Iw}}\right)(1)[-1]\end{aligned}$$

[3]This statement should still hold true if one does not Nygaard complete both sides. This should follow from a version of Theorem A for the decompleted THH constructed in the upcoming work of Devalapurkar-Raksit-Hahn-Yuan [DHRY].

where the transition maps on the limit on the left hand side are given by the trace maps in syntomic cohomology (see Section 1.2 below), the tensor products on the right are $(p, q-1)$-adically completed, (1) denotes the Tate twist, and where $\Phi_X\{r-1\}$ is the map given by the composition

$$\mathrm{Fil}^{\geq r-1}_{\mathrm{Nyg}} \Delta^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}\{r-1\} \otimes_{\mathbb{Z}_p[\![q-1]\!]} \varprojlim_{\tau} \mathbb{Z}_p[\![q-1]\!] \xrightarrow{\varphi_X\{r-1\}\otimes \mathrm{shift}} \Delta_{X/\mathbb{Z}_p[\![q-1]\!]}\{r-1\} \otimes_{\mathbb{Z}_p[\![q-1]\!]} \varprojlim_{\tau} \mathbb{Z}_p[\![q-1]\!]$$
$$\longrightarrow \Delta^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}\{r-1\} \otimes_{\mathbb{Z}_p[\![q-1]\!]} \varprojlim_{\tau} \mathbb{Z}_p[\![q-1]\!]$$

with the second map induced by

$$\mathrm{id} \otimes 1 : \Delta_{X/\mathbb{Z}_p[\![q-1]\!]} \to \Delta_{X/\mathbb{Z}_p[\![q-1]\!]} \otimes_{\mathbb{Z}_p[\![q-1]\!],\varphi} \mathbb{Z}_p[\![q-1]\!] =: \Delta^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}.$$

This equivalence is moreover functorial in X both contravariantly for the pull-back on cohomology, but also covariantly for Gysin maps associated to regular closed immersions and for transfers along finite flat regular maps.

Remark 1.5. One recovers Coleman's isomorphism from Corollary C by taking $X = \mathrm{Spf}\,\mathbb{Z}_p[\zeta_p]$ and $r = 1$, and using that

$$\mathrm{Fil}^0_{\mathrm{Nyg}} \widehat{\Delta}^{(1)}_{\mathbb{Z}_p[\zeta_p]/\mathbb{Z}_p[\![q-1]\!]} \simeq \widehat{\Delta}^{(1)}_{\mathbb{Z}_p[\zeta_p]/\mathbb{Z}_p[\![q-1]\!]} \simeq \mathbb{Z}_p[\![q-1]\!]$$

as well as the isomorphisms

$$\mathrm{fib}\left(\varprojlim_{\tau} \mathbb{Z}_p[\![q-1]\!] \xrightarrow{\mathrm{shift}-\mathrm{id}} \varprojlim_{\tau} \mathbb{Z}_p[\![q-1]\!]\right) \simeq \mathrm{fib}(\mathbb{Z}_p[\![q-1]\!] \xrightarrow{\tau-\mathrm{id}} \mathbb{Z}_p[\![q-1]\!]) \xleftarrow[q\frac{\mathrm{d}}{\mathrm{d}q}\log]{\simeq} \left(\left(\mathbb{Z}_p[\![q-1]\!]^\times\right)^{N_\varphi=\mathrm{id}}\right)^\wedge_p.$$

One can also prove a statement about p-complete (derived) de Rham cohomology dR_X, and its Hodge filtration $\mathrm{Fil}^{\geq r}_H \mathrm{dR}_X$. This requires a little extra effort.

Corollary D. *Suppose X is a proper regular p-adic formal scheme over $\mathbb{Z}_p[\zeta_p]$, and write $X_n := X \times_{\mathrm{Spf}\,\mathbb{Z}_p[\zeta_p]} \mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^n}]$. Then, there are $\mathbb{Z}_p^\times$-equivariant equivalences*

$$\varprojlim_n \mathrm{Fil}^{\geq r}_H \widehat{\mathrm{dR}}_{X_n} \simeq \mathrm{Fil}^{\geq r-1}_H \widehat{\mathrm{dR}}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]} \widehat{\otimes}_{\mathbb{Z}_p[\![q-1]\!]} \varprojlim_{\tau} \mathbb{Z}_p[\![q-1]\!](1)[-1],$$
$$\varprojlim_n \widehat{\mathrm{dR}}_{X_n} \simeq \widehat{\mathrm{dR}}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]} \widehat{\otimes}_{\mathbb{Z}_p[\![q-1]\!]} \varprojlim_{\tau} \mathbb{Z}_p[\![q-1]\!](1)[-1],$$

where the left hand side is the limit over trace maps in derived de Rham cohomology, the right hand side of all of these equivalences are $(p, q-1)$-adically completed, the $\widehat{(-)}$ on dR_- denotes Hodge-completion[4], the $(-)^{(1)}$ denote Frobenius twists over $\mathbb{Z}_p[\![q-1]\!]$, and the remaining notation is as in Corollary B

Moreover, in addition to the usual contravariant functoriality in X, these equivalences are covariantly functorial for Gysin maps associated to regular closed immersions and for transfers along finite regular maps.

1.2 Transfer Maps

A key technical input is the existence of trace maps in prismatic cohomology in the first place: indeed, the reason the trace maps (aka corestriction maps)

$$\mathrm{R}\Gamma_{\text{ét}}(L, -) \to \mathrm{R}\Gamma_{\text{ét}}(K, -)$$

exist in étale cohomology for finite field extensions L/K is that $\mathrm{Spec}\,L \to \mathrm{Spec}\,K$ is a finite étale map. However, $\mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^n}] \to \mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^{n-1}}]$ is not étale. Thus, one needs to do more work in order to construct the trace.

Relatedly, on the homotopy theoretic side, we have glossed over a key technical detail: that the transfer maps, obtained by applying THH to the map

$$\left(\mathrm{Perf}(X) \xrightarrow{f_*} \mathrm{Perf}(Y)\right) \in \mathrm{Cat}^{\mathrm{perf}}_\infty$$

for f a finite flat regular map, preserve the motivic filtration.

[4] As with Corollary B, one should be able to get Hodge-decompleted statements by using a decompleted version of HH as in [DHRY].

Both these issues are resolved by the following, which is one of the main technical results of this paper and may be of independent interest. In order to express the theorem in full generality[5], we use the theory of prismatic F-gauges, for which we refer the reader to [Bha22]. In the following, for a map $f : X \to Y$ of p-adic formal schemes we denote by $f^{\mathrm{syn}} : X^{\mathrm{Syn}} \to Y^{\mathrm{Syn}}$ the induced map on syntomifications. We will use $f^{\mathrm{syn},*}$ and f^{syn}_* for the *derived* pull-backs and pushforwards respectively.

Theorem E. *Suppose $f : X \to Y$ is a finite locally free regular map of quasi-syntomic qcqs p-adic formal schemes. Then, for any perfect prismatic F-gauge $\mathcal{E} \in \mathrm{Perf}(Y^{\mathrm{Syn}})$, there is a 'trace' map of prismatic F-gauges*

$$\mathrm{tr}^{\mathrm{syn}}_f(\mathcal{E}) : f^{\mathrm{syn}}_*(f^{\mathrm{syn},*}\mathcal{E}) \to \mathcal{E}$$

functorial in $\mathcal{E}$ as well as functorial in base-change in f and composition in f (the latter only up to non-canonical homotopy), satisfying the following properties:

1. *The composition*
$$\mathcal{E} \to f^{\mathrm{syn}}_*(f^{\mathrm{syn},*}\mathcal{E}) \xrightarrow{\mathrm{tr}^{\mathrm{syn}}_f(\mathcal{E})} \mathcal{E}$$
is multiplication by $\deg f$.

2. *Under the identification provided by the projection formula $f^{\mathrm{syn}}_*(f^{\mathrm{syn},*}\mathcal{E}) \simeq f^{\mathrm{syn}}_*\mathcal{O}_{X^{\mathrm{Syn}}} \otimes \mathcal{E}$, one has $\mathrm{tr}^{\mathrm{syn}}_f(\mathcal{E}) \simeq \mathrm{tr}^{\mathrm{syn}}_f(\mathcal{O}_{X^{\mathrm{Syn}}}) \otimes \mathrm{id}_{\mathcal{E}}$.*

3. *(*Étale comparison*) For any $\mathcal{E} \in \mathrm{Perf}(X^{\mathrm{Syn}})$, étale realisation of the trace map on F-gauges*
$$f^{\mathrm{syn}}_* f^{\mathrm{syn},*}\mathcal{E} \to \mathcal{E}$$
coincides with the étale trace map
$$f_{\eta,*}f^*_\eta T_{\text{ét}}(\mathcal{E}) \to T_{\text{ét}}(\mathcal{E})$$
on $\mathbb{Z}_p$-local systems constructed in [LRZ24], where $T_{\text{ét}}$ denotes étale realisation.

4. *If f is finite étale, then this coincides with the trace map for finite étale extensions discussed in [GR24, Section 7.2].*

5. *When $X = \mathrm{Spf}\, S$ and $Y = \mathrm{Spf}\, R$ are both p-torsion free integral perfectoids with S/R finite locally free, the trace map*
$$A_{\mathrm{inf}}(S) \to A_{\mathrm{inf}}(R)$$
is induced by the usual trace map associated to the finite locally free extension $A_{\mathrm{inf}}(R) \to A_{\mathrm{inf}}(S)$.

Remark 1.6. This theorem is conditional on the forthcoming work [DHRY].

Remark 1.7. We have glossed over a subtlety in statement (3) above. In [Bha22], étale realisation of an F-gauge is only well-defined for perfect F-gauges. However, $f^{\mathrm{syn}}_*\mathcal{O}_{X^{\mathrm{Syn}}}$ for f a finite map that is not étale will not be perfect, and so *a priori* étale realisation as defined in [Bha22] is not well-defined. We address this issue in Remark 3.26.

The key point in the proof of Theorem E is that transfer maps exist for free for localising invariants, so all one needs to do is to check that the associated motivic filtrations are preserved by the transfer map. Following a suggestion of Clausen, we do this by factoring our finite regular map $f : X \to Y$ through a regular closed immersion $X \hookrightarrow \mathbb{P}^n_Y$ into projective space, and then using the Gysin maps of Longke Tang [Tan24; Tan26] as well as the projective bundle formula. Once we know that the motivic filtrations are preserved, the rest of Theorem E is then straightforward using standard techniques in the theory of prismatic cohomology.

[5]To actually prove Coleman's isomorphism, we only really need to know that the transfer maps on THH preserve the motivic filtrations. The rest of Theorem E is simply a sanity check to ensure that the transfer maps induced on the associated graded of the motivic filtration satisfy expected properties.

1.3 Idea Of Proof of Theorem A

Theorem A is essentially a consequence of an explicit description of the filtered transfer map in motivically filtered THH, stated in Theorem 4.32. Partly because stating the result rigorously will require introducing a lot more notation, and partly for the convenience of those readers who may not be as familiar with THH, we instead write the following more concrete consequence in prismatic cohomology.

Proposition F (Consequence of Theorem 4.32)**.** *Let X be any bounded quasi-syntomic p-adic formal scheme over $\mathbb{Z}_p[\zeta_p]$, and set $X_n := X \times_{\operatorname{Spf}\mathbb{Z}_p[\zeta_p]} \operatorname{Spf}\mathbb{Z}_p[\zeta_{p^n}]$. Then, there are commuting diagrams*

$$\begin{tikzcd}
\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}_{X_{n+1}} \arrow[r] \arrow[d, "\mathrm{tr}^{\mathbbl{\Delta}}"] & \mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}^{(1)}_{X_{n+1}/\mathbb{Z}_p[\![q_n-1]\!]}\{r\} \arrow[r, "\partial_{n+1}"] \arrow[d, "T\{r\}"] & \mathrm{Fil}^{\geq r-1}_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}^{(1)}_{X_{n+1}/\mathbb{Z}_p[\![q_n-1]\!]}\{r\} \arrow[d, "\mathrm{proj}_{X,n}\{r\}"] \\
\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}_{X_n} \arrow[r] & \mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}^{(1)}_{X_n/\mathbb{Z}_p[\![q_{n-1}-1]\!]}\{r\} \arrow[r, "\partial_n"] & \mathrm{Fil}^{\geq r-1}_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}^{(1)}_{X_n/\mathbb{Z}_p[\![q_{n-1}-1]\!]}\{r\}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\widehat{\mathbbl{\Delta}}_{X_{n+1}} \arrow[r] \arrow[d, "\mathrm{tr}^{\mathbbl{\Delta}}"] & \widehat{\mathbbl{\Delta}}^{(1)}_{X_{n+1}/\mathbb{Z}_p[\![q_n-1]\!]}\{r\} \arrow[r, "\partial_{n+1}"] \arrow[d, "T\{r\}"] & \widehat{\mathbbl{\Delta}}^{(1)}_{X_{n+1}/\mathbb{Z}_p[\![q_n-1]\!]}\{r\} \arrow[d, "\mathrm{proj}_{X,n}\{r\}"] \\
\widehat{\mathbbl{\Delta}}_{X_n} \arrow[r] & \widehat{\mathbbl{\Delta}}^{(1)}_{X_n/\mathbb{Z}_p[\![q_{n-1}-1]\!]}\{r\} \arrow[r, "\partial_n"] & \widehat{\mathbbl{\Delta}}^{(1)}_{X_n/\mathbb{Z}_p[\![q_{n-1}-1]\!]}\{r\}
\end{tikzcd}$$

where

- $\mathrm{tr}^{\mathbbl{\Delta}}_n$ *are the prismatic trace maps* $\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}_{X_{n+1}} \to \mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}_{X_n}$ *and* $\widehat{\mathbbl{\Delta}}_{X_{n+1}} \to \widehat{\mathbbl{\Delta}}_{X_n}$ *constructed in Theorem E,*
- *the horizontal rows are fibre sequences,*
- $\partial_n = \frac{\gamma_n - 1}{q-1}$ *for* γ_n *a topological generator of* $(1+p^n\mathbb{Z}_p)$ *(chosen so that* $\gamma_{n+1} = \gamma_n^p$*) acting on* $\widehat{\mathbbl{\Delta}}_{X_n/\mathbb{Z}_p[\![q_{n-1}-1]\!]}$ *via the usual action on the prism* $(\mathbb{Z}_p[\![q_{n-1}-1]\!], [p]_q)$*,*
- $d_r := [p]^r_{q_1}$ *if* $r \geq 0$*, and* $d_r = 0$ *if* $r < 0$*,*
- $\mathrm{proj}_{X,n}$ *is the composition*

$$\mathrm{proj} : \mathrm{Fil}^r_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}^{(1)}_{X_{n+1}/\mathbb{Z}_p[\![q_n-1]\!]} \simeq \mathrm{Fil}^r_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}^{(1)}_{X_n/\mathbb{Z}_p[\![q_{n-1}-1]\!]} \otimes_{\mathbb{Z}_p[\![q_n-1]\!]} \mathbb{Z}_p[\![q_{n+1}-1]\!] \xrightarrow{\mathrm{id}\otimes\mathrm{proj}} \mathrm{Fil}^r_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}^{(1)}_{X_n/\mathbb{Z}_p[\![q_{n-1}-1]\!]}$$

  with $\mathrm{proj} : \mathbb{Z}_p[\![q_{n+1}-1]\!] \to \mathbb{Z}_p[\![q_n-1]\!]$ *given by* $q^i_{n+1} \mapsto 0$ *if* $p \nmid i$ *and* $q^{pi}_{n+1} \mapsto q^i_n$*, and*
- $T_n := \sum_{i=0}^{p-1} \gamma_n^i \circ \mathrm{proj} : \mathrm{Fil}^r_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}^{(1)}_{X_{n+1}/\mathbb{Z}_p[\![q_n-1]\!]} \to \mathrm{Fil}^r_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}^{(1)}_{X_n/\mathbb{Z}_p[\![q_{n-1}-1]\!]}$.

> **Remark 1.8.** The identification of $\mathbbl{\Delta}_{X_n}$ implicit in the above theorem is essentially a corollary of [Liu25, Theorem 1.0.2] (see Remark 4.10 for the precise relationship), though the proof method is completely different. The identification of $\mathrm{Fil}^{\geq *}_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}_{X_n}$ in terms of the Nygaard filtration on relative prismatic cohomology is however new.
>
> The identifications of $\widehat{\mathbbl{\Delta}}_{X_n}$ and $\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\mathbbl{\Delta}}_{X_n}$ given above are essentially corollaries of Devalapurkar-Raksit's calculation of $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}])$ combined with the symmetric monoidality of THH and an analysis of motivic filtrations; see Proposition 4.26.
>
> Apart from the description of the Nygaard filtration, all of the new content here is the calculation of the transfer maps which is carried out in Theorem 4.32.

Proposition F (or rather its homotopy theoretic upgrade Theorem 4.32) is proven by a calculation using the explicit description of $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}])$ and $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q_1-1]\!])$ as cyclotomic spectra due to Devalapurkar-Raksit (recalled in Section 2.4.2) and the explicit calculation of the free loop transfer map $\mathrm{THH}(\mathbb{S}[\mathbb{Z}]) \to \mathrm{THH}(\mathbb{S}[p\mathbb{Z}])$ due to Schlichtkrull (recalled in Section 2.4.1), followed by an analysis of motivic filtrations. Apart from Theorem E, this is the only other place in the paper that the homotopy theoretic input is essential, since we use in an essential way the structure of THH of spherical group algebras. Moreover, the reduction to the case of a point uses symmetric monoidality of THH, whereas an analogous Künneth type formula for absolute prismatic cohomology fails.

Once one has the description of the prismatic trace map as in Proposition F (respectively the homotopy theoretic result in Theorem 4.32), one can prove Corollary B (respectively Theorem A) directly. The key point is that upon taking limits in n, the middle term in the fibre sequence of Proposition F vanishes since the $1+p^n\mathbb{Z}_p$ action on $\Delta_{-/\mathbb{Z}_p[\![q_{n-1}-1]\!]}$ is trivial modulo $q-1$; see [BL22a, Section 3.8] for the case $n=1$ and [Liu25, Theorem 1.0.6] in general. Indeed, once the action is trivialised, the sum $\sum_{i=0}^{p-1}\gamma_n^i$ identifies with multiplication by p, and taking the limit over this is zero in the p-complete category.

The above suffices to give a description of $\varprojlim_n \mathrm{THH}(X_n)$ as a spectrum with S^1-action. The description of Frobenius in Theorem A is then obtained by a slightly tedious diagram chase; the key point however is to understand how the map τ interacts with Frobenius, which uses an incarnation of τ in terms of the ∞-semiadditive structure of $K(1)$-local spectra.

1.4 Future Work

1. In [CC99] (see also [Ber03] and [LLZ12]), Coleman's isomorphism and its generalisations are used to construct the Perrin-Riou p-adic regulator map, which interpolates between inverses of the Bloch-Kato exponential map and the dual exponential map. Recently, in [FKM26], a prismatic interpretation of the Bloch-Kato exponential map was studied. In future work, we will study the relation between the work carried out here and the work of [FKM26], in order to construct an integral version of Perrin-Riou's p-adic regulator.

2. As previously mentioned, the Coleman isomorphism can be used to construct the Kubota-Leopoldt p-adic L-function (see [CS06] for instance). However, it has already been observed in [AHR10] that the Kubota-Leopoldt p-adic L-function has a description in $K(1)$-local homotopy theory as a certain endomorphism of p-complete topological complex K-theory KU_p. One can then ask a rather obvious question: how are these two related? It should be the case that Theorem A connects these two observations. In future work, we will explore this connection further.

3. One could ask for a description of the inverse limit of prismatic/syntomic cohomology over towers other than the cyclotomic tower.

   Coleman's original theorem was for towers associated to any Lubin-Tate formal group (not just $\widehat{\mathbb{G}}_m$) in which he replaces what we now call the q-de Rham prism with the ring of functions on the Lubin-Tate formal group. However, such generalisations—where we replace the q-de Rham prism with rings equipped with Frobenius lifts behaving like the p-power map in higher height formal groups—will require a theory of prismatic cohomology associated to that formal group (where the usual prismatic cohomology is the one associated to $\widehat{\mathbb{G}}_m$). Such a generalisation of prismatic cohomology to Lubin-Tate formal groups has been started in [Mar25]. Once such theories are developed, and once one has a spectral level description of THH of each level of the tower, one should be able to give a generalisation of Coleman's result using the methods of this paper.

   On the other hand, one could also ask for generalisations to more general perfectoid towers, still using the usual prismatic cohomology. For the strategy of this paper, this will require a detailed understanding of the THH of this perfectoid tower, as well as a computation of the transfer in THH. This is likely to succeed if the perfectoid tower is built from the p-power torsion points $G_n := G[p^n]$ of a sufficiently nice p-divisible group G, for then one could try and calculate the free loop transfer

$$\mathbb{S}[LBG_{n+1}] \simeq \mathrm{THH}(\mathbb{S}[G_{n+1}]) \to \mathrm{THH}(\mathbb{S}[G_n]) \simeq \mathbb{S}[LBG_n].$$

4. There is also a relationship to the theory of higher cyclotomic extensions [CSY24]. Indeed, by [CSY24, Proposition 4.11] applied to the ∞-semiadditive category $L_{K(1)}\,\mathrm{Sp}$, there is a decomposition

$$L_{K(1)}\mathbb{S}[B^2\mathbb{Z}_p] \simeq L_{K(1)}\mathbb{S}[B^2\mathbb{Z}_p] \oplus L_{K(1)}\mathbb{S}[\zeta_{p^\infty}^{(1)}]$$

   where $L_{K(1)}\mathbb{S}[\zeta_{p^\infty}^{(1)}]$ is the infinite cyclotomic extension of $L_{K(1)}\mathbb{S}$. Then, the map $\tau : \mathbb{Z}_p[\![q-1]\!] \to \mathbb{Z}_p[\![q-1]\!]$ is $\mathrm{KU}_p^0(-)$ of the inclusion map[6]

$$L_{K(1)}\mathbb{S}[B^2\mathbb{Z}_p] \xrightarrow{\mathrm{id}\oplus 0} L_{K(1)}\mathbb{S}[B^2\mathbb{Z}_p] \oplus L_{K(1)}\mathbb{S}[\zeta_{p^\infty}^{(1)}] \simeq L_{K(1)}\mathbb{S}[B^2\mathbb{Z}_p].$$

   It would be interesting to try and understand the calculations in this paper from the perspective of the theory of higher cyclotomic extensions. Of course, *a priori*, we do not work in an ∞-semiadditive category, so one cannot directly generalise. However, this framework is likely to give a more conceptual understanding of the calculations carried out in Section 4.

[6] To see this, one should compare Construction 4.15 with the discussion in [DR25, Section 2.2].

1.5 Summary of Sections

Section 2 gives some preliminaries on cyclotomic spectra, localising invariants, and motivic filtrations that we will need. In Section 3, we prove Theorem E: in Section 3.1 and Section 3.2 we show that the transfer map on localising invariants respects the corresponding motivic filtrations, Section 3.3 proves the existence part of Theorem E, Section 3.4 gives a calculation of the trace map for perfectoids, and finally Section 3.5 establishes the comparison of our transfer maps with the étale traces of [LRZ24].

The bulk of this paper is Section 4; the plan for this section is as follows. In Section 4.1, we list a collection of technical lemmas that will be used in the following; the reader may safely skip this on a first reading and refer back to lemmas as necessary. In Section 4.2, we prove Proposition F and its homotopy theoretic upgrade for the case $X = \operatorname{Spf} \mathbb{Z}_p$, using the results in Section 2.4.1 and Section 2.4.2. Using symmetric monoidality of THH and an analysis of its motivic filtration, we may then obtain a description (see Theorem 4.32) of the transfer map for motivically filtered THH whose associated graded yields Proposition F; this is the subject of Section 4.3. With this description in hand, we then prove Theorem A in Section 4.4. Finally, in Section 4.5 we prove Corollary D.

Notation and Conventions

1. p is an odd prime number.

2. Throughout, we will set $\mathbb{Q}_p^{\mathrm{cyc}} := \mathbb{Z}_p[\mu_{p^\infty}]_p^\wedge[\frac{1}{p}]$ to be the $\mathbb{Z}_p^\times$-Galois perfectoid extension of $\mathbb{Q}_p$ containing all p-power roots of unity. We fix a compatible choice of primitive p-power roots of unity ζ_{p^n} (so $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ and $\zeta_p \neq 1$) inside $\mathbb{Q}_p^{\mathrm{cyc}}$. We have the $(\mathbb{Z}/p^n\mathbb{Z})^\times$-Galois extension $\mathbb{Q}_p(\zeta_{p^n})$ of $\mathbb{Q}_p$. We will denote the integral subrings of these fields by $\mathbb{Z}_p^{\mathrm{cyc}}$ and $\mathbb{Z}_p[\zeta_{p^n}]$ respectively.

3. For R a classical commutative ring, we write $\mathcal{D}(R)$ for the stable ∞-category of R-modules. If R is furthermore p-complete, then we write $\mathcal{D}(R)_p^\wedge$ for the full subcategory of p-complete R-modules.

4. For any ∞-category $\mathcal{C}$, write $\mathrm{gr}\mathcal{C} := \mathrm{Fun}(\mathbb{Z}^\delta, \mathcal{C})$ for the category of graded objects of $\mathcal{C}$, and write $\mathrm{Fil}\mathcal{C} := \mathrm{Fun}((\mathbb{Z}, \leq), \mathcal{C})$ for the category of decreasingly filtered objects. For more on filtered objects see Section 2.3.

5. For G a topological group, we will write $\mathcal{C}^{BG} := \mathrm{Fun}(BG, \mathcal{C})$ for the category of $\mathcal{C}$-valued local systems on BG, equivalently the category of objects of $\mathcal{C}$ with G-action, equivalently Borel G-objects in $\mathcal{C}$. Some more notation related to spectra with G-action is given in Section 2.1.1.

6. Write Sp for the symmetric monoidal category of spectra, with symmetric monoidal product written as $\otimes$ and unit the sphere spectrum $\mathbb{S}$. Write $\mathrm{map}(-,-)$ for the internal mapping spectrum in Sp. One can, and should, think of Sp as the "derived category $\mathcal{D}(\mathbb{S})$ of $\mathbb{S}$-modules". For $X \in \mathcal{D}(\mathbb{Z})$, we will abuse notation by writing $X \in \mathrm{Sp}$ for the Eilenberg-MacLane spectrum associated to X.

   Throughout, CAlg will denote the category of E_∞-ring spectra, and CAlg_R the category of E_∞-R-algebras for R an E_∞-ring. We will need to consider $\mathrm{CAlg}_{\mathbb{Z}}^\heartsuit$, the 1-category of classical commutative $\mathbb{Z}$-algebras.

   Write $(-)[1] : \mathrm{Sp} \to \mathrm{Sp}$ for the suspension functor in Sp, which induces the canonical shift functor on the corresponding triangulated category. One has that
$$\pi_n(X[1]) \simeq \pi_{n-1}X$$
   for all $X \in \mathrm{Sp}$. Cohomologically, we have $H^n(C[1]) = H^{n+1}(C)$ for $C \in \mathcal{D}(R)$.

   Write $\tau_{\geq n}$ for the truncation functor on Sp, so that
$$\pi_i \tau_{\geq n} X := \begin{cases} 0 & i < n, \\ \pi_i X & i \geq n. \end{cases}$$

7. Write Sp_p for the full-subcategory of Sp consisting of p-complete spectra. Unless otherwise specified, in Section 4 we will always be working in the p-complete category, and so we will suppress notation. This is particularly relevant for all tensor products showing up in Section 4: we use $\otimes$ instead of $\widehat{\otimes}$ to denote the p-complete tensor product. Similarly, for the Tate construction for a group G (see Section 2.1.1), we write $(-)^{tG}$ to mean the p-completion of the usual Tate construction on G.

8. For any topological space A, we will write $\mathbb{S}_p[A]$ for the p-complete suspension spectrum $(\Sigma^\infty_+ A)_p^\wedge$. As usual, we are suppressing the p-completion.

9. For any E_∞-ring R in a symmetric monoidal stable ∞-category $\mathcal{C}$, write $\mathrm{Mod}_R(\mathcal{C})$ for the category of R-module objects in $\mathcal{C}$. If R is a classical commutative ring, then there is an identification $\mathrm{Mod}_R(\mathrm{Sp}) \simeq \mathcal{D}(R)$.

10. We will write the n'th Frobenius twist of $\mathbb{Z}_p[\![q-1]\!]$ as $\mathbb{Z}_p[\![q_n-1]\!]$, where $q_n = q^{p^{-n}}$. Thus, the non-linear Frobenius map $\mathbb{Z}_p[\![q-1]\!] \to \mathbb{Z}_p[\![q-1]\!]$ can be viewed as a $\mathbb{Z}_p[\![q-1]\!]$-algebra map
$$\mathbb{Z}_p[\![q-1]\!] \to \mathbb{Z}_p[\![q_1-1]\!].$$
Of course, Frobenius induces an isomorphism $\mathbb{Z}_p[\![q_{n+1}-1]\!] \xrightarrow{\simeq} \mathbb{Z}_p[\![q_n-1]\!]$ where $q_{n+1} \mapsto q_n$.

11. We reserve the older notation $\varprojlim_n$ for specifically sequential limits, while denoting limits over any other indexing diagram with $\lim_i$.

12. Let KU_p denote the p-adic completion of topological complex K-theory. There is a continuous action of $\mathbb{Z}_p^\times$ via Adams operations on KU_p, which will be denoted by $\psi^g : \mathrm{KU}_p \to \mathrm{KU}_p$ for $g \in \mathbb{Z}_p^\times$. We write $\mathrm{ku}_p := \tau_{\geq 0}\mathrm{KU}_p$. The Adams operations act on ku_p, and we will also write ψ^g for this.

13. For any module M over ku_p, write
$$M(1)[2] := \tau_{\geq 2}\mathrm{ku}_p \otimes_{\mathrm{ku}_p} M.$$
This notation makes sense since multiplication by the Bott class yields the equivalence $\tau_{\geq 2}\mathrm{ku}_p \simeq \mathrm{ku}_p[2]$ on underlying spectra, and the Adams operations act on the Bott class via multiplication.

14. Recall that a map $R \to S$ of classical commutative p-complete rings is p-quasi-syntomic (resp. a p-quasi-syntomic cover) if S/p is flat (resp. faithfully flat) over R/p and $\mathbb{L}_{S/R}/p \in \mathcal{D}(S/p)$ has Tor amplitude in $[-1,0]$ (cf. [BMS19, Definition 4.10]). A ring R is p-quasi-syntomic if $\mathbb{Z}_p \to R$ is p-quasi-syntomic. As usual, we will abuse notation by only writing quasi-syntomic rather than p-quasi-syntomic.

    We write QSyn_R for the Grothendieck site of quasi-syntomic maps $R \to S$, where the covers are the quasi-syntomic covers. This site has a basis by the quasi-regular semi-perfectoids (see [BMS19, Section 4.4]), which we will always abbreviate by QRSP.

    We also comment on the notion of regularity that we are assuming in Theorem E. We say a finite map $R \to S$ of classical commutative p-complete rings is regular if it is Koszul-regular in the sense of [SP, Tag068E], i.e. if it is a locally complete intersection.

15. For $f : X \to Y$ a map of p-adic formal schemes, we denote by $f^{\mathrm{syn}} : X^{\mathrm{Syn}} \to Y^{\mathrm{Syn}}$ the induced map on the syntomic stacks of [Bha22]. We will use $f^{\mathrm{syn},*}$ and f^{syn}_* for the derived pull-backs and pushforwards respectively as we shall not need their underived versions. However, to prevent confusion, we still use the notation $\mathrm{R}\Gamma(-)$ to denote derived global sections.

Acknowledgements

TO DO

2 Preliminaries

2.1 Cyclotomic Spectra and THH

We follow the streamlined approach to cyclotomic spectra and THH by Nikolaus-Scholze [NS18].

2.1.1 Group Actions in Homotopy Theory

Suppose G is a topological group, and $\mathcal{C}$ is a stable ∞-category. Consider a subgroup $H \subset G$, and let $\pi : BH \to BG$ and $\rho : BG \to B(G/H)$ be the canonical maps on classifying spaces induced by the inclusion $H \hookrightarrow G$ and the projection $G \to G/H$. Pull-back $\pi^* : \mathcal{C}^{BG} \to \mathcal{C}^{BH}$ corresponds to the restriction functor, where an object with G-action is sent to the object with H-action obtained by forgetting the rest of the G-action. Similarly, $\rho^* : \mathcal{C}^{B(G/H)} \to \mathcal{C}^{BG}$ implements the functor sending an object with G/H-action to the object with G-action where G acts via the quotient G/H.

The functor ρ^* has left and right adjoints, namely
$$\rho_! : \mathcal{C}^{BG} \to \mathcal{C}^{B(G/H)} \quad \text{and} \quad \rho_* : \mathcal{C}^{BG} \to \mathcal{C}^{B(G/H)}$$

obtained by taking the colimit and limit along H respectively. These functors are often denoted by $\rho_! =: (-)_{hH}$ and $\rho_* =: (-)^{hH}$, and are referred to as the homotopy orbits and homotopy fixed points respectively. On homotopy groups, these functors give group homology and group cohomology respectively.

Now, if G is a compact Lie group of dimension d, and $\mathcal{C}$ is stable, then there is a norm map

$$\mathrm{Nm}_G : (-)_{hG}[d] \to (-)^{hG}$$

obtained informally by integrating along fibres. See [NS18, Section I.1, I.4] for a quick overview.

Definition. The cofibre of Nm_G is defined to be *Tate-cohomology* $(-)^{tG}$.

If G is finite and so $d = 0$, then the usual norm map Nm_G coincides with the map

$$\sum_{g \in G} g : (-)_{hG} \to (-)^{hG}$$

from orbits to fixed points, and the resulting cofibre $(-)^{tG}$ coincides with the usual notion of Tate cohomology.

Since $(-)_{hG}$ and $(-)^{hG}$ are both exact functors of stable ∞-categories, so too is $(-)^{tG}$. However, not much more can be said about the commutation of $(-)^{tG}$ with limits or colimits in great generality. With connectivity conditions, we have the following technical lemma.

Lemma 2.1 ([AN21, Lemma 2.11(b)])**.** *Suppose that $F : I \to \mathrm{Sp}^{BS^1}$ is an I-diagram in spectra with S^1-action whose limit is $X := \lim_i F(i)$, such that there exists $N \in \mathbb{Z}$ satisfying $F(i) \in \mathrm{Sp}_{\geq N}$ for all i. Suppose I satisfies the following property: there exists $d \in \mathbb{Z}$ such that the functor $\lim_I : \mathrm{Sp}^I \to \mathrm{Sp}$ sends $\mathrm{Sp}^I_{\geq 0}$ to $\mathrm{Sp}_{\geq -d}$. Then,*

$$X_{hC_p} \simeq \lim_i (F(i)_{hC_p}) \quad \text{and} \quad X^{tC_p} \simeq \lim_i (F(i)^{tC_p}).$$

In particular, countable products and $\mathbb{N}$-indexed sequential limits in Sp^{BS^1} commute with $(-)_{hC_p}$ and $(-)^{tC_p}$.

2.1.2 Cyclotomic Spectra

Definition. A *(p-typical) cyclotomic spectrum* is the following data:

- a spectrum X with S^1-action, and
- a S^1-equivariant map
$$\varphi_p : X \to X^{tC_p}.$$

The stable ∞-category of cyclotomic spectra is denoted CycSp.

Construction 2.2. There is a functor

$$(-)^{\mathrm{triv}} : \mathrm{Sp} \to \mathrm{CycSp}$$

which sends a spectrum X to the cyclotomic spectrum X^{triv} whose underlying spectrum is X equipped with the trivial S^1-action, and whose cyclotomic Frobenius is the S^1-equivariant composition

$$X \to X^{hC_p} \to X^{tC_p}.$$

The first map is induced from the fact that X has trivial circle action.

Cyclotomic spectra have a symmetric monoidal structure as well, with unit $\mathbb{S}^{\mathrm{triv}}$. For a precise ∞-categorical construction of this structure, see [NS18, Construction IV.2.1].

Construction 2.3. Suppose X and Y are cyclotomic spectra with cyclotomic Frobenii $\varphi_X : X \to X^{tC_p}$ and $\varphi_Y : Y \to Y^{tC_p}$. We define the tensor product of X and Y as a cyclotomic spectrum with underlying spectrum with S^1-action given simply by $X \otimes Y$ with the diagonal circle action, and with the cyclotomic Frobenius given by

$$X \otimes Y \xrightarrow{\varphi_X \otimes \varphi_Y} X^{tC_p} \otimes Y^{tC_p} \to (X \otimes Y)^{tC_p}$$

where the second arrow is from the lax symmetric monoidal structure on $(-)^{tC_p}$. We abuse notation and write $X \otimes Y$ viewed as a cyclotomic spectrum.

We will use the following fact implicitly.

Lemma 2.4. *The functor*

$$(-)^{\mathrm{triv}} : \mathrm{Sp} \to \mathrm{CycSp}$$

is symmetric monoidal.

Definition. For $X \in \mathrm{CycSp}$, write

$$\mathrm{TC}(X) := \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X).$$

This defines a functor

$$\mathrm{TC} : \mathrm{CycSp} \to \mathrm{Sp}\,.$$

Essentially by definition, one checks that TC is right adjoint to the functor $(-)^{\mathrm{triv}} : \mathrm{Sp} \to \mathrm{CycSp}$, and that for $X \in \mathrm{CycSp}$

$$\mathrm{TC}(X) \simeq \mathrm{fib}(X^{hS^1} \xrightarrow{\mathrm{can}-\varphi_X^{hS^1}} (X^{tC_p})^{hS^1})$$

Here are some more examples of cyclotomic spectra.

Construction 2.5. Suppose X is a connective cyclotomic spectrum. We define a new cyclotomic spectrum $X^{(-1)}$ whose underlying S^1-spectrum is $\tau_{\geq 0}(X^{tC_p})$ (where $C_p \subset S^1$ is the subgroup of p'th roots of unity in $\mathbb{C}^\times$). There is a canonical S^1-equivariant map $\widetilde{\phi}_X : X \to \tau_{\geq 0}(X^{tC_p})$ factoring the cyclotomic Frobenius $\phi_X : X \to X^{tC_p}$ of X (since X is connective). The cyclotomic Frobenius on $X^{(-1)}$ is then the composition

$$\tau_{\geq 0}X^{tC_p} \to X^{tC_p} \xrightarrow{\widetilde{\phi}_X^{tC_p}} (\tau_{\geq 0}X^{tC_p})^{tC_p}.$$

One checks that the map $X \to \tau_{\geq 0}(X^{tC_p})$ upgrades to a map

$$\phi_X^0 : X \to X^{(-1)}$$

of cyclotomic spectra.

It turns out that the functor $(-)^{(-1)} : \mathrm{CycSp}_{\geq 0} \to \mathrm{CycSp}$ is lax symmetric monoidal.

If X is an S^1-equivariant spectrum endowed with the trivial cyclotomic structure, then we write $X^{(-1)}$ for the cyclotomic spectrum $(X^{\mathrm{triv}})^{(-1)} := \tau_{\geq 0}(X^{\mathrm{triv}})^{tC_p}$.

Construction 2.6. Suppose A is an E_∞-ring. Let A^{Tate} be the cyclotomic spectrum whose underlying spectrum with circle action is A equipped with the trivial circle action, and whose cyclotomic Frobenius is given by the Tate-valued Frobenius $\varphi_A^{\mathrm{Tate}} : A \to A^{\mathrm{Tate}}$ defined in [NS18, Section IV.1], given by

$$A \xrightarrow{\Delta_p} (A \otimes \cdots \otimes A)^{tC_p} \xrightarrow{\mathrm{mult}} A^{tC_p}.$$

One has $\mathbb{S}^{\mathrm{Tate}} \simeq \mathbb{S}^{\mathrm{triv}}$ (see [NS18, Example IV.1.2(ii)]). In general, the cyclotomic Frobenius on A^{Tate} can be computed via power operations on A.

We end with a technical lemma. In general, the forgetful functor $\mathrm{CycSp} \to \mathrm{Sp}^{BS^1}$ does not commute with arbitrary limits. However, in certain special cases, it does.

Lemma 2.7. *Suppose $F : I \to \mathrm{CycSp}_{\geq 0}$ is an I-diagram in connective cyclotomic spectra (i.e. the underlying spectrum is connective) where I is a diagram as in Lemma 2.1. Then, $\lim_I F$ exists in $\mathrm{CycSp}_{\geq 0}$, and the underlying spectrum with circle action of $\lim_I F$ is given by $\lim_i F(i) \in \mathrm{Sp}^{BS^1}$.*

Proof. This is clear by Lemma 2.1, since we can define the cyclotomic Frobenius $\varphi_p : \lim_I F \to (\lim_I F)^{tC_p}$ by the composition

$$\lim_i F(i) \xrightarrow{\lim_i \varphi_{F(i)}} \lim_i F(i)^{tC_p} \simeq \left(\lim_i F(i)\right)^{tC_p}.$$

That this satisfies the correct universal property is easy to see, using the fact that the forgetful functor $\mathrm{Sp}^{BS^1} \to \mathrm{Sp}$ commutes with all limits. ☐

2.1.3 THH and Related Functors

Definition. For an E_∞-ring spectrum A, the *topological Hochschild homology* $\mathrm{THH}(A)$ is a spectrum with S^1-action defined as the geometric realization of the cyclic bar construction

$$\mathrm{THH}(A) \simeq A \otimes_{A \otimes A^{\mathrm{op}}} A.$$

The cyclic bar construction may equivalently be expressed as the tensor of A with the circle S^1 in the ∞-category of E_∞-ring spectra:

$$\mathrm{THH}(A) \simeq A \otimes S^1.$$

This endows $\mathrm{THH}(A)$ with a natural action of the circle group S^1. For A an E_∞-ring, the definition of THH essentially yields the following universal property.

Proposition 2.8 ([NS18, Proposition IV.2.2])**.** *The (non-S^1-equivariant) map $A \to \mathrm{THH}(A)$ is initial among all maps $A \to R$ where R is equipped with an S^1-action.*

The circle action on $\mathrm{THH}(A)$ extends to a canonical cyclotomic structure

$$\varphi_A : \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$$

making $\mathrm{THH}(A)$ a ring object in cyclotomic spectra. The following is essentially [NS18, Corollary IV.2.4].

Lemma 2.9. *There is a functorial map $\pi_A : \mathrm{THH}(A) \to A^{\mathrm{Tate}}$ of cyclotomic spectra that, on underlying spectra, is a retract of the canonical (non-S^1-equivariant) map $A \to \mathrm{THH}(A)$.*

Proposition 2.10. $\mathrm{THH} : \mathrm{CAlg} \to \mathrm{CycSp}$ *is symmetric monoidal, i.e. for E_∞-rings A, B, C, we have*

$$\mathrm{THH}(B \otimes_A C) \simeq \mathrm{THH}(B) \otimes_{\mathrm{THH}(A)} \mathrm{THH}(C).$$

Corollary 2.11. $\mathrm{THH}(A \otimes_{\mathbb{S}} B) \simeq \mathrm{THH}(A) \otimes_{\mathbb{S}^{\mathrm{triv}}} \mathrm{THH}(B)$.

One can also define relative THH, at least as spectra with S^1-action.

Definition. For $R \to S$ a map of E_∞-rings, define the E_∞-R-algebra with S^1-action given by

$$\mathrm{THH}(S/R) := S \otimes_{S \otimes_R S} S.$$

In general, one has an equivalence of spectra with S^1-action

$$\mathrm{THH}(S/R) \simeq \mathrm{THH}(S) \otimes_{\mathrm{THH}(R)} R^{\mathrm{triv}}.$$

If R is a $\mathbb{Z}$-algebra, one often writes $\mathrm{HH}(S/R) := \mathrm{THH}(S/R)$. If $R = \mathbb{Z}$, then we simply write $\mathrm{HH}(S) := \mathrm{THH}(S/\mathbb{Z})$.

The cyclotomic structure is much more subtle; see [BMY24] for more. However, we will be interested only in the case $R = \mathbb{S}_p[\![q-1]\!]$, where the latter is defined as follows.

Notation. Write $\mathbb{S}[\![x-1]\!]$ for the $(x-1)$-adic completion of $\mathbb{S}[x] := \mathbb{S}[\mathbb{N}]$, i.e.

$$\mathbb{S}[\![x-1]\!] \simeq \varprojlim_n \mathrm{cofib}((x-1)^n : \mathbb{S}[x] \to \mathbb{S}[x]).$$

Upon p-completion, we equivalently have

$$\mathbb{S}_p[\![x-1]\!] \simeq \varprojlim_n \mathbb{S}_p[\mathbb{Z}/p^n].$$

Thus, we can think of $\mathbb{S}_p[\![x-1]\!]$ as equivalently the completed group algebra $\mathbb{S}[\![\mathbb{Z}_p]\!]$.

Definition. The cyclotomic structure on $\mathrm{THH}(S/\mathbb{S}[\![x-1]\!])$ is given by the following tensor product in CycSp

$$\mathrm{THH}(S/\mathbb{S}[\![x-1]\!]) := \mathrm{THH}(S) \otimes_{\mathrm{THH}(\mathbb{S}[\![x-1]\!])} \mathbb{S}[\![x-1]\!]^{\mathrm{Tate}}.$$

Remark 2.12. Under p-completion, and for S a $\mathbb{Z}_p[\zeta_p]$-algebra, there is an equivalence

$$\mathrm{THH}(S/\mathbb{S}_p[\![x-1]\!])^\wedge_p \simeq \mathrm{THH}(S/\mathbb{S}_p[x^{\pm 1}])^\wedge_p.$$

The base reason is that in Sp_p, $\mathbb{S}[B\mathbb{Z}_p] \simeq \mathbb{S}[B\mathbb{Z}] \simeq \mathbb{S}[S^1]$.

We also have other functors built out of THH.

Definition. Suppose R is an E_∞-ring. Then, we have the functors $\mathrm{CAlg} \to \mathrm{Sp}$ given by

$$\mathrm{TC}^-(R) := \mathrm{THH}(R)^{hS^1} \quad \text{and } \mathrm{TP}(R) := \mathrm{THH}(R)^{tS^1}.$$

There are two maps $\mathrm{TC}^-(R) \to \mathrm{TP}(R)^\wedge_p$ when R is connective.

1. For any spectrum with S^1-action, there is always a canonical map $(-)^{hS^1} \to (-)^{tS^1}$. This induces a map
$$\mathrm{can} : \mathrm{TC}^-(R) \to \mathrm{TP}(R)^\wedge_p.$$
2. We have the cyclotomic Frobenius $\varphi_{\mathrm{THH}(R)} : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$. Taking $(-)^{hS^1}$ yields the map
$$\varphi^{hS^1} : \mathrm{TC}^-(R) \to (\mathrm{THH}(R)^{tC_p})^{hS^1} \simeq \mathrm{TP}(R)^\wedge_p$$
where we use [NS18, Lemma II.4.2] for the last equivalence.

One then has
$$\mathrm{TC}(X) \simeq \mathrm{fib}(\mathrm{TC}^-(R) \xrightarrow{\mathrm{can}-\varphi} \mathrm{TP}(R)^\wedge_p).$$

2.1.4 Decompleted Cyclotomic Spectra

We would like to get Nygaard decompleted TC^- and TP. An *ad hoc* approach is given in [Man24]. However, the following construction that will appear in forthcoming work [DHRY]; an exposition without proofs is given in the thesis [Dev25]. Throughout, C_p denotes the finite cyclic group of order p.

Definition. A *decompleted cyclotomic spectrum* $(X, \Phi X)$ is the data of

1. a cyclotomic spectrum X in the sense of [NS18] (i.e. a spectrum X with S^1-action along with an S^1-equivariant map $\varphi : X \to X^{tC_p}$),
2. another spectrum ΦX with S^1-action, and
3. maps $X \to \Phi X$ and $\Phi X \to X^{tC_p}$ that factor φ.

The cyclotomic spectrum X will be referred to as the *underlying cyclotomic spectrum* and the data of ΦX with the maps $X \to \Phi X$ and $\Phi X \to X^{tC_p}$ will be referred to as a *decompleted structure* on X.

The category of decompleted cyclotomic spectra will be denoted CycSp_{Δ}.

Of course, there is a fully faithful functor

$$\beta : \mathrm{CycSp} \to \mathrm{CycSp}_{\Delta}$$

which endows a cyclotomic spectrum X with the decompleted structure $\Phi X := X^{tC_p}$ with the maps $\varphi : X \to \Phi X$ and $\mathrm{id} : \Phi X \to X^{tC_p}$.

Definition. (cf. [Dev25, Example 7.1.3]) Let R be a $\mathbb{Z}_p$-algebra. Denote by $\mathrm{THH}_{\Delta}(R)^\wedge_p$ the decompleted cyclotomic E_∞-ring whose underlying cyclotomic spectrum is $\mathrm{THH}(R)^\wedge_p$, and whose decompleted structure is given by declaring

$$\Phi\mathrm{THH}_{\Delta}(R)^\wedge_p := \left(\mathrm{THH}(R) \otimes_{\mathrm{THH}(\mathbb{Z}_p)} \mathrm{THH}(\mathbb{Z}_p)^{tC_p}\right)^\wedge_p.$$

The map $\mathrm{THH}(R)^\wedge_p \to \Phi\mathrm{THH}_{\Delta}(R)^\wedge_p$ is obtained by tensoring the unit map $\mathbb{S}_p \to \left(\mathrm{THH}(\mathbb{Z}_p)^{tC_p}\right)^\wedge_p$, while the map $\Phi\mathrm{THH}_{\Delta}(R)^\wedge_p \to \left(\mathrm{THH}(R)^{tC_p}\right)^\wedge_p$ is induced by the commuting square

$$\begin{tikzcd}
\mathrm{THH}(\mathbb{Z}_p)^\wedge_p \arrow[r] \arrow[d] & \left(\mathrm{THH}(\mathbb{Z}_p)^{tC_p}\right)^\wedge_p \arrow[d] \\
\mathrm{THH}(R)^\wedge_p \arrow[r] & \left(\mathrm{THH}(R)^{tC_p}\right)^\wedge_p
\end{tikzcd}$$

coming from functoriality of the cyclotomic Frobenius.

Example 2.13. $\mathrm{THH}_{\Delta}(\mathbb{Z}_p)^{\wedge}_p = \beta\mathrm{THH}(\mathbb{Z}_p)^{\wedge}_p$.

For $(X, \Phi X)$ a decompleted cyclotomic spectrum, we write X^{C_p} for the ring spectrum with S^1-action defined by the following pull-back square

$$\begin{tikzcd} X^{C_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X^{hC_p} \arrow[d, "\mathrm{can}"] \\ \Phi X \arrow[r] & X^{tC_p}. \end{tikzcd}$$

In particular, the projection map yields a canonical map

$$\mathrm{can} : (X^{C_p})^{hS^1} \to (\Phi X)^{hS^1}.$$

There is also a Frobenius map $\varphi : (X^{C_p})^{hS^1} \to (\Phi X)^{hS^1}$ defined as the composition

$$(X^{C_p})^{hS^1} \to (X^{hC_p})^{hS^1} \simeq X^{hS^1} \to (\Phi X)^{hS^1}.$$

This sits in a Cartesian square

$$\begin{tikzcd} (X^{C_p})^{hS^1} \arrow[r] \arrow[d, "\varphi"] \arrow[dr, phantom, "\lrcorner"] & (X^{hC_p})^{hS^1} \simeq X^{hS^1} \arrow[d, "\varphi^{hS^1}"] \\ (\Phi X)^{hS^1} \arrow[r] & (X^{tC_p})^{hS^1}. \end{tikzcd}$$

The decompleted cyclotomic spectrum $\mathrm{THH}_{\Delta}(R)$ is related to prismatic cohomology by the following.

Proposition 2.14 ([DHRY])**.** *Suppose R is a QRSP. Then,*

$$\pi_{2n}\left(\left(\mathrm{THH}(R)^{C_p}\right)^{hS^1}\right) \simeq (\mathrm{Fil}^n_{\mathrm{Nyg}}\Delta_R)\{n\}$$
$$\pi_{2n}\left((\Phi\mathrm{THH}(R))^{hS^1}\right) \simeq \Delta_R\{n\}$$

for any n. All the odd homotopy groups of the above spectra vanish.

Moreover, the canonical map $\mathrm{can} : (\mathrm{THH}(R)^{C_p})^{hS^1} \to (\Phi\mathrm{THH}(R))^{hS^1}$ *induces on homotopy groups the canonical map*

$$(\mathrm{Fil}^n_{\mathrm{Nyg}}\Delta_R)\{n\} \hookrightarrow \Delta_R\{n\},$$

and the Frobenius map $\varphi : (\mathrm{THH}(R)^{C_p})^{hS^1} \to (\Phi\mathrm{THH}(R))^{hS^1}$ *induces on homotopy groups the divided Frobenius map*

$$\varphi_n : (\mathrm{Fil}^n_{\mathrm{Nyg}}\Delta_R)\{n\} \to \Delta_R\{n\}.$$

Corollary 2.15. *The even filtration (in the sense of [HRW22]) on* $\left(\mathrm{THH}(R)^{C_p}\right)^{hS^1}$ *(resp.* $(\Phi\mathrm{THH}(R))^{hS^1}$*) has associated graded given by* $(\mathrm{Fil}^n_{\mathrm{Nyg}}\Delta_R)\{n\}[2n]$ *(resp.* $\Delta_R\{n\}[2n]$*).*

Thus, one can think of

$$\mathrm{TC}^-_{\Delta}(R) := \left(\mathrm{THH}(R)^{C_p}\right)^{hS^1}$$

as being Nygaard-decompleted $\mathrm{TC}^-(R)$ and

$$\mathrm{TP}_{\Delta}(R) := (\Phi\mathrm{THH}(R))^{hS^1}$$

as being Nygaard decompleted $\mathrm{TP}(R)$. Moreover, the Cartesian square defining $\mathrm{THH}(R)^{C_p}$ yields the following pull-back square

$$\begin{tikzcd} \mathrm{Fil}^i_{\mathrm{Nyg}}\Delta_R\{n\} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fil}^i_{\mathrm{Nyg}}\widehat{\Delta}_R\{n\} \arrow[d, hook] \\ \Delta_R\{n\} \arrow[r, hook] & \widehat{\Delta}_R\{n\} \end{tikzcd}$$

for all $n \geq 0$.

Remark 2.16. Since THH satisfies quasi-syntomic descent, we can globalize the above constructions to define a sheaf in decompleted cyclotomic spectra $\mathrm{THH}_{\mathbb{A}}$ on the quasi-syntomic site of X. In particular, we have the quasi-syntomic sheaves $\mathrm{TC}^-_{\mathbb{A}}$ and $\mathrm{TP}_{\mathbb{A}}$ whose even filtrations recover Nygaard-filtered prismatic cohomology and usual prismatic cohomology of X respectively.

Remark 2.17. Suppose R is perfectoid. Then, $\mathbb{A}_R = A_{\mathrm{inf}}(R)$ is already Nygaard-complete: indeed, $\mathrm{Fil}^i_{\mathrm{Nyg}} A_{\mathrm{inf}}(R) = (\varphi^{-1}(\xi))^i A_{\mathrm{inf}}(R)$ where ξ generates the prismatic ideal, and $A_{\mathrm{inf}}(R)$ is $\varphi^{-1}(\xi)$-adically complete. In this case, the equality $\widehat{\mathbb{A}}_R = \mathbb{A}_R$ implies that

$$\mathrm{TP}_{\mathbb{A}}(R) = \mathrm{TP}(R) \quad \text{and} \quad \mathrm{TC}^-_{\mathbb{A}}(R) = \mathrm{TC}^-(R).$$

In fact, it turns out that $\Phi\mathrm{THH}(R) = \mathrm{THH}(R)^{tC_p}$ so that $\mathrm{THH}_{\mathbb{A}}(R) = \beta\mathrm{THH}(R)$, generalising Example 2.13.

Finally, we can define the motivic filtration on $\Phi\mathrm{THH}, \mathrm{TC}^-_{\mathbb{A}}$, and $\mathrm{TP}_{\mathbb{A}}$ as in [BMS19].

Definition. For $\mathcal{F} \in \{\Phi\mathrm{THH}_{\mathbb{A}}, \mathrm{TC}^-_{\mathbb{A}}, \mathrm{TP}_{\mathbb{A}}\}$, we define the motivic filtration by

$$\mathrm{Fil}^i\mathcal{F}(X) := \mathrm{R}\Gamma(\mathrm{QSyn}_X, \tau_{\geq 2i}\mathcal{F}(-)).$$

Since QRSPs form a basis for QSyn, we see that

$$\mathrm{Fil}^i\mathcal{F}(X) \simeq \varprojlim_{R \text{ QRSP, } \mathrm{Spf}\, R \to X} \tau_{\geq 2i}\mathcal{F}(R).$$

Remark 2.18. In [HRW22], the authors defined the even filtration $\mathrm{Fil}^\bullet_{\mathrm{ev}}$ on $\mathrm{THH}, \mathrm{TP}, \mathrm{TC}^-$, and TC. Moreover, they showed that for any quasi-syntomic ring R, the even filtration coincides with the BMS filtration

$$\mathrm{Fil}^*_{\mathrm{ev}}\mathcal{F}(R) \simeq \mathrm{Fil}^*\mathcal{F}(R)$$

for $\mathcal{F} \in \{\mathrm{THH}, \mathrm{TP}, \mathrm{TC}^-, \mathrm{TC}\}$. It formally follows that the even filtration coincides with the above defined motivic filtration on affine p-adic formal schemes, for $\mathcal{F} \in \{\Phi\mathrm{THH}, \mathrm{TC}^-_{\mathbb{A}}, \mathrm{TP}_{\mathbb{A}}\}$.

However, the even filtration does not coincide with the BMS/motivic filtration on non-affine inputs. Since we will need to care about non-affine p-adic formal schemes, we thus use the BMS filtration rather than the even filtration.

Remark 2.19. One can also define a motivic filtration on $\mathrm{THH}_{\mathbb{A}}(-/\mathbb{S}_p[\![q_1 - 1]\!])$ in the same way.

2.2 A Zoo of Cohomology Theories

Many cohomology theories, and certainly all cohomology theories that we will consider, are in fact the associated graded of a multiplicative 'motivic' filtration on a certain nice Zariski sheaf $\mathcal{F} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Sp})$ (resp. $\mathrm{FSch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Sp}_p)$ for FSch the category of p-adic formal schemes) valued in E_∞-rings in spectra (resp. p-complete spectra). These sheaves are usually localising invariants.

Definition. Let $\mathrm{Cat}^{\mathrm{perf}}_\infty$ denote the category of idempotent-complete small stable ∞-categories, and maps are exact functors (i.e. those that preserve finite limits and colimits). Note that $\mathrm{Perf}(X) \in \mathrm{Cat}^{\mathrm{perf}}_\infty$ for any (p-adic formal) scheme X.

A sequence

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

in $\mathrm{Cat}^{\mathrm{perf}}_\infty$ is exact if the composite is zero, the functor f is fully faithful, and the map $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence.

A *localising invariant* is a functor $\mathcal{F} : \mathrm{Cat}^{\mathrm{perf}}_\infty \to \mathcal{D}$ valued in a stable ∞-category $\mathcal{D}$ that inverts Morita equivalences and sends any exact sequence of categories to a (co)fibre sequence in $\mathcal{D}$.

We will write $\mathcal{F}(X) := \mathcal{F}(\mathrm{Perf}(X))$ for any functor $\mathcal{F} : \mathrm{Cat}^{\mathrm{perf}}_\infty \to \mathcal{D}$.

Lemma 2.20. *Each of* $\mathrm{THH}, \Phi\mathrm{THH}, \mathrm{TP}, \mathrm{TP}_{\mathbb{A}}, \mathrm{TC}^-, \mathrm{TC}^-_{\mathbb{A}}, \mathrm{TC}, \mathrm{HH}, \mathrm{HP}, \mathrm{HC}^-, L_{K(1)}\mathrm{TC}$ *are all localising invariants.*

Proof. For THH, TP, TC^-, and TC, this is well known: for THH this is [BGT13, Proposition 10.2]. In fact, THH refines to a localising invariant $\mathrm{THH}:\mathrm{Cat}_\infty^{\mathrm{perf}}\to\mathrm{CycSp}$, and it thus follows for $\mathrm{TC},\mathrm{TP},\mathrm{TC}^-$ as each of these are exact functors $\mathrm{CycSp}\to\mathrm{Sp}$. That $\Phi\mathrm{THH}$ is a localising invariant is [Mao24, Theorem 1.9]. The claim then follows for

$$\mathrm{TP}_{\mathbb{\Delta}}=(\Phi\mathrm{THH})^{hS^1}:=\lim_{BS^1}\Phi\mathrm{THH},$$

and thus for $\mathrm{TC}^-_{\mathbb{\Delta}}:=\mathrm{TP}_{\mathbb{\Delta}}\times_{\mathrm{TP}}\mathrm{TC}^-$. Throughout, we are using that finite colimits in a stable ∞-category are finite limits and thus commute with arbitrary limits. Since $K(1)$-localisation also commutes with finite (co)limits, we see that $L_{K(1)}K$ is also an localising invariant. Finally, since $\mathrm{HH}(-)\simeq\mathrm{THH}(-)\otimes_{\mathrm{THH}(\mathbb{Z}_p)}\mathbb{Z}_p$, the claim follows for $\mathrm{HH}(-)$ from the case of HH. $\square$

By [BGT13, Theorem 1.12] we know that for any localising invariant $\mathcal{F}$, we have

$$\mathrm{Map}(K,\mathcal{F})\simeq\mathcal{F}(\mathbb{S})$$

where K is (non-connective) algebraic K-theory and $\mathbb{S}$ is the sphere spectrum. Suppose moreover that $\mathcal{F}$ is a lax symmetric monoidal functor into Sp; then the unit map for the E_∞-ring $\mathcal{F}(\mathbb{S})$ yields a canonical (up to homotopy) choice of natural transformation $T:K\to\mathcal{F}$. If $\mathcal{F}=\mathrm{THH}$ (resp. $\mathcal{F}=\mathrm{TC}$), then T is the Dennis trace (resp. the cyclotomic trace) [BGT13, Section 10].

We now list some motivic filtrations on various localising invariants, where throughout the input X is a p-quasi-syntomic p-adic formal scheme and the output is implicitly p-completed.

1. For any discrete commutative ring k, there is a motivic filtration on $\mathrm{HH}(X/k)$, $\mathrm{HP}(X/k)$, and $\mathrm{HC}^-(X/k)$, whose associated graded yields, respectively, p-complete derived Hodge cohomology $L\Omega^*_{X/k}[2*]$, p-complete Hodge-completed derived de Rham cohomology $\widehat{\mathrm{dR}}_{X/k}[2*]$, and the p-complete Hodge filtration $\widehat{\mathrm{dR}}^{\geq *}_{X/k}[2*]$ thereon. For more on this filtration see [BMS19; Ant19; Rak20; HRW22].

2. There is a motivic filtration on $\mathrm{TC}(X)$, $\mathrm{TP}(X)$, $\mathrm{TC}^-(X)$, and $\mathrm{THH}(X)$, whose associated graded is respectively syntomic cohomology $\mathbb{Z}_p^{\mathrm{syn}}(*)(X)[2*]$, Breuil-Kisin twisted Nygaard completed prismatic cohomology $\widehat{\mathbb{\Delta}}_X\{*\}[2*]$, the Nygaard filtration $(\mathrm{Fil}^{\geq *}_{\mathrm{Nyg}}\widehat{\mathbb{\Delta}}_X)\{*\}[2*]$ thereon, and its associated graded $\mathrm{gr}^*_{\mathrm{Nyg}}\widehat{\mathbb{\Delta}}_X\{*\}[2*]$. For more on this filtration see [BMS19], [BS22], [HRW22].

3. As described in the previous section, there are motivic filtrations on the decompleted variants of the above localising invariants $\mathrm{TP}_{\mathbb{\Delta}},\mathrm{TC}^-_{\mathbb{\Delta}}$, and $\mathrm{THH}_{\mathbb{\Delta}}$ [DHRY]. The associated motivic graded identifies respectively with Breuil-Kisin twisted prismatic cohomology $\mathbb{\Delta}_X\{*\}[2*]$, the Nygaard filtration thereon $\mathrm{Fil}^{\geq *}_{\mathrm{Nyg}}\mathbb{\Delta}_X\{*\}[2*]$, and its associated graded $\mathrm{gr}^*_{\mathrm{Nyg}}\mathbb{\Delta}_X\{*\}[2*]$.

4. Endow $\mathbb{Z}_p[\zeta_p]$ with a $\mathbb{S}_p[\![q-1]\!]$-algebra structure via the map $q\mapsto\zeta_p$. For X a p-adic formal scheme over $\mathbb{Z}_p[\zeta_p]$, there is a motivic filtration on $\mathrm{TC}(X/\mathbb{S}_p[\![q-1]\!])$, $\mathrm{TP}(X/\mathbb{S}_p[\![q-1]\!])$, $\mathrm{TC}^-(X/\mathbb{S}_p[\![q-1]\!])$, and $\mathrm{THH}(X/\mathbb{S}_p[\![q-1]\!])$; see [Wag25, Appendix A] and [HRW22]. This motivic filtration agrees with the Hahn-Raksit-Wilson even filtration on affines. The associated graded of this motivic filtration identifies with relative syntomic cohomology $\mathbb{Z}_p^{\mathrm{syn}}(*)(X/\mathbb{Z}_p[\![q-1]\!])[2*]$, Frobenius twisted Nygaard completed relative prismatic cohomology $\widehat{\mathbb{\Delta}}^{(1)}_{X_1/\mathbb{Z}_p[\![q-1]\!]}\{*\}[2*]$, the Nygaard filtration thereon $\mathrm{Fil}^{\geq *}_{\mathrm{Nyg}}\widehat{\mathbb{\Delta}}^{(1)}_{X_1/\mathbb{Z}_p[\![q-1]\!]}\{*\}[2*]$, and its associated graded $\mathrm{gr}^*_{\mathrm{Nyg}}\widehat{\mathbb{\Delta}}^{(1)}_{X_1/\mathbb{Z}_p[\![q-1]\!]}\{*\}[2*]$.

5. The localising invariant $L_{K(1)}K(-)$, when viewed as a functor from $\mathbb{Z}[\frac{1}{p}]$-algebras to ($K(1)$-local) spectra, has the Thomason filtration given by the étale sheafification of $\tau_{\geq 2*}L_{K(1)}K(-)$, whose associated graded is

$$\mathrm{R}\Gamma_{\text{ét}}(-,\mathbb{Z}_p(*))[2*].$$

This filtration is moreover exhaustive, but not necessarily complete. For any ring R, one has

$$L_{K(1)}\mathrm{TC}(R_p^\wedge)\simeq L_{K(1)}\mathrm{TC}(R)\simeq L_{K(1)}K(R)\simeq L_{K(1)}K(R[\tfrac{1}{p}])$$

by [BCM20], and so one can view the functor $\mathrm{Fil}^\bullet_{\mathrm{mot}}L_{K(1)}\mathrm{TC}$ as being a functor from arbitrary commutative $\mathbb{Z}$-algebras to filtered spectra, with associated graded given by

$$\mathrm{R}\Gamma_{\text{ét}}((-)_p^\wedge[\tfrac{1}{p}],\mathbb{Z}_p(*))[2*]$$

the étale cohomology of the rigid generic fibre. In [Kim25], it is shown that the Thomason filtration on $L_{K(1)}K(R[\frac{1}{p}])\simeq L_{K(1)}\mathrm{TC}(R)$ is compatible with the motivic filtration on TC. Further facts about the motivic filtration on $L_{K(1)}\mathrm{TC}$ that we will need are recorded in Proposition 3.6.

All of the usual comparison maps between these cohomology theories are realised by maps at the level of the corresponding localising invariant. Moreover, all of the above filtered localising invariants except for filtered $K(1)$-local K-theory satisfy quasi-syntomic descent [BMS19] and are all quasi-syntomic locally even.

All of the above filtrations, at least on affines, are obtained in two ways:

1. Via descent where, on a basis of a certain topology, the motivic filtration is the double speed Postnikov filtration (using that the value of the localising invariant at such a basis is always even).
2. Wariants of the even filtration as in [HRW22].

We will discuss the latter approach in a little more detail in the next section.

2.3 Cyclotomic Synthetic Spectra

In order to keep track of motivic filtrations, we need to use synthetic spectra which were introduced in [Pst23], as well as cyclotomic synthetic spectra from [AR24]. We will follow the latter reference. This also gives us a chance to recall some fact about filtrations.

> **Definition.** For a stable category $\mathcal{C}$, write
>
> $$\mathrm{Fil}\mathcal{C} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$$
>
> for the ∞-category of filtered objects, where $\mathbb{Z}^{\mathrm{op}}$ is the poset of decreasing integers.

We will write an object of $\mathrm{Fil}\mathcal{C}$ as $F^{\geq *}M$. Evaluation at $i \in \mathbb{Z}$ yields a limit and colimit preserving functor

$$F^{\geq k} : \mathrm{Fil}\mathcal{C} \to \mathcal{C}, \quad F^{\geq *}M \mapsto F^{\geq k}M.$$

For each $k \in \mathbb{Z}$, there is also a shifting functor

$$- \langle k \rangle : \mathrm{Fil}\mathcal{C} \to \mathrm{Fil}\mathcal{C}, \quad F^{\geq *}(M \langle k \rangle) := F^{*-k}M.$$

If $\mathcal{C}$ is symmetric monoidal with tensor product $\otimes_{\mathcal{C}}$, then so is $\mathrm{Fil}\mathcal{C}$ via Day convolution, so that

$$F^{\geq k}(F^{\geq *}M \otimes_{\mathrm{Fil}\mathcal{C}} F^{\geq *}N) := \operatorname*{colim}_{i+j \geq k} F^{\geq i}M \otimes_{\mathcal{C}} F^{\geq j}N.$$

If $\mathcal{C}$ admits all sequential colimits, we have the underlying object functor

$$F^{\geq -\infty} : \mathrm{Fil}\mathcal{C} \to \mathcal{C}, \quad F^{\geq *}M \mapsto F^{\geq -\infty}M := \operatorname*{colim}_{k \to -\infty} F^{\geq k}M.$$

We say that an object $F^{\geq *}M \in \mathrm{Fil}\mathcal{C}$ is complete if $\lim_i F^{\geq i}M \simeq 0$, and we will denote the full subcategory of complete filtered objects of $\mathcal{C}$ by $\widehat{\mathrm{Fil}}\mathcal{C}$.

The basic example of filtered spectra we will need is the following construction, see [HRW22, Section 2].

Lemma 2.21. *There is a lax symmetric monoidal functor*

$$\mathrm{CAlg}(\mathrm{Sp}_p) \to \mathrm{CAlg}(\widehat{\mathrm{Fil}}\,\mathrm{Sp}_p), \quad R \mapsto F^*_{\mathrm{ev}}R$$

which is right Kan extended from even rings and satisfies $F^{\geq *}_{\mathrm{ev}}R := \tau_{\geq 2*}R$ whenever R is even.

In particular, we set $\mathbb{S}_{\mathrm{ev}} := F^*_{\mathrm{ev}}\mathbb{S}_p \in \mathrm{Fil}\,\mathrm{Sp}_p$. We abuse notation by not including the dependence of p in $\mathbb{S}_{\mathrm{ev}}$.

> **Definition.** The category of (p-complete) synthetic spectra is the symmetric monoidal stable ∞-category
>
> $$\mathrm{SynSp}_p := \mathrm{Mod}_{\mathbb{S}_{\mathrm{ev}}}(\mathrm{Fil}\,\mathrm{Sp}_p).$$

Note that $F^{\geq *}_{\mathrm{ev}} : \mathrm{CAlg}(\mathrm{Sp}_p) \to \mathrm{CAlg}(\widehat{\mathrm{Fil}}\,\mathrm{Sp}_p)$ factors through a lax-symmetric monoidal functor

$$F^{\geq *}_{\mathrm{ev}} : \mathrm{CAlg}(\mathrm{Sp}_p) \to \mathrm{CAlg}(\mathrm{SynSp}_p).$$

Set $\mathbb{T}_{\mathrm{ev}} := F^{\geq *}_{\mathrm{ev}}\mathbb{S}_p[S^1]$. The bicommutative bialgebra structure on $\mathbb{S}_p[S^1]$ naturally upgrades to a bicommutative bialgebra structure on $\mathbb{T}_{\mathrm{ev}}$ in the category SynSp_p; see [AR24] for more details.

Definition. A synthetic spectrum with synthetic circle action is a $\mathbb{T}_{\mathrm{ev}}$-module in SynSp_p. Write

$$\mathrm{SynSp}_{\mathbb{T}_{\mathrm{ev}}} := \mathrm{Mod}_{\mathbb{T}_{\mathrm{ev}}}(\mathrm{SynSp}_p),$$

where as usual we leave the p implicit.

Since $\mathbb{T}_{\mathrm{ev}}$ is a bicommutative bialgebra, it in particular has an E_∞-augmentation $\mathbb{T}_{\mathrm{ev}} \to \mathbb{S}_{\mathrm{ev}}$, and so restriction of scalars along this augmentation yields a functor

$$(-)^{\mathrm{triv}} : \mathrm{SynSp}_p \to \mathrm{SynSp}_{\mathbb{T}_{\mathrm{ev}}}.$$

This functor has left and right adjoints

$$(-)_{h\mathbb{T}_{\mathrm{ev}}} := \mathbb{S}_{\mathrm{ev}} \otimes_{\mathbb{T}_{\mathrm{ev}}} - : \mathrm{SynSp}_{\mathbb{T}_{\mathrm{ev}}} \to \mathrm{SynSp}_p \quad \text{and} \quad (-)^{h\mathbb{T}_{\mathrm{ev}}} := \mathrm{Hom}_{\mathbb{T}_{\mathrm{ev}}}(\mathbb{S}_{\mathrm{ev}}, -) : \mathrm{SynSp}_{\mathbb{T}_{\mathrm{ev}}} \to \mathrm{SynSp}_p.$$

This is a filtered version of the homotopy orbits and homotopy fixed points under a circle action, whence the notation. There is a synthetic norm map

$$\mathrm{Nm}_{\mathbb{T}_{\mathrm{ev}}} : (-)_{h\mathbb{T}_{\mathrm{ev}}}(-1)[1] \to (-)^{h\mathbb{T}_{\mathrm{ev}}}$$

and we define

$$(-)^{t\mathbb{T}_{\mathrm{ev}}} := \mathrm{cofib}(\mathrm{Nm}_{\mathbb{T}_{\mathrm{ev}}}).$$

The p^k-power map $S^1 \to S^1$ induces a map $\rho(k) : \mathbb{T}_{\mathrm{ev}} \to \mathbb{T}_{\mathrm{ev}}$. There are left and right adjoints of the restriction-of-scalars along $\rho(k)$ functor $\rho(k)_* : \mathrm{SynSp}_{\mathbb{T}_{\mathrm{ev}}} \to \mathrm{SynSp}_{\mathbb{T}_{\mathrm{ev}}}$ given by

$$(-)_{hC_{p^k,\mathrm{ev}}} := \rho(n)_*\mathbb{T}_{\mathrm{ev}} \otimes_{\mathbb{T}_{\mathrm{ev}}} - : \mathrm{SynSp}_{\mathbb{T}_{\mathrm{ev}}} \to \mathrm{SynSp}_{\mathbb{T}_{\mathrm{ev}}}, \text{ and}$$
$$(-)^{hC_{p^k,\mathrm{ev}}} := \mathrm{Hom}_{\mathbb{T}_{\mathrm{ev}}}(\rho(k)_*\mathbb{T}_{\mathrm{ev}}, -) : \mathrm{SynSp}_{\mathbb{T}_{\mathrm{ev}}} \to \mathrm{SynSp}_{\mathbb{T}_{\mathrm{ev}}}.$$

Again, there is a norm functor

$$\mathrm{Nm}_{C_{p^k,\mathrm{ev}}} : (-)_{hC_{p^k,\mathrm{ev}}}(-1)[1] \to (-)^{hC_{p^k,\mathrm{ev}}}$$

and we define

$$(-)^{tC_{p^k,\mathrm{ev}}} : \mathrm{cofib}(\mathrm{Nm}_{C_{p^k,\mathrm{ev}}}).$$

We finally define cyclotomic synthetic spectra.

Definition. A (p-complete) cyclotomic synthetic spectrum is a p-complete synthetic spectrum $F^{\geq *}M$ with synthetic circle action equipped with a map $\varphi_M : F^{\geq *}M \to (F^{\geq *}M)^{tC_{p,\mathrm{ev}}}$. Let CycSyn_p denote the category of p-complete cyclotomic synthetic spectra.

This is a symmetric monoidal stable ∞-category with monoidal unit $\mathbb{S}_{\mathrm{ev}}^{\mathrm{triv}}$.

The following example is [AR24, Proposition 3.25].

Example 2.22. Suppose $X \in \mathrm{CycSp}$ is an E_∞-ring in cyclotomic spectra such that the underlying spectrum of X is even. Then $F_{\mathrm{ev}}^{\geq *}X$ naturally admits the structure of an E_∞-ring object in CycSyn. This essentially follows because $F_{\mathrm{ev}}^{\geq *}X \simeq \tau_{\geq 2*}X$ and $(F_{\mathrm{ev}}^{\geq *}X)^{tC_{p,\mathrm{ev}}} \simeq \tau_{\geq 2*}X^{tC_p}$, and so the cyclotomic Frobenius $\varphi_X : X \to X^{tC_p}$ naturally upgrades to a filtered map. See [AR24, Proposition 3.25] for an upgrade of this construction to an equivalence of categories.

Given $F^{\geq *}M \in \mathrm{CycSyn}_p$, we define

$$\mathrm{TC}(F^{\geq *}M) := \mathrm{map}_{\mathrm{CycSyn}}(\mathbb{S}_{\mathrm{ev}}^{\mathrm{triv}}, F^{\geq *}M) \simeq \mathrm{fib}\left(\mathrm{can} - \varphi_M^{h\mathbb{T}_{\mathrm{ev}}} : (F^{\geq *}M)^{h\mathbb{T}_{\mathrm{ev}}} \to \left((F^{\geq *}M)^{tC_{p,\mathrm{ev}}}\right)^{h\mathbb{T}_{\mathrm{ev}}}\right).$$

With all of this notation out of the way, we finally come to the result we will need.

Theorem 2.23 ([AR24])**.** *There is a functor*

$$F_{\mathrm{ev}}^{\geq *}\mathrm{THH}(-) : \mathrm{QSyn}_p \to \mathrm{CAlg}(\mathrm{CycSyn}_p)$$

such that for any $R \in \mathrm{QSyn}$, there are natural equivalences of E_∞-rings in p-complete filtered spectra:

$$F_{\mathrm{ev}}^{\geq *}\mathrm{THH}(R) \simeq \mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(R), \qquad \left(F_{\mathrm{ev}}^{\geq *}\mathrm{THH}(R)\right)^{h\mathbb{T}_{\mathrm{ev}}} \simeq \mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{TC}^-(R),$$
$$\left(F_{\mathrm{ev}}^{\geq *}\mathrm{THH}(R)\right)^{t\mathbb{T}_{\mathrm{ev}}} \simeq \mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{TP}(R), \qquad \mathrm{TC}\left(F_{\mathrm{ev}}^{\geq *}\mathrm{THH}(R)\right) \simeq \mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{TC}(R).$$

Moreover, the underlying cyclotomic spectrum of $F_{\mathrm{ev}}^{\geq *}\mathrm{THH}$ identifies canonically with THH.

2.4 Computations in THH

In this subsection we record some computations of THH of certain E_∞-rings that will be crucial in the proof of Theorem A.

2.4.1 THH of Group Algebras

For G a topological group, write $\mathbb{S}[G] := \Sigma^\infty_+ G$ for the corresponding E_1-ring. If G is abelian, which we will assume from now on, then $\mathbb{S}[G]$ is an E_∞-ring. In this case, the cyclotomic spectrum $\mathbb{S}[G]^{\mathrm{Tate}}$ is particularly nice.

Lemma 2.24. *For G either a discrete abelian group or a topological abelian group that is a finite CW complex, we have $\mathbb{S}_p[G]^{tC_p} \simeq \mathbb{S}_p[G]^{\mathrm{triv}}$ of S^1-equivariant spectra, and the Tate-valued Frobenius*

$$\varphi^{\mathrm{Tate}}_{\mathbb{S}_p[G]} : \mathbb{S}_p[G] \to \mathbb{S}_p[G]^{tC_p} \simeq \mathbb{S}_p[G]$$

is induced by the multiplication-by-p map $G \to G$.

Proof. That $\mathbb{S}_p[G]^{tC_p} \simeq \mathbb{S}_p[G]$ follows in the discrete case from [Yua23, Lemma 6.7] and in the finite CW complex case from [BS24, Theorem 1.2 and Example 4.4]; these follow from the now proven Segal conjecture $\mathbb{S}_p^{tC_p} \simeq \mathbb{S}_p$. The second part of the statement then follows from the definition of the Tate-valued Frobenius. ☐

We summarise everything we need to know about $\mathrm{THH}(\mathbb{S}[G])$ for G an abelian topological group. All of these results are classical, but for a modern reference see [NS18].

Theorem 2.25. *Let G be an abelian topological group.*

1. *There is a natural S^1-equivariant equivalence*

$$\mathrm{THH}(\mathbb{S}[G]) \simeq \mathbb{S}[LBG]$$

where $LBG := \mathrm{Map}_{\mathscr{S}}(S^1, BG)$ is the free loop space of the classifying space BG, with the natural S^1-action.

2. *The cyclotomic Frobenius is given by*

$$\mathrm{THH}(\mathbb{S}[G]) \simeq \mathbb{S}[LBG] \to \mathbb{S}[LBG]^{hC_p} \to \mathbb{S}[LBG]^{tC_p} \simeq \mathrm{THH}(\mathbb{S}[G])^{tC_p}$$

where the first map is induced by the map

$$LBG \to LBG^{hC_p} \simeq \mathrm{Map}(S^1, BG)^{hC_p} \simeq \mathrm{Map}(S^1/C_p, BG)$$

given by taking $\mathrm{Map}(-, BG)$ of the p-power covering map $[p] : S^1 \to S^1$ (this latter map factors through S^1/C_p and induces the equivalence $S^1/C_p \cong S^1$).

3. *The assembly map $\mathbb{S}[BG]^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{S}[G])$, under the equivalence $\mathrm{THH}(\mathbb{S}[G]) \simeq \mathbb{S}[LBG]$, is induced by the S^1-equivariant map $BG \to LBG$ sending a point of BG to the constant map valued at that point.*

4. *We have an equivalence of cyclotomic spectra*

$$\mathbb{S}^{\mathrm{triv}} \otimes_{\mathbb{S}[BG]^{\mathrm{triv}}} \mathrm{THH}(\mathbb{S}[G]) \simeq \mathbb{S}[G]^{\mathrm{Tate}}.$$

5. *For H a subgroup of G, the canonical map $\mathrm{THH}(\mathbb{S}[H]) \to \mathrm{THH}(\mathbb{S}[G])$ induced by the ring map $\mathbb{S}[H] \to \mathbb{S}[G]$ is induced by the map $LBH \to LBG$ given by post-composing with the projection $BH \to BG$.*

Remark 2.26. For $G = \mathbb{Z}$, we can write the S^1-action on $S^1 \times \mathbb{Z} \simeq LB\mathbb{Z}$ as $t \cdot (s, n) = (t^n s, n)$ for $t \in S^1$ and $(s, n) \in S^1 \times \mathbb{Z}$.

Under the S^1-equivariant equivalence $\mathrm{THH}(\mathbb{S}[\mathbb{Z}]) \simeq \mathbb{S}[S^1 \times \mathbb{Z}]$, the cyclotomic Frobenius lift $\mathrm{THH}(\mathbb{S}[\mathbb{Z}]) \to \mathrm{THH}(\mathbb{S}[\mathbb{Z}])^{hC_p}$ is induced by the map

$$S^1 \times \mathbb{Z} \simeq \mathrm{Map}(S^1, B\mathbb{Z}) \to \mathrm{Map}(S^1, B\mathbb{Z})^{hC_p} \simeq \mathrm{Map}(S^1/C_p, B\mathbb{Z}) \simeq S^1 \times \mathbb{Z}$$

given by $(s, n) \mapsto (s, pn)$. Indeed, the map $z \mapsto sz^n$ upon pre-composition with the p-power map goes to $z \mapsto sz^{pn}$. This fixes a typo in [BMS19, Proposition 11.3].

We now make a few remarks about the transfer map in THH for group algebras of abelian groups, which is the key input in the proof of Proposition F. Let G be an abelian topological group and H an open subgroup of finite index. Then $\mathbb{S}[G] \in \mathrm{Perf}(\mathbb{S}[H])$ so that we have the restriction of scalars map

$$\mathrm{Perf}(\mathbb{S}[G]) \to \mathrm{Perf}(\mathbb{S}[H]).$$

This map is moreover $\mathrm{Perf}(\mathbb{S}[H])$ linear. In particular, we get the $\mathrm{THH}(\mathbb{S}[H])$-linear transfer map

$$\mathrm{tr}_{G/H} : \mathrm{THH}(\mathbb{S}[G]) \to \mathrm{THH}(\mathbb{S}[H]).$$

This map is explicitly described in [Sch98; Sch06]. Rather than stating the result in full generality, we specialise to the case of infinite cyclic groups.

Proposition 2.27 (Schlichtkrull)**.** *Under the S^1-equivariant equivalences of Remark 2.26, the transfer map*

$$\mathrm{tr}_{\mathbb{Z}/n\mathbb{Z}} : \mathrm{THH}(\mathbb{S}[\mathbb{Z}]) \to \mathrm{THH}(\mathbb{S}[n\mathbb{Z}])$$

is given by

$$\mathbb{S}[\mathbb{Z} \times S^1] \to \mathbb{S}[n\mathbb{Z} \times S^1], \quad (a, s) \mapsto \begin{cases} 0 & n \nmid a, \\ \sum_{j=0}^{n-1} (a, s_j) & n | a, \end{cases}$$

where $\{s_0, ..., s_{n-1}\}$ is the preimage of $s \in S^1$ under the n-power map $S^1 \to S^1$.

> **Remark 2.28.** Here, the S^1-action on the right hand side of the decomposition $\mathrm{THH}(\mathbb{S}[n\mathbb{Z}]) \simeq \mathbb{S}[n\mathbb{Z} \times S^1]$ is given informally by
>
> $$t \cdot (a, s) = (a, t^{a/n} s)$$
>
> where $a \in n\mathbb{Z}$ and $s, t \in S^1$.

One checks rather easily that the above map as written is S^1-equivariant and commutes with Frobenius.

2.4.2 THH of Cyclotomic Extensions

In this subsection, we give an explicit description of the cyclotomic spectrum $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}])$, due to Devalapurkar-Raksit [DR25]. These results are one of the key inputs in the proof of Theorem A. In order to give this explicit description, we need to recall some notation from [Dev25, Remark 6.1.5].

> **Construction 2.29.** For $n \geq 0$, define
>
> $$\mathrm{ku}_{p,n} := \tau_{\geq 0}(\mathrm{ku}_p^{hC_{p^n}})$$
>
> where ku_p has the trivial C_{p^n}-action. Here, $\mathrm{ku}_{p,0} := \mathrm{ku}_p$. There is a residual action of $S^1/C_{p^n} \cong S^1$ on $\mathrm{ku}_{p,n}$ (it is the trivial circle action on $\mathrm{ku}_{p,0} = \mathrm{ku}_p$). This S^1-action on $\mathrm{ku}_{p,n}$ refines to a cyclotomic structure on $\mathrm{ku}_{p,n}$ as follows.
>
> Since ku_p has the trivial S^1-action, we have an S^1-equivariant E_∞-ring map $\mathrm{ku}_p \to \mathrm{ku}_p^{hC_p}$. Taking hC_{p^n} fixed points followed by connective covers yields a map
>
> $$\mathrm{ku}_{p,n} = \tau_{\geq 0}\mathrm{ku}_p^{hC_{p^n}} \to \tau_{\geq 0}(\mathrm{ku}_p^{hC_p})^{hC_{p^n}} \simeq \tau_{\geq 0}(\mathrm{ku}_p^{hC_{p^n}})^{hC_p}.$$
>
> Since $\tau_{\geq 0} : \mathrm{Sp} \to \mathrm{Sp}_{\geq 0}$ is a right adjoint and so commutes with limits (computed in the appropriate category), we have
>
> $$\tau_{\geq 0}(\mathrm{ku}_p^{hC_{p^n}})^{hC_p} \simeq \tau_{\geq 0}\mathrm{ku}_{p,n}^{hC_p}.$$
>
> We then have the composition
>
> $$\mathrm{ku}_{p,n} \to \tau_{\geq 0}\mathrm{ku}_{p,n}^{hC_p} \to \mathrm{ku}_{p,n}^{hC_p} \to \mathrm{ku}_{p,n}^{tC_p}$$
>
> which defines the cyclotomic Frobenius on $\mathrm{ku}_{p,n}$.

Notice that the Adams operations induce an action of $\mathbb{Z}_p^\times$ on $\mathrm{ku}_{p,n}$, which is moreover compatible with the given cyclotomic structure. It is also clear that each $\mathrm{ku}_{p,n}$ is a $\mathrm{ku}_{p,n-1}$-algebra in cyclotomic spectra: indeed, we simply take $\tau_{\geq 0}(-)^{hC_{p^{n-1}}}$ of the S^1-equivariant map of E_∞-rings $\mathrm{ku}_p \to \mathrm{ku}_p^{hC_p}$. This $\mathrm{ku}_{p,n-1}$-algebra structure on $\mathrm{ku}_{p,n}$ is also $\mathbb{Z}_p^\times$-equivariant.

We now define the cyclotomic spectrum $\widetilde{j}_n$ as follows.

Construction 2.30. Define the cyclotomic spectrum

$$\widetilde{j}_n := \tau_{\geq 0}\left(\mathrm{ku}_{p,n}^{h(1+p^{n+1}\mathbb{Z}_p)}\right).$$

This has an S^1-action induced by the S^1-action on $\mathrm{ku}_{p,n}$. The cyclotomic Frobenius of $\widetilde{j}_n$ is constructed as follows: recall that the cyclotomic Frobenius on $\mathrm{ku}_{p,n}$ factors through $\mathrm{ku}_{p,n}^{hC_p}$. Taking $(1+p^{n+1}\mathbb{Z}_p)$ fixed points and then taking connective covers yields

$$\widetilde{j}_n \to \tau_{\geq 0}(\mathrm{ku}_{p,n}^{hC_p})^{h(1+p^{n+1}\mathbb{Z}_p)} \simeq \tau_{\geq 0}\left(\tau_{\geq 0}\mathrm{ku}_{p,n}^{h(1+p^{n+1}\mathbb{Z}_p)}\right)^{hC_p} \simeq \tau_{\geq 0}\widetilde{j}_n^{hC_p} \to \widetilde{j}_n^{hC_p} \to \widetilde{j}_n^{tC_p}.$$

Clearly, $\widetilde{j}_0 \simeq j_0$. Since homotopy fixed points and connective covers are lax-symmetric monoidal, the canonical map $\widetilde{j}_n \to \mathrm{ku}_{p,n}$ is a map of cyclotomic E_∞-rings. We also get a map of cyclotomic E_∞-rings $\widetilde{j}_{n-1} \to \widetilde{j}_n$ given by taking connective covers of the composition

$$\mathrm{ku}_{p,n-1}^{h(1+p^n\mathbb{Z}_p)} \to \mathrm{ku}_{p,n-1}^{h(1+p^{n+1}\mathbb{Z}_p)} \to \mathrm{ku}_{p,n}^{h(1+p^{n+1}\mathbb{Z}_p)},$$

where the first map is induced by restricting to fixed points of a smaller subgroup, and the second map is induced by the $\mathrm{ku}_{p,n-1}$-algebra structure map on $\mathrm{ku}_{p,n}$. Note that the $\mathbb{Z}_p^\times$-action on $\mathrm{ku}_{p,n}$ induces a residual $\mathbb{Z}_p^\times/(1+p^n\mathbb{Z}_p) \cong (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ action on $\widetilde{j}_n$.

The main result we will need is the following result, stated in [Dev25, Remark 6.1.5, Theorem 6.4.1].

Theorem 2.31 (Devalapurkar-Raksit)**.** *For $n \geq 1$, there is an equivalence of $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$-equivariant cyclotomic E_∞-ring spectra*

$$\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \simeq \widetilde{j}_{n-1}^{(-1)}.$$

Moreover, the E_∞-cyclotomic ring map $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \to \mathrm{THH}(\mathbb{Z}_p[\zeta_{p^{n+1}}])$ induced by functoriality of THH *is the $(-)^{(-1)}$-functor of Construction 2.5 applied to the E_∞-ring map $\widetilde{j}_{n-1} \to \widetilde{j}_n$.*

One also has the following statement on relative THH, the case $n=1$ of which is already [Dev25, Theorem 6.4.1].

Corollary 2.32 (Devalapurkar)**.** *For any $n \geq 1$, view $\mathbb{Z}_p[\zeta_{p^n}]$ as a $\mathbb{S}_p[\![q_n-1]\!]$-algebra via the map $q_n \mapsto \zeta_{p^n}$. Then, there is an equivalence of $\mathbb{Z}_p^\times$-equivariant cyclotomic E_∞-ring spectra*

$$\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n-1]\!]) \simeq \mathrm{ku}_{p,n-1}^{(-1)}.$$

Remark 2.33. The observant reader may have noticed that we have switched convention from the introduction, as we now consider $\mathbb{Z}_p[\zeta_{p^n}]$ as a $\mathbb{S}_p[\![q_n-1]\!]$-algebra via $q_n \mapsto \zeta_{p^n}$, unlike in the introduction (especially the statement of Theorem A and its corollaries) where we use the convention that $\mathbb{Z}_p[\zeta_p]$ is a $\mathbb{S}_p[\![q-1]\!]$-algebra via $q \mapsto \zeta_p$. This is related to Frobenius twists.

Indeed, the Frobenius map $\varphi : \mathbb{Z}_p[\![q-1]\!] \to \mathbb{Z}_p[\![q-1]\!]$ as also the inclusion $\mathbb{Z}_p[\![q-1]\!] \to \mathbb{Z}_p[\![q_1-1]\!]$. The Frobenius twist $\mathbb{Z}_p[\![q-1]\!]^{(1)} = \varphi^*\mathbb{Z}_p[\![q-1]\!]$, after restricting scalars along Frobenius (i.e. applying φ_*), can thus be thought of as $\mathbb{Z}_p[\![q_1-1]\!]$. One has

$$\pi_0\mathrm{ku}_{p,0}^{(-1),tS^1} \simeq {\mathbb{\Delta}}^{(1)}_{\mathbb{Z}_p[\zeta_p]/\mathbb{Z}_p[\![q-1]\!]}.$$

In particular, it is a $\mathbb{Z}_p[\![q-1]\!]^{(1)}$-algebra.

Since we will need to often apply $\varphi_*\varphi^*$, we just make the convention that $\mathbb{Z}_p[\![q-1]\!]^{(1)} =: \mathbb{Z}_p[\![q_1-1]\!]$. In particular, from here on out (especially in Section 4) we will use the convention that $\mathbb{Z}_p[\zeta_{p^n}]$ is a $\mathbb{S}_p[\![q_n-1]\!]$-algebra via $q_n \mapsto \zeta_{p^n}$

One can also calculate $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{cyc}})$ as a cyclotomic spectrum. This is in fact much easier than the preceding theorem and follows by the exact same argument as in [BMS19, Proposition 11.7, Corollary 11.8], using that $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]) \simeq \mathrm{ku}_p^{(-1)}$.

Corollary 2.34 (Devalapurkar-Raksit)**.** *There is a $\mathbb{Z}_p^\times$-equivariant equivalence of cyclotomic E_∞-ring spectra*

$$\mathrm{THH}(\mathbb{Z}_p^{\mathrm{cyc}}) \simeq \mathrm{ku}_p^{(-1)} \otimes_{\mathbb{S}_p[\![q_1-1]\!]^{\mathrm{Tate}}} \mathbb{S}_p[\![q^{1/p^\infty}-1]\!]^{\mathrm{Tate}}$$

compatible with the equivalences $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \simeq \widetilde{j}_{n-1}^{(-1)}$. Here,

$$\mathbb{S}_p[\![q^{1/p^\infty}-1]\!] := \left(\operatorname*{colim}_n \mathbb{S}_p[\![q_n-1]\!]\right)^\wedge_{(p,q-1)}$$

We end with some remarks on the decompleted cyclotomic structure and the motivic filtrations.

Remark 2.35. We have

$$\mathrm{THH}_{\Delta}(\mathbb{Z}_p[\zeta_{p^n}]) \simeq \beta\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \simeq \beta\widetilde{j}_{n-1}^{(-1)}.$$

This follows from the fact that the Nygaard filtration on $\Delta_{\mathbb{Z}_p[\zeta_{p^n}]}$ is already complete. One can actually check this using the description of $\mathrm{Fil}^*_{\mathrm{Nyg}}\Delta_{\mathbb{Z}_p[\zeta_{p^n}]}$ in terms of

$$\mathrm{Fil}^*_{\mathrm{Nyg}}\Delta^{(1)}_{\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{Z}_p[\![q_{n-1}-1]\!]} \simeq [p]^*_{q_1}\mathbb{Z}_p[\![q_n-1]\!]$$

(the latter filtration obviously being complete) given by Corollary 4.6 below.

Remark 2.36. The motivic filtration on $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n-1]\!]) \simeq \mathrm{ku}_{p,n}^{(-1)}$ is given by the even filtration, which identifies with the double-speed Postnikov filtration.

For absolute THH, the motivic filtration is given by

$$\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \simeq \tau_{2*-1}\widetilde{j}_{n-1}^{(-1)}$$

by [Mor24, Corollary 1.4].

3 Transfer Maps for Prismatic Cohomology

In this section, specifically in Section 3.1 and Section 3.2, we prove that the transfer maps on the localising invariants considered in Section 2.2 respect the motivic filtration when $f : X \to Y$ is a finite flat regular map of p-adic formal schemes. As a consequence, we obtain the existence result in Theorem E. The rest of this section is then devoted to proving Theorem E.

3.1 The Gysin Map

We put ourselves in the following situation.

Assumption 3.1. Let $\mathcal{F}$ be a localising invariant $\mathrm{Cat}^{\mathrm{perf}}_{\infty} \to \mathrm{Sp}$ equipped with a functorial exhaustive filtration

$$\mathrm{Fil}^\bullet\mathcal{F} : \mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}} \to \mathrm{CAlg}(\mathrm{Fil}\,\mathrm{Sp})$$

satisfying the following conditions:

1. $\mathcal{F}$ and $\mathrm{Fil}^\bullet\mathcal{F}$ satisfy descent for the Nisnevich topology; in particular, we can view $\mathrm{Fil}^\bullet\mathcal{F} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Fil}\,\mathrm{Sp})$.
2. The spectrum $\mathrm{gr}^r\mathcal{F}(X)[-2r]$ is coconnective (i.e. concentrated in homotopy degrees ≤ 0)
3. There is a 'Chern class' map
$$c : \mathrm{Pic} \to \mathrm{Fil}^1\mathcal{F}$$
such that the following diagram commutes
$$\begin{tikzcd} \mathrm{Pic} \arrow[r, "\mathcal{L}\mapsto\mathcal{O}-\mathcal{L}^\vee"] \arrow[d, "c"] & K \arrow[d] \\ \mathrm{Fil}^1\mathcal{F} \arrow[r] & \mathcal{F} \end{tikzcd}$$
where the right vertical arrow is the canonical map $K \to \mathcal{F}$ coming from universality of algebraic K-theory as a localizing invariant [BGT13].
4. For all $R \in \mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}}$ and all $r \geq 0$, the $\mathrm{Fil}^\bullet\mathcal{F}(R)$-linear map
$$\sum_{i=0}^{r} c(\mathcal{O}_{\mathbb{P}^n_R}(1))^i : \bigoplus_{i=0}^{r}\mathrm{Fil}^{\bullet-i}\mathcal{F}(R) \to \mathrm{Fil}^\bullet\mathcal{F}(\mathbb{P}^r_R)$$
is an equivalence; in other words, $\mathrm{Fil}^\bullet\mathcal{F}$ satisfies a filtered projective bundle formula.

Remark 3.2. Over here, we have not required the input rings R to be p-completed. However, in all of the examples of $\mathcal{F}$ that we will care about, one has an equivalence $\mathrm{Fil}^\bullet\mathcal{F}(R) \simeq \mathrm{Fil}^\bullet\mathcal{F}(R_p^\wedge)$.

We also consider the following condition.

Condition 3.3. For $\mathrm{Fil}^\bullet\mathcal{F}$ as in Assumption 3.1, there exists a topology $\mathscr{T}$ (at least as fine as the Zariski topology) for which $\mathrm{Fil}^\bullet\mathcal{F}$ satisfies descent, such that there is a basis $\mathscr{B}$ of $\mathscr{T}$ satisfying

$$\mathrm{Fil}^\bullet\mathcal{F}(X) \simeq \tau_{\geq 2\bullet}\mathcal{F}(X)$$

for all $X \in \mathscr{B}$.

Notice that Condition 3.3 implies Assumption 3.1(2): the condition implies that

$$\mathrm{gr}^r\mathcal{F}(Y)[-2r] \simeq \lim_{X\in\mathscr{B}, X\to Y} \pi_{2r}\mathcal{F}(X),$$

but limits only increase in coconnectivity so that $\mathrm{gr}^r\mathcal{F}[-2r]$ is coconnective.

Notice that all of the filtered localising invariants listed in Section 2.2 satisfy Assumption 3.1:

1. All of the filtrations are obtained by the sheafification in a topology finer than the Nisnevich topology (étale in the case of $L_{K(1)}\mathrm{TC}(-) \simeq L_{K(1)}K(-[\frac{1}{p}])$, quasi-syntomic for the rest) of the double-speed Postnikov filtration, so that Assumption 3.1 (1) holds.

2. For Assumption 3.1 (2), note that $\mathrm{gr}^r\mathcal{F}$ is the sheafification of $\pi_{2r}\mathcal{F}[2r]$. Since sheafification is a limit, it only increases coconnectivity, and so $\mathrm{gr}^r\mathcal{F}[-2r]$ is coconnective.

3. Assumption 3.1 (3) holds for $\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{TC}$ essentially by construction (see also [AHI24, Proposition 6.7]). The remaining examples of $\mathrm{Fil}^\bullet\mathcal{F}$ in Section 2.2 are all algebras over $\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{TC}$, and so trivially satisfy Assumption 3.1 (3).

4. For Assumption 3.1 (4), and for all $\mathcal{F} \neq L_{K(1)}\mathrm{TC}$, the filtration is complete and on associated graded the projective bundle formula is given in [BL22a, Section 9.1].

   It remains to check for $L_{K(1)}\mathrm{TC}$. On associated graded, we have

   $$\mathrm{R}\Gamma_{\text{ét}}(X_\eta, \mathbb{Z}_p(r)) \simeq \mathrm{fib}\left(\varphi - \mathrm{id} : \Delta_X\{r\}[\tfrac{1}{\mathcal{I}_\Delta}]_p^\wedge \to \Delta_X\{r\}[\tfrac{1}{\mathcal{I}_\Delta}]_p^\wedge\right)$$

   and since prismatic cohomology satisfies the projective bundle formula, étale cohomology must too. At the level of underlying objects, this follows from $K(1)$-localisation of the projective bundle formula for TC (which holds by completeness of the motivic filtration on TC and the fact that it holds on associated graded by [BL22a, Theorem 9.1.1]). Since the associated graded functor together with the underlying object functor are jointly conservative, it follows that $\mathrm{Fil}^\bullet_{\mathrm{mot}}L_{K(1)}\mathrm{TC}$ satisfies it too.

In fact, all of the $\mathrm{Fil}^\bullet\mathcal{F}$ listed in Section 2.2 except for $\mathrm{Fil}^\bullet\mathcal{F} = \mathrm{Fil}^\bullet_{\mathrm{mot}}L_{K(1)}\mathrm{TC}$ satisfy Condition 3.3.

Lemma 3.4. *Suppose* $\mathrm{Fil}^\bullet\mathcal{F}$ *is as in Assumption 3.1. Then, for all* $m < n$ *and all schemes* X*,*

$$\pi_*\,\mathrm{cofib}\left(\mathrm{Fil}^n\mathcal{F}(X) \to \mathrm{Fil}^m\mathcal{F}(X)\right) = 0$$

for all $* \geq 2n-1$*. In particular,*

$$\pi_*\,\mathrm{cofib}\left(\mathrm{Fil}^n\mathcal{F}(X) \to \mathcal{F}(X)\right) = 0$$

for $* \geq 2n-1$*.*

Proof. For $n = m+1$, we have $\pi_*\mathrm{gr}^m\mathcal{F}(X) = 0$ for $* \geq 2m+1 = 2n-1$ by Assumption 3.1(2). Setting $k = n-m$, consider the fibre sequence

$$\frac{\mathrm{Fil}^{n-(k-1)}\mathcal{F}(X)}{\mathrm{Fil}^n\mathcal{F}(X)} \to \frac{\mathrm{Fil}^{n-k}\mathcal{F}(X)}{\mathrm{Fil}^n\mathcal{F}(X)} \to \mathrm{gr}^{n-k}\mathcal{F}(X).$$

By induction on $k \geq 1$, noting that $\pi_*\mathrm{gr}^{n-k}\mathcal{F}(X) = 0$ for $* > 2n-1 \geq 2(n-k)-1$, the first statement follows. The final statement follows from the fact that filtered colimits commute with taking homotopy groups. $\square$

Corollary 3.5. *Suppose* $\mathrm{Fil}^\bullet\mathcal{F}$ *as in Assumption 3.1 also satisfies Condition 3.3. Then, for any scheme* Y *and for any* $n \in \mathbb{Z}$*, any functorial-in-*X *map*

$$\mathrm{Fil}^n\mathcal{F}(X) \to \frac{\mathcal{F}(X \times Y)}{\mathrm{Fil}^n\mathcal{F}(X \times Y)}$$

must be uniquely null-homotopic.

Proof. By functoriality, we reduce to $X \in \mathscr{B}$. In this case, Condition 3.3 implies that $\mathrm{Fil}^n\mathcal{F}(X) \simeq \tau_{\geq 2n}\mathcal{F}(X)$ is concentrated in degrees $\geq 2n$, whereas by the lemma $\frac{\mathcal{F}(X\times Y)}{\mathrm{Fil}^n\mathcal{F}(X\times Y)}$ is concentrated in degrees $< 2n-1$. □

We need a version of this corollary for $L_{K(1)}\mathrm{TC}$ too. We record the following result on the motivic filtration on $L_{K(1)}\mathrm{TC}$. For a pre-sheaf $\mathcal{G}$ on p-adic formal schemes we denote by $L_{\mathrm{arc}_p}\mathcal{G}$ the arc-p hyper-sheafification of $\mathcal{G}$.

Proposition 3.6. *For any* r*, there is a pull-back square*

$$\begin{tikzcd} \mathrm{Fil}^r_{\mathrm{mot}}L_{K(1)}\mathrm{TC} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{\mathrm{arc}_p}\left(\tau_{\geq 2r}L_{K(1)}\mathrm{TC}\right) \arrow[d] \\ L_{K(1)}\mathrm{TC} \arrow[r] & L_{\mathrm{arc}_p}L_{K(1)}\mathrm{TC} \end{tikzcd}$$

of Zariski sheaves. The filtration

$$\mathrm{Fil}^\bullet_{\mathrm{mot}}L_{\mathrm{arc}_p}L_{K(1)}\mathrm{TC} := L_{\mathrm{arc}_p}\left(\tau_{\geq 2\bullet}L_{K(1)}\mathrm{TC}\right)$$

is moreover a complete exhaustive filtration on $L_{\mathrm{arc}_p}L_{K(1)}\mathrm{TC}$*, and in fact the map*

$$\mathrm{Fil}^\bullet_{\mathrm{mot}}L_{K(1)}\mathrm{TC} \to \mathrm{Fil}^\bullet_{\mathrm{mot}}L_{\mathrm{arc}_p}L_{K(1)}\mathrm{TC}$$

exhibits the target as the filtration-completion of the source, i.e.

$$\mathrm{Fil}^r_{\mathrm{mot}}L_{\mathrm{arc}_p}L_{K(1)}\mathrm{TC} \simeq \varprojlim_n \frac{\mathrm{Fil}^r_{\mathrm{mot}}L_{K(1)}\mathrm{TC}}{\mathrm{Fil}^{r+n}_{\mathrm{mot}}L_{K(1)}\mathrm{TC}}.$$

Moreover, $(L_{\mathrm{arc}_p}L_{K(1)}\mathrm{TC})(X) \simeq L_{K(1)}\mathrm{TC}(X)$ *and* $(\mathrm{Fil}^\bullet_{\mathrm{mot}}L_{K(1)}\mathrm{TC})(X) \simeq (\mathrm{Fil}^\bullet_{\mathrm{mot}}L_{\mathrm{arc}_p}L_{K(1)}\mathrm{TC})(X)$ *for a qcqs* p*-adic formal scheme* X *that is Zariski-locally* $X = \mathrm{Spf}\,R$ *where* R *is one of the following forms:*

1. R *is a QRSP* $\mathcal{O}_C$*-algebra for* C *a complete algebraically closed field containing* $\mathbb{Q}_p$ *such that* R *is* p*-completely flat over* $\mathbb{Z}_p$*; or*
2. R *is a finitely generated* p*-torsion free quasi-syntomic ring that is* p*-completely finitely generated over either a perfectoid* $\mathcal{O}_C$*-algebra or a complete discrete valuation ring whose residue field* κ *is characteristic* p *with* $[\kappa : \kappa^p] < \infty$*; or*
3. $R[\frac{1}{p}]$ *has finite Krull dimension and admits a uniform bound on the mod* p *virtual cohomological dimensions of the residue field.*

In particular, the filtration $\mathrm{Fil}^\bullet_{\mathrm{mot}}L_{K(1)}\mathrm{TC}(X)$ *is complete if* X *satisfies one of the above conditions.*

Proof. The pull-back square is by [Kim25, Construction 3.29]. The filtration $\mathrm{Fil}^\bullet_{\mathrm{mot}}L_{\mathrm{arc}_p}L_{K(1)}\mathrm{TC}$ being complete is clear, and the fact that it is the filtration-completion of $\mathrm{Fil}^\bullet_{\mathrm{mot}}L_{K(1)}\mathrm{TC}$ is [Kim25, Lemma A.2(2)] using the fact that

$$\mathrm{gr}^\bullet_{\mathrm{mot}}L_{K(1)}\mathrm{TC} \simeq L_{\mathrm{arc}_p}\left(\mathrm{gr}^\bullet_{\mathrm{mot}}L_{K(1)}\mathrm{TC}\right) \simeq \mathrm{gr}^\bullet_{\mathrm{mot}}L_{\mathrm{arc}_p}L_{K(1)}\mathrm{TC}$$

since étale cohomology satisfies arc-p-descent [Kim25, Construction 3.12(4)]. The equivalence $(L_{\mathrm{arc}_p}L_{K(1)}\mathrm{TC})(X) \simeq L_{K(1)}\mathrm{TC}(X)$ holds for X in (1) by [Kim25, Proposition 3.11], which says that $L_{K(1)}\mathrm{TC}$ is an arc-p-hypersheaf when restricted to QRSP $\mathcal{O}_C$-algebras that are p-completely flat over $\mathbb{Z}_p$. For X as in (2), the claimed equivalence is either [Kim25, Proposition 3.20] for finite generation over a perfectoid, and [Kim25, Proposition 3.22] for finite generation over DVRs. For X as in (3), this is [Kim25, Corollary 3.39]. □

Corollary 3.7. *For any scheme* Y *over* $\mathbb{Z}$ *and any* $n \in \mathbb{Z}$*, any map*

$$\mathrm{Fil}^n_{\mathrm{mot}}L_{K(1)}\mathrm{TC}(X) \to \frac{L_{K(1)}\mathrm{TC}(X \times Y)}{\mathrm{Fil}^n_{\mathrm{mot}}L_{K(1)}\mathrm{TC}(X \times Y)}$$

functorial in X *is uniquely null-homotopic.*

Proof. Note that the right hand side is an arc-p-hypersheaf in X, since by Proposition 3.6 one has

$$\frac{L_{K(1)}\mathrm{TC}}{\mathrm{Fil}^n_{\mathrm{mot}} L_{K(1)}\mathrm{TC}} \simeq \frac{L_{\mathrm{arc}_p} L_{K(1)}\mathrm{TC}}{\mathrm{Fil}^n_{\mathrm{mot}} L_{\mathrm{arc}_p} L_{K(1)}\mathrm{TC}}.$$

In particular, any functorial map in X as in the statement of Corollary 3.7 must factor through the arc-p-hypersheafification of its source, so it suffices to show that there are no non-zero functorial-in-X maps

$$\mathrm{Fil}^n_{\mathrm{mot}} L_{\mathrm{arc}_p} L_{K(1)}\mathrm{TC}(X) \to \frac{L_{\mathrm{arc}_p} L_{K(1)}\mathrm{TC}(X\times Y)}{\mathrm{Fil}^n_{\mathrm{mot}} L_{\mathrm{arc}_p} L_{K(1)}\mathrm{TC}(X\times Y)}.$$

Lemma 3.4 implies that

$$\frac{L_{K(1)}\mathrm{TC}(X\times Y)}{\mathrm{Fil}^n_{\mathrm{mot}} L_{K(1)}\mathrm{TC}(X\times Y)} \simeq \frac{L_{\mathrm{arc}_p} L_{K(1)}\mathrm{TC}(X\times Y)}{\mathrm{Fil}^n_{\mathrm{mot}} L_{\mathrm{arc}_p} L_{K(1)}\mathrm{TC}(X\times Y)}$$

is concentrated in homotopy degrees $< 2n-1$, for all X. However, a basis for the arc-p topology is given by QRSP $\mathcal{O}_C$-algebras R (in fact, R can even be assumed to be a valuation ring, though we will not need this). For such R, [Kim25, Construction 3.12] implies that

$$\mathrm{Fil}^n_{\mathrm{mot}} L_{\mathrm{arc}_p} L_{K(1)}\mathrm{TC}(R) \simeq \mathrm{fib}\left(\tau_{\geq 2\bullet} L_{K(1)}\mathrm{TC}^-(R) \xrightarrow{\varphi-\mathrm{id}} \tau_{\geq 2\bullet} L_{K(1)}\mathrm{TP}\right)$$

is concentrated in degrees $\geq 2n-1$. The map must thus be uniquely null-homotopic. □

The following is a generalisation of [Tan24, Theorem 4.2].

Proposition 3.8. *Suppose* $\mathrm{Fil}^\bullet\mathcal{F}$ *is as in Assumption 3.1. For any stack* Y *and any rank* r *locally free sheaf* $\mathcal{E}$ *on* Y*, there is a functorial-in-*Y *fibre sequence*

$$\mathrm{Fil}^{\bullet-r}\mathcal{F}(Y\times B\mathbb{G}_m) \to \mathrm{Fil}^\bullet\mathcal{F}(\mathbb{V}(\mathcal{E})/\mathbb{G}_m) \to \mathrm{Fil}^\bullet\mathcal{F}(\mathbb{P}(\mathcal{E})).$$

Here, $\mathbb{V}(\mathcal{E}) \simeq \underline{\mathrm{Spec}}_Y(\mathrm{Sym}^*\mathcal{E})$ *is the vector bundle over* Y *associated to* $\mathcal{E}$*, and* $\mathbb{P}(\mathcal{E}) := (\mathbb{V}(\mathcal{E})\setminus 0)/\mathbb{G}_m \simeq \underline{\mathrm{Proj}}_Y(\mathrm{Sym}^*\mathcal{E})$ *is the associated projective bundle.*

Remark 3.9. In the proof, we do not use Assumption 3.1(2).

Proof. By Zariski descent on Y, we may assume that $\mathcal{E} \simeq \mathcal{O}_Y^r$ is the trivial vector bundle. In particular, $\mathbb{V}(\mathcal{E}) \simeq \mathbb{A}^r_Y$ and $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}^{r-1}_Y$.

Note that $\mathrm{Fil}^\bullet\mathcal{F}$ satisfies the conditions of [AHI24, Construction 6.5] (in fact, only Assumption 3.1(4) is used), and so is represented by an oriented E_∞-ring object $f \in \mathrm{MS}$ with f-linear orientation $c : \mathrm{Pic}\otimes f \to \mathbb{P}^1\otimes f$ also equipped with an f-linear map $\beta : \Sigma_{\mathbb{P}^1} f \to f$. Here are using the $\mathbb{P}^1$-invariant stable motivic category MS over $\mathbb{Z}$ of [AHI25; AHI24] (we use the Nisnevich topology), and by represented we mean that for any stack X, there are equivalences

$$\begin{aligned}
\mathrm{Fil}^n\mathcal{F}(X) &\simeq \mathrm{map}_{\mathrm{MS}}(\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^n_{\mathbb{P}^1} f),\\
\mathrm{gr}^n\mathcal{F}(X) &\simeq \mathrm{map}_{\mathrm{MS}}(\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^n_{\mathbb{P}^1}(f/\beta)),\\
\mathcal{F}(X) &\simeq \mathrm{map}_{\mathrm{MS}}(\Sigma^\infty_{\mathbb{P}^1} X_+, f[\beta^{-1}]).
\end{aligned}$$

We thus need to show that for every n, there is a functorial-in-Y fibre sequence

$$\mathrm{map}_{\mathrm{MS}}(\Sigma^\infty_{\mathbb{P}^1}(Y\times B\mathbb{G}_m)_+, \Sigma^{n-r}_{\mathbb{P}^1} f) \to \mathrm{map}_{\mathrm{MS}}(\Sigma^\infty_{\mathbb{P}^1}(\mathbb{A}^r_Y/\mathbb{G}_m)_+, \Sigma^n_{\mathbb{P}^1} f) \to \mathrm{map}_{\mathrm{MS}}(\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^{r-1}_Y)_+, \Sigma^n_{\mathbb{P}^1} f).$$

Now, there is a commutative diagram

$$\begin{tikzcd}
\mathbb{P}^{r-1} \arrow[r] \arrow[d] & \mathbb{A}^r/\mathbb{G}_m \arrow[d] \\
\mathbb{P}^\infty \arrow[r] & B\mathbb{G}_m
\end{tikzcd}$$

where the right vertical arrow is an equivalence in MS by [AHI25, Lemma 5.2] and the bottom horizontal arrow is an equivalence in MS by [AHI25, Theorem 5.3] (for notational simplicity we are dropping the $\Sigma^\infty_{\mathbb{P}^1}$). In particular, we have an identification

$$\mathrm{cofib}(\mathbb{P}^{r-1}\to\mathbb{A}^r/\mathbb{G}_m) \simeq \mathrm{cofib}(\mathbb{P}^{r-1}\to\mathbb{P}^\infty) \simeq \operatorname*{colim}_n \mathrm{cofib}(\mathbb{P}^{r-1}\to\mathbb{P}^{nr-1}).$$

For any oriented $E \in \mathrm{MS}$, writing $E(-) := \mathrm{map}_{\mathrm{MS}}(-, E)$ for notational simplicity, the projective bundle formula yields

$$E(\mathrm{cofib}(\mathbb{P}^{r-1} \to \mathbb{P}^{nr-1})) \simeq \bigoplus_{i=r}^{nr-1} (\Sigma^{-i}_{\mathbb{P}^1} E)(Y) \simeq (\Sigma^{-r}_{\mathbb{P}^1} E)(\mathbb{P}^{(n-1)r-1}).$$

Here, we have used the fact that the inclusion $\mathbb{P}^{r-1} \hookrightarrow \mathbb{P}^{nr-1}$ under the projective bundle formula induces the projection onto the first r summands. In particular, we see that

$$E\left(\operatorname*{colim}_n \mathrm{cofib}(\mathbb{P}^{r-1} \to \mathbb{P}^{nr-1})\right) \simeq \lim_n (\Sigma^{-r}_{\mathbb{P}^1} E)(\mathbb{P}^{(n-1)r-1}) \simeq (\Sigma^{-r}_{\mathbb{P}^1} E)\left(\operatorname*{colim}_n \mathbb{P}^{(n-1)r-1}\right) \simeq (\Sigma^{-r}_{\mathbb{P}^1} E)(\mathbb{P}^\infty).$$

Using the identification $\mathbb{P}^\infty \simeq Y \times B\mathbb{G}_m$ again, we obtain

$$\mathrm{fib}\left(E(\mathbb{A}^r/\mathbb{G}_m) \to E(\mathbb{P}^{r-1})\right) \simeq E(\mathrm{cofib}(\mathbb{P}^{r-1} \to \mathbb{P}^\infty)) \simeq (\Sigma^{-r}_{\mathbb{P}^1} E)(\mathbb{P}^\infty).$$

Taking $E = \Sigma^n_{\mathbb{P}^1} f$ yields the proposition. $\square$

The following is the main result of this subsection.

Theorem 3.10. *Let* $\mathrm{Fil}^\bullet \mathcal{F}$ *satisfy Assumption 3.1. Suppose also that* $\mathrm{Fil}^\bullet \mathcal{F}$ *satisfies either Condition 3.3 or is* $\mathrm{Fil}^\bullet \mathcal{F} = \mathrm{Fil}^\bullet_{\mathrm{mot}} L_{K(1)}\mathrm{TC}$. *Let X be a bounded quasi-syntomic p-adic formal scheme and $i : Z \hookrightarrow X$ a regular closed immersion of codimension r.*

Then, the space of filtered maps

$$\mathrm{Fil}^\bullet \mathcal{F}(Z) \to \mathrm{Fil}^{\bullet+r} \mathcal{F}(X) \tag{2}$$

that lift the transfer map $\mathcal{F}(\iota_*) : \mathcal{F}(Z) \to \mathcal{F}(X)$ is non-empty and contractible. Informally, the transfer map upgrades uniquely to a map of filtered spectra $\mathrm{Fil}^\bullet \mathcal{F}(Z) \to \mathrm{Fil}^{\bullet+r} \mathcal{F}(X)$.

Moreover, the filtered map (2) is linear over $\mathrm{Fil}^\bullet \mathcal{F}(X)$.

> **Remark 3.11.** By uniqueness, the above filtered maps matches with the Gysin map for oriented motivic spectra constructed in [Tan26].

Proof. The following proof proceeds exactly as in [Tan24, Theorem 5.38]. First, by Zariski descent we may suppose X and Z are affine p-adic formal schemes. As in the proof of Proposition 3.8, we will work in MS, with f denoting the oriented motivic spectrum represented by $\mathrm{Fil}^\bullet \mathcal{F}$, equipped with an f-linear Bott map $\beta : \Sigma_{\mathbb{P}^1} f \to f$. Since $X = \mathrm{Spf}\, R$ and $Z = \mathrm{Spf}(R/I)$ are affine, we can consider $U := \mathrm{Spec}(R) \setminus \mathrm{Spec}(R/I)$ as the scheme-theoretic (not p-adic formal scheme theoretic!) open complement of Z in X (see also Remark 3.2).

Before we get to the proof, we need to make a small digression on the deformation to the normal cone. Let $N = \mathbb{V}(\mathcal{N}_i)$ be the rank r vector bundle over Z associated to the normal bundle $\mathcal{N}_i$ of the regular closed immersion $i : Z \to X$. Let D_i denote the deformation to the normal cone of i (see for instance [Tan24, Definition 5.11]). Recall that D_i is a stack over $\mathbb{A}^1/\mathbb{G}_m$ sitting in the following commutative diagram [HKR25, (4.6)].

$$\begin{tikzcd}
\mathbb{P}(\mathcal{N}_i) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Bl}_i \arrow[d, hook] \arrow[dr, phantom, "\llcorner"] & U \arrow[l, hook'] \arrow[d, hook] \\
N/\mathbb{G}_m \arrow[r, hook] & D_i & X \arrow[l, hook', "\tilde{j}"] \\
Z \times B\mathbb{G}_m \arrow[r, hook] \arrow[u, hook, "0"] \arrow[ur, phantom, "\urcorner"] & Z \times \mathbb{A}^1/\mathbb{G}_m \arrow[u, hook, "\tilde{i}"] \arrow[ur, phantom, "\ulcorner"] & Z \arrow[l, hook', "j"] \arrow[u, hook, "i"]
\end{tikzcd} \tag{3}$$

Here, Bl_i is the blow-up of X in Z, and $\mathbb{P}(\mathcal{N}_i) \simeq E_i$ is the exceptional divisor of the blow-up. The diagram is obtained as follows: the closed immersion $Z \hookrightarrow X$ lifts to a closed immersion $Z \times \mathbb{A}^1/\mathbb{G}_m \hookrightarrow D_i$, with open complement Bl_i. This explains the middle column. The middle row is obtained by $D_i \times_{\mathbb{A}^1/\mathbb{G}_m} \mathbb{G}_m/\mathbb{G}_m \simeq X$ and $D_i \times_{\mathbb{A}^1/\mathbb{G}_m} B\mathbb{G}_m \simeq N/\mathbb{G}_m$, which is the defining feature of the deformation to the normal cone. The bottom row and top rows are then obtained by looking at the canonical closed immersions and their corresponding open complements. The arrow labelled 0 is the $\mathbb{G}_m$-quotient of the inclusion of the zero section $Z \hookrightarrow N$. By [Tan, Proposition 5], applying $\Sigma^\infty_{\mathbb{P}^1}(-)_+$ to the top left square yields the following Cartesian square in MS.

$$\begin{tikzcd}
\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}(\mathcal{N}_i))_+ \arrow[r] \arrow[d] & \Sigma^\infty_{\mathbb{P}^1}(\mathrm{Bl}_i)_+ \arrow[d] \\
\Sigma^\infty_{\mathbb{P}^1}(N/\mathbb{G}_m)_+ \arrow[r] & \Sigma^\infty_{\mathbb{P}^1}(D_i)_+
\end{tikzcd}$$

In particular, applying $\mathrm{map}_{\mathrm{MS}}(-, \mathrm{cofib}(\Sigma^{n+r}_{\mathbb{P}^1} f \to f[\beta^{-1}]))$ yields an equivalence

$$\frac{\mathcal{F}(D_i, \mathrm{Bl}_i)}{\mathrm{Fil}^{n+r}\mathcal{F}(D_i, \mathrm{Bl}_i)} \simeq \frac{\mathcal{F}(N/\mathbb{G}_m, \mathbb{P}(\mathcal{N}_i))}{\mathrm{Fil}^{n+r}\mathcal{F}(N/\mathbb{G}_m, \mathbb{P}(\mathcal{N}_i))}. \tag{4}$$

We are now ready to prove the theorem. Note that the composition

$$\mathrm{Perf}(Z) \xrightarrow{i_*} \mathrm{Perf}(X) \xrightarrow{(-)|_U} \mathrm{Perf}(U)$$

is functorially zero. In particular, the map $\mathcal{F}(i_*) : \mathcal{F}(Z) \to \mathcal{F}(X)$ factors through the canonical map $\mathcal{F}(X; U) \to \mathcal{F}(X)$, where we write

$$\mathcal{F}(X; U) := \mathrm{fib}(\mathcal{F}(X) \to \mathcal{F}(U)).$$

It thus suffices to show that, for any $n \in \mathbb{Z}$, the map

$$\mathrm{Fil}^n\mathcal{F}(Z) \to \mathcal{F}(Z) \to \mathcal{F}(X; U) \to \frac{\mathcal{F}(X; U)}{\mathrm{Fil}^{n+r}\mathcal{F}(X; U)} \tag{5}$$

has a unique functorial null-homotopy. Base change in the lower right square of (3) yields $i_* \circ j^* \simeq \widetilde{j}^* \circ \widetilde{i}_*$. If $\mathrm{pr} : Z \times \mathbb{A}^1/\mathbb{G}_m \to Z$ is the projection map, then $\mathrm{pr} \circ j = \mathrm{id}_Z$ implies that $i_* \simeq \widetilde{j}^* \circ \widetilde{i}_* \circ \mathrm{pr}^*$. Using this as well as base-change formulas in the commutative diagram (3), one checks that (5) can be rewritten as the composite

$$\mathrm{Fil}^n\mathcal{F}(Z) \to \mathrm{Fil}^n\mathcal{F}(Z \times B\mathbb{G}_m) \to \frac{\mathcal{F}(N/\mathbb{G}_m, \mathbb{P}(\mathcal{N}_i))}{\mathrm{Fil}^{n+r}\mathcal{F}(N/\mathbb{G}_m, \mathbb{P}(\mathcal{N}_i))} \simeq \frac{\mathcal{F}(D_i, \mathrm{Bl}_i)}{\mathrm{Fil}^{n+r}\mathcal{F}(D_i, \mathrm{Bl}_i)} \to \frac{\mathcal{F}(X; U)}{\mathrm{Fil}^{n+r}\mathcal{F}(X; U)}$$

where the first arrow is induced by the projection $Z \times B\mathbb{G}_m \to Z$ (which is compatible with pr under the inclusion $Z \times B\mathbb{G}_m \hookrightarrow Z \times \mathbb{A}^1/\mathbb{G}_m$), the second arrow is the composition (5) instead for the regular closed immersion $Z \times B\mathbb{G}_m \hookrightarrow N/\mathbb{G}_m$, the equivalence is (4), and the last arrow is restriction to the open fibre.

It thus suffices to show that the functorial map

$$\mathrm{Fil}^n\mathcal{F}(Z) \to \frac{\mathcal{F}(N/\mathbb{G}_m, \mathbb{P}(N))}{\mathrm{Fil}^{n+r}\mathcal{F}(N/\mathbb{G}_m, \mathbb{P}(\mathcal{N}_i))}$$

has a unique null-homotopy. Proposition 3.8 functorially identifies the target with

$$\frac{\mathcal{F}(Z \times B\mathbb{G}_m, \mathbb{P}(N))}{\mathrm{Fil}^n\mathcal{F}(Z \times B\mathbb{G}_m)},$$

and so we need to check that the composite

$$\mathrm{Fil}^n\mathcal{F}(Z) \to \frac{\mathcal{F}(Z \times B\mathbb{G}_m)}{\mathrm{Fil}^n\mathcal{F}(Z \times B\mathbb{G}_m)}$$

is uniquely null-homotopic. By looking at the Čech nerve for the cover $* \to B\mathbb{G}_m$, it suffices to show that any functorial-in-Z map

$$\mathrm{Fil}^n\mathcal{F}(Z) \to \frac{\mathcal{F}(Z \times \mathbb{G}_m^k, \mathbb{P}(\mathcal{N}_i))}{\mathrm{Fil}^n\mathcal{F}(Z \times \mathbb{G}_m^k)}$$

is uniquely null-homotopic, for any $k \geq 0$. By our assumption on $\mathrm{Fil}^\bullet\mathcal{F}$, this is true by Corollary 3.5 in case $\mathrm{Fil}^\bullet\mathcal{F}$ satisfies Condition 3.3, or by Corollary 3.7 in case $\mathrm{Fil}^\bullet\mathcal{F} \simeq \mathrm{Fil}^\bullet_{\mathrm{mot}} L_{K(1)}\mathrm{TC}$.

It remains to prove linearity. We again work in MS; we denote by f the oriented motivic spectrum represented by $\mathrm{Fil}^\bullet\mathcal{F}$ coming from [AHI24, Construction 6.5]. By [AHI24, Remark 6.6], $\Sigma^\bullet_{\mathbb{P}^1} f$ is a $\mathbb{Z}$-graded E_∞-ring. The f-linear map $\beta : \Sigma_{\mathbb{P}^1} f \to f$ upgrades this to a filtered E_∞-ring object. Let

$$\mathrm{Th}_Z(\mathcal{N}_i) := \mathrm{cofib}\left(\mathbb{P}(\mathcal{N}_i) \to \mathbb{P}(\mathcal{N}_i \oplus \mathcal{O})\right)$$

be the Thom space. By the discussion at the beginning of [AHI25, Section 6], for any oriented $E \in \mathrm{CAlg}(\mathrm{MS})$, the projective bundle formula gives a $E^{Z_+} := \mathrm{map}_{\mathrm{MS}}(\Sigma^\infty_{\mathbb{P}^1} Z_+, E)$-linear equivalence

$$t : \mathrm{map}_{\mathrm{MS}}(\Sigma^{\infty+r}_{\mathbb{P}^1} Z_+, E) \simeq \mathrm{map}_{\mathrm{MS}}(\Sigma^\infty_{\mathbb{P}^1} \mathrm{Th}_Z(\mathcal{N}_i), E).$$

By functoriality of the Gysin map (constructed in [Tan26])

$$\mathrm{gys}_i : \Sigma^\infty_{\mathbb{P}^1} X_+ \to \Sigma^\infty_{\mathbb{P}^1} \mathrm{Th}_Z(\mathcal{N}_i) \in \mathrm{MS}$$

under base-change in i, the uniqueness assertion implies that the filtered map $\mathrm{Fil}^{\bullet-r}\mathcal{F}(Z) \to \mathrm{Fil}^{\bullet}\mathcal{F}(X)$ identifies with the composition

$$\mathrm{map}_{\mathrm{MS}}(\Sigma^{\infty+r}_{\mathbb{P}^1} Z_+, \Sigma^{\bullet}_{\mathbb{P}^1} f) \xrightarrow[t_i]{\simeq} \mathrm{map}_{\mathrm{MS}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathrm{Th}_Z(\mathcal{N}_i), \Sigma^{\bullet}_{\mathbb{P}^1} f) \xrightarrow{\mathrm{gys}_i^*} \mathrm{map}_{\mathrm{MS}}(\Sigma^{\infty}_{\mathbb{P}^1} X_+, \Sigma^{\bullet}_{\mathbb{P}^1} f).$$

Now, t_i is $(\Sigma^{\bullet}_{\mathbb{P}^1} f)^{Z_+}$-linear, so in particular is $(\Sigma^{\bullet}_{\mathbb{P}^1} f)^{X_+}$-linear. The Gysin map gys_i^* is also $(\Sigma^{\bullet}_{\mathbb{P}^1} f)^{X_+}$-linear essentially by symmetric monoidality of both the Thom space construction (see [AHI25, Section 3]) and the Gysin map [Tan26, Theorem 1.3]. We conclude using $(\Sigma^{\bullet}_{\mathbb{P}^1} f)^{X_+} \simeq \mathrm{Fil}^{\bullet}\mathcal{F}(X)$. $\square$

3.2 Transfer Maps for Finite Extensions

Theorem 3.12. *Suppose $f : X \to Y$ is some regular finite locally free map of bounded quasi-syntomic p-adic formal schemes. Suppose $\mathrm{Fil}^{\bullet}\mathcal{F}$ satisfies Assumption 3.1 and satisfies either Condition 3.3 or $\mathrm{Fil}^{\bullet}\mathcal{F} \simeq \mathrm{Fil}^{\bullet}_{\mathrm{mot}} L_{K(1)}\mathrm{TC}$.*

Then, there is a $\mathrm{Fil}^{\bullet}\mathcal{F}(Y)$-linear filtered map

$$\mathrm{tr}^{\mathcal{F}}_f : \mathrm{Fil}^{\bullet}\mathcal{F}(X) \to \mathrm{Fil}^{\bullet}\mathcal{F}(Y)$$

satisfying the following properties:

1. *on passing to underlying objects of the filtered object, it identifies canonically with the transfer map $\mathcal{F}(X) \to \mathcal{F}(Y)$;*
2. *it is functorial in base-change in f;*
3. *it is functorial in maps $\mathrm{Fil}^{\bullet}\mathcal{F} \to \mathrm{Fil}^{\bullet}\mathcal{F}$ that also preserve the Chern class map, where both $\mathrm{Fil}^{\bullet}\mathcal{F}$ and $\mathrm{Fil}^{\bullet}\mathcal{G}$ satisfy the conditions in the theorem;*
4. *if $g : Y \to Z$ is another regular finite locally free map of bounded quasi-syntomic p-adic formal schemes, then there is a non-canonical homotopy*
$$\mathrm{tr}^{\mathcal{F}}_g \circ \mathrm{tr}^{\mathcal{F}}_f \simeq \mathrm{tr}^{\mathcal{F}}_{g\circ f};$$
5. *if $X \overset{\iota}{\hookrightarrow} \mathbb{A}^n_Y \subset \mathbb{P}^n_Y \overset{\pi}{\twoheadrightarrow} Y$ is a factorisation of f through a regular closed immersion ι, then there is a homotopy*
$$\begin{tikzcd} \mathrm{Fil}^{\bullet+n}_{\mathrm{mot}}\mathcal{F}(\mathbb{P}^n_Y) \arrow[r, phantom, "\simeq"] & \bigoplus_{i=0}^n \mathrm{Fil}^{\bullet+n-i}_{\mathrm{mot}}\mathcal{F}(Y)\cdot c(\mathcal{O}(1))^i \arrow[d, "\text{coeff of } c(\mathcal{O}(1))^n"] \\ \mathrm{Fil}^{\bullet}_{\mathrm{mot}}\mathcal{F}(X) \arrow[u, "\mathrm{gys}_\iota"] \arrow[r, "\mathrm{tr}^{\mathcal{F}}_f"'] \arrow[ur, phantom, "\Downarrow"] & \mathrm{Fil}^{\bullet}_{\mathrm{mot}}\mathcal{F}(Y) \end{tikzcd}$$
where the map marked gys_ι is the filtered map of Theorem 3.10 for the regular closed immersion ι. This homotopy is functorial in $\mathcal{F}$ as in (3) above, as well as functorial in base-change in Y.

The following lemma explains why statement (5) above makes sense.

Lemma 3.13. *Let $\iota : \mathrm{Spf}\, S \hookrightarrow \mathbb{A}^n_R \subset \mathbb{P}^n_R$ be a regular closed immersion of codimension n, $\pi : \mathbb{P}^n_R \to \mathrm{Spf}\, R$ the projection map, and $f = \pi \circ \iota : \mathrm{Spf}\, S \to \mathrm{Spf}\, R$. Suppose f is finite flat. Suppose $\mathrm{Fil}^{\bullet}\mathcal{F}$ is as in Assumption 3.1. Then, the following hold.*

1. *the map $\mathcal{F}(\pi_*) : \mathcal{F}(\mathbb{P}^n_R) \to \mathcal{F}(R)$ sends $c(\mathcal{O}(1))^i$ to 1 for all $0 \le i \le n$;*
2. *the composition*
$$\mathcal{F}(S) \xrightarrow{\mathcal{F}(\iota_*)} \mathcal{F}(\mathbb{P}^n_R) \simeq \bigoplus_{i=0}^{n} \mathcal{F}(R)\cdot c(\mathcal{O}(1))^i \xrightarrow{\text{coefficient of } c(\mathcal{O}(1))^i} \mathcal{F}(R)$$
is canonically zero unless $i = n$, in which case it is canonically homotopic to $\mathcal{F}(f_*) : \mathcal{F}(S) \to \mathcal{F}(R)$.

Proof. By Assumption 3.1(3), and since the lemma is about the underlying unfiltered localising invariant, it suffices to restrict to $\mathcal{F} = K$. For the first statement, since the projective bundle formula for $K(\mathbb{P}^n_R)$ is $K(R)$-linear, it suffices to check that $\pi_* c^i$ is 1 for the class $c := 1 - [\mathcal{O}(-1)] \in K_0(\mathbb{P}^n_R)$. By [SP, Tag01XT], one computes that

$$\pi_*\mathcal{O}(-i) = \mathrm{R}\Gamma(\mathbb{P}^n_R, \mathcal{O}(-i)) = 0$$

for $1 \leq i \leq n$. The binomial formula then yields that $\pi_* c^i = 1$.

We now prove the second statement. Since ι_* and $\mathrm{R}\Gamma(\mathbb{P}^n_R, -)$ are $\mathrm{Perf}(R)$-linear functors (i.e. the projection formula holds), the maps $K(S) \to K(R)$ are all $K(R)$-linear. It thus suffices to prove the claim at the level of K_0. Let $M \in \mathrm{Perf}(S)$. By the filtered projective bundle formula as well as the proof of [BM12, Theorem 1.5], we can express $[\iota_* M] \in K_0(\mathbb{P}^n_R)$ as a linear combination

$$[\iota_* M] = \sum_{i=0}^{n} c_i u^i$$

of powers of $u = 1 - c(\mathcal{O}(1)) = [\mathcal{O}(-1)] \in K_0(R)$, where $c_i \in K_0(R)$ are defined by $c_0 = [\mathrm{R}\Gamma(\mathbb{P}^n_R, \iota_* M)]$ and

$$c_k = [\mathrm{R}\Gamma(\mathbb{P}^n_R, (\iota_* M)(k))] - \sum_{i=0}^{k-1} \binom{n+k-i}{n} c_i.$$

By the projection formula, we have

$$(\iota_* M)(k) \simeq \iota_*\big(M \otimes \iota^* \mathcal{O}_{\mathbb{P}^n_R}(k)\big).$$

Since $\mathrm{Spf}\, S$ does not intersect the hyperplane at ∞ by assumption, we have $\iota^* \mathcal{O}_{\mathbb{P}^n_R}(k) \cong S$, the trivial line bundle, and so

$$\mathrm{R}\Gamma(\mathbb{P}^n_R, (\iota_* M)(k)) \simeq \mathrm{R}\Gamma(\mathbb{P}^n_R, \iota_* M) \simeq f_* M.$$

Hence, we have $c_k = [f_* M] a_k \in K_0(R)$ where $a_k \in \mathbb{Z}$ satisfy the recurrence relation $a_0 = 1$ and

$$a_k = 1 - \sum_{i=0}^{k-1} \binom{n+k-i}{n} a_i.$$

One checks that $a_k = (-1)^k \binom{n}{k}$, which can be proven by induction and an application of Lemma 3.14 below.

Thus, we see that

$$[\iota_* M] = [f_* M] \sum_{k=0}^{n} (-1)^k \binom{n}{k} u^k \in K_0(\mathbb{P}^n_R).$$

Changing base, we see that

$$[\iota_* M] = [f_* M] \beta^n.$$

This proves the second statement. □

Lemma 3.14. *For all integers $0 \leq k < n$, one has the identity*

$$(-1)^{k+1} \binom{n}{k+1} = 1 - \sum_{i=0}^{k} (-1)^i \binom{n+k+1-i}{n} \binom{n}{i}.$$

Proof. The generating function of the right hand side is

$$\begin{aligned}
G(z) &:= \sum_{k \geq 0} \left(1 - \sum_{i=0}^{k} (-1)^i \binom{n+k+1-i}{n} \binom{n}{i}\right) z^k = \frac{1}{1-z} - \sum_{k \geq i \geq 0} (-1)^i \binom{n+k+1-i}{n} \binom{n}{i} z^k \\
&= \frac{1}{1-z} - \sum_{i,j \geq 0} (-1)^i \binom{n+1+j}{n} \binom{n}{i} z^{i+j} = \frac{1}{1-z} - \left(\sum_{i \geq 0} (-1)^i \binom{n}{i} z^i\right) \left(\sum_{j \geq 0} \binom{n+1+j}{n} z^j\right) \\
&= \frac{1}{1-z} - (1-z)^n \left(\frac{(1-z)^{-n-1} - 1}{z}\right) = \frac{(1-z)^n - 1}{z}
\end{aligned}$$

and the coefficient of z^k is $(-1)^{k+1}\binom{n}{k+1}$, as expected. □

The following is an immediate consequence of Lemma 3.13 and the filtered projective bundle formula.

Corollary 3.15. *Keep notation as in Lemma 3.13. There is a canonical choice of null-homotopy, functorial in* $\mathrm{Fil}^\bullet \mathcal{F}$ *(as in Theorem 3.12(3) above) and depending only on the factorisation $f = \pi \circ i$, for the composition*

$$\mathrm{Fil}^\bullet \mathcal{F}(S) \to \mathcal{F}(S) \xrightarrow{\mathcal{F}(i_*)} \mathcal{F}(\mathbb{P}^n_R) \xrightarrow{\mathcal{F}(\pi_*)} \mathcal{F}(R) \to \mathcal{F}(R)/\mathrm{Fil}^\bullet \mathcal{F}(R).$$

We are now in a position to prove Theorem 3.12.

Proof of Theorem 3.12. By Zariski descent, we may suppose that $Y = \operatorname{Spf} R$ and thus $X = \operatorname{Spf} S$ are affine. Corollary 3.15 gives the entire theorem for free, as long as we can do the following:

1. choose a factorisation $X \overset{\iota}{\hookrightarrow} \mathbb{A}^n_Y \subset \mathbb{P}^n_Y \twoheadrightarrow Y$ that is functorial in base-change in f, where ι is a regular closed immersion, and

2. show that any two such factorisations $X \overset{\iota}{\hookrightarrow} \mathbb{P}^n_Y \twoheadrightarrow Y$ and $X \overset{\iota'}{\hookrightarrow} \mathbb{P}^{n'}_Y \twoheadrightarrow Y$ have a common refinement $X \overset{\widetilde{\iota}}{\hookrightarrow} \mathbb{P}^m_Y \twoheadrightarrow Y$, in the sense that the following diagram commutes

$$\begin{tikzcd}
 & \mathbb{A}^n_Y \arrow[r, hook] \arrow[d, hook, "\sigma"] & \mathbb{P}^n_Y \arrow[d, hook] \arrow[dr, two heads] & \\
X \arrow[ur, hook, "\iota"] \arrow[r, hook, "\widetilde{\iota}"] \arrow[dr, hook, "\iota'"'] & \mathbb{A}^m_Y \arrow[r, hook] & \mathbb{P}^m_Y \arrow[r, two heads] & Y \\
 & \mathbb{A}^n_Y \arrow[u, hook, "\sigma'"] \arrow[r, hook] & \mathbb{P}^{n'}_Y \arrow[u, hook] \arrow[ur, two heads] &
\end{tikzcd}$$

for some choice of regular closed immersions $\sigma : \mathbb{A}^n_Y \hookrightarrow \mathbb{A}^m_Y$ and $\sigma' : \mathbb{A}^{n'}_Y \hookrightarrow \mathbb{A}^m_Y$.

For the first, since $f_*\mathcal{O}_X$ is a vector bundle on Y by assumption, we take ι to be the canonical map

$$X \hookrightarrow \mathbb{V}_Y(f_*\mathcal{O}_X);$$

this is obtained by taking Spf of the map $\operatorname{Sym}_R(S) \to S$ induced by the multiplication on S. That this ι is a regular closed immersion follows from [SP, Tag069G]. This is also clearly functorial in pull-backs along arbitrary maps $Y' \to Y$.

For the second, we can simply take $m = n + n'$ and $\widetilde{\iota} : X \hookrightarrow \mathbb{A}^m_Y$ to be the factorisation

$$X \hookrightarrow X \times_Y X \overset{\iota \times \iota'}{\hookrightarrow} \mathbb{A}^n_Y \times_Y \mathbb{A}^{n'}_Y \simeq \mathbb{A}^{n+n'}_Y,$$

with σ and σ' the inclusion of the first n and last n' coordinates respectively. $\square$

Remark 3.16. We expect there to be a canonical choice of null-homotopy of the composite

$$\mathrm{Fil}^\bullet \mathcal{F}(X) \to \mathcal{F}(X) \xrightarrow{\mathcal{F}(f_*)} \mathcal{F}(Y) \to \frac{\mathcal{F}(Y)}{\mathrm{Fil}^\bullet \mathcal{F}(Y)}.$$

For instance, if $\mathscr{I}$ denotes the 1-category of regular closed immersions $\iota : X \hookrightarrow \mathbb{A}^n_Y$ factorising f (morphisms given by maps $\mathbb{A}^n_Y \to \mathbb{A}^m_Y$ compatible with f), then Corollary 3.15 yields a functor from $\mathscr{I}$ to the space of null-homotopies of the above composition. If one can show that this functor admits a colimit, then one immediately gets a stronger result than Theorem 3.12, where the choice of map $\mathrm{tr}^{\mathcal{F}}_f$ is canonically functorial in composition in f too. Unfortunately $\mathscr{I}$ is not filtered, so it is non-trivial to check whether such a colimit exists without digging deeper into the space of null-homotopies of the above composition.

3.3 Transfer Maps and Prismatic F-Gauges

We now show Theorem 3.12 yields a construction of transfer maps for F-gauges, thus proving the existence part of Theorem E.

Lemma 3.17. *Suppose $f : X \to Y$ is a map of bounded quasi-syntomic p-adic formal schemes such that f_* induces transfer maps*

$$\mathrm{Fil}^\bullet \mathrm{TC}^-_{\mathbb{\Delta}}(X) \to \mathrm{Fil}^{\bullet+k} \mathrm{TC}^-_{\mathbb{\Delta}}(Y) \quad \text{and} \quad \mathrm{Fil}^\bullet \mathrm{TP}_{\mathbb{\Delta}}(X) \to \mathrm{Fil}^{\bullet+k} \mathrm{TP}_{\mathbb{\Delta}}(Y)$$

for some $k \in \mathbb{Z}$, functorial in change-of-base Y and composition in f (the latter only up to possibly non-canonical homotopy), and compatible with the two maps $\mathrm{can}, \varphi : \mathrm{TC}^-_{\mathbb{\Delta}} \to \mathrm{TP}_{\mathbb{\Delta}}$. *Then, there is a map of prismatic F-gauges*

$$f^{\mathrm{syn}}_* \mathcal{O}_{X^{\mathrm{syn}}} \to \mathcal{O}_{Y^{\mathrm{syn}}}\{k\}[2k]$$

again functorial in base-change in Y and composition in f (the latter only up to possibly non-canonical homotopy).

Proof. By quasi-syntomic descent, we may assume that $Y = \mathrm{Spf}\, R$ is a QRSP. In this case, recall that the category of prismatic F-gauges over Y a QRSP is equivalent to the category of (p, I)-complete $\mathbb{Z}$-indexed filtered modules $\mathrm{Fil}^\bullet M$ over the filtered ring $\mathrm{Fil}^\bullet_{\mathrm{Nyg}} \Delta_R$ equipped with a $\mathrm{Fil}^\bullet_{\mathrm{Nyg}} \Delta_R$-linear map

$$\varphi : \mathrm{Fil}^\bullet M \to I^\bullet M := I^\bullet \Delta_R \otimes_{\Delta_R} M$$

such that the map

$$\mathrm{Fil}^\bullet M \widehat{\otimes}_{\mathrm{Fil}^\bullet_{\mathrm{Nyg}} \Delta_R, \varphi} I^\bullet \Delta_R \to I^\bullet M$$

is an isomorphism of filtered $I^\bullet \Delta_R$-modules [Mon24, Remark 3.22]; here, the tensor product on the left is (p, I)-completed. In particular, the prismatic F-gauge associated to $\mathcal{O}_{Y^{\mathrm{Syn}}}\{r\}[2r]$ is simply $(\mathrm{Fil}^{\bullet + r}_{\mathrm{Nyg}} \Delta_R)\{r\}[2r]$ itself, while the prismatic F-gauge associated to $f^{\mathrm{syn}}_* \mathcal{O}_{X^{\mathrm{Syn}}}$ is $\mathrm{Fil}^\bullet_{\mathrm{Nyg}} \mathrm{R}\Gamma_{\Delta}(X)$.

Taking the i'th associated graded of the filtered transfer maps in the statement of the lemma, we get a map

$$\mathrm{Fil}^i_{\mathrm{Nyg}} \mathrm{R}\Gamma_{\Delta}(X, \mathcal{O}\{i\})[2i] \to (\mathrm{Fil}^{i+k}_{\mathrm{Nyg}} \Delta_R)\{i + k\}[2i + 2k].$$

Since $\mathrm{R}\Gamma_{\Delta}(X)$ is an algebra over Δ_R and $\mathrm{R}\Gamma_{\Delta}(X, \mathcal{O}\{i\})$ is a module over $\mathrm{R}\Gamma_{\Delta}(X)$, it follows that we can pull the Breuil-Kisin twist out of the cohomology. We thus get a map

$$\mathrm{Fil}^i_{\mathrm{Nyg}} \mathrm{R}\Gamma_{\Delta}(X) \to (\mathrm{Fil}^{i+k}_{\mathrm{Nyg}} \Delta_R)\{k\}[2k]$$

for all i. This map is compatible with the canonical and Frobenius maps by assumption. Hence, the map

$$\mathrm{Fil}^\bullet_{\mathrm{Nyg}} \mathrm{R}\Gamma_{\Delta}(X) \to (\mathrm{Fil}^{\bullet + k}_{\mathrm{Nyg}} \Delta_R)\{k\}[2k]$$

is actually a map of prismatic F-gauges on Y^{Syn}. This gives us the required map

$$f^{\mathrm{syn}}_* \mathcal{O}_{X^{\mathrm{Syn}}} \to \mathcal{O}_{Y^{\mathrm{Syn}}}\{k\}[2k].$$

□

Lemma 3.18. *For $f : X \to Y$ any quasi-syntomic map of qcqs quasi-syntomic p-adic formal schemes, and any $\mathcal{E} \in \mathcal{D}_{\mathrm{qcoh}}(X^{\mathrm{Syn}})$ and $\mathcal{F} \in \mathrm{Perf}(Y^{\mathrm{Syn}})$, the projection formula holds:*

$$(f^{\mathrm{syn}}_* \mathcal{E}) \otimes \mathcal{F} \simeq f^{\mathrm{syn}}_* (\mathcal{E} \otimes f^{\mathrm{syn}, *} \mathcal{F}).$$

Proof. Note that there is always a canonical map

$$(f^{\mathrm{syn}}_* \mathcal{E}) \otimes \mathcal{F} \to f^{\mathrm{syn}}_* (\mathcal{E} \otimes f^{\mathrm{syn}, *} \mathcal{F})$$

adjoint to the map

$$f^{\mathrm{syn},*}((f^{\mathrm{syn}}_* \mathcal{E}) \otimes \mathcal{F}) \to (f^{\mathrm{syn},*} f^{\mathrm{syn}}_* \mathcal{E}) \otimes f^{\mathrm{syn},*} \mathcal{F} \to \mathcal{E} \otimes f^{\mathrm{syn},*} \mathcal{F}.$$

By a thick subcategory argument, we reduce to the case $\mathcal{F} = \mathcal{O}_{Y^{\mathrm{Syn}}}$ in which case the projection formula is a tautology. □

We finally prove the existence of the prismatic trace map.

Proof of Existence of Traces and Statement (2) in Theorem E. By Theorem 3.12 and Lemma 3.17 for $f : X \to Y$ finite flat and regular, we get a trace map of prismatic F-gauges

$$\mathrm{tr}^{\mathrm{syn}}_f : f^{\mathrm{syn}}_* \mathcal{O}_{X^{\mathrm{Syn}}} \to \mathcal{O}_{Y^{\mathrm{Syn}}}.$$

Using the projection formula

$$f^{\mathrm{syn}}_* \mathcal{O}_{X^{\mathrm{Syn}}} \otimes \mathcal{E} \simeq f^{\mathrm{syn}}_* f^{\mathrm{syn},*} \mathcal{E}$$

for any $\mathcal{E} \in \mathrm{Perf}(Y^{\mathrm{Syn}})$, we get a trace map

$$\mathrm{tr}^{\mathrm{syn}}_f(\mathcal{E}) : f^{\mathrm{syn}}_* f^{\mathrm{syn},*} \mathcal{E} \to \mathcal{E}$$

for any prismatic F-Gauge $\mathcal{E}$ on Y. This also establishes Theorem E(2). □

Remark 3.19. Combining Theorem 3.10 and [Tan24, Theorem 5.38], we see that the Gysin map

$$\mathrm{Gys}^{\mathrm{syn}}_{\mathcal{O}\{r\}} : i_*\mathcal{O}_{Z^{\mathrm{Syn}}} \to \mathcal{O}_{X^{\mathrm{Syn}}}\{r\}[2r]$$

coming from Lemma 3.17 applied to Theorem 3.10 upon taking coherent cohomology over X^{Syn} coincides with the Gysin map

$$\mathbb{Z}_p(i)(Z) \to \mathbb{Z}_p(i+r)(X)[2r]$$

on syntomic cohomology given in [Tan24, Definition 5.32].

Lemma 3.20. *Suppose $f : X \to Y$ is a finite flat regular map of quasi-syntomic qcqs p-adic formal schemes. Then, for any prismatic F-gauge $\mathcal{E} \in \mathcal{D}_{\mathrm{qcoh}}(Y^{\mathrm{Syn}})$, the composition*

$$\mathcal{E} \to f^{\mathrm{syn}}_* f^{\mathrm{syn},*}\mathcal{E} \xrightarrow{\mathrm{tr}^{\mathrm{syn}}_f(\mathcal{E})} \mathcal{E}$$

is multiplication by $\deg f$.

In other words, Theorem E(1) is true.

Proof. By Zariski descent, we may suppose $X = \operatorname{Spf} S$ and $Y = \operatorname{Spf} R$ are affine and that S is a free R-module of rank d. By construction of the prismatic trace, it suffices to check that the composition

$$\mathcal{F}(R) \to \mathcal{F}(S) \xrightarrow{\text{transfer}} \mathcal{F}(R)$$

is multiplication by d, for $\mathcal{F} \in \{\mathrm{TC}^-_{\Delta}, \mathrm{TP}_{\Delta}\}$. Since this composition is $\mathcal{F}(R)$-linear and so $K(R)$-linear, it suffices to check that the above composition is multiplication by d in the 'universal' case $\mathcal{F} = K$. By linearity again, it suffices to check that $1 \in \pi_0 K(R)$ is sent to $d \in \pi_0 K(R)$. However, this is clear, since the image of 1 in $K_0(R)$ under the above composition is by definition $[S] = [R^d] = d \in K_0(R)$. $\square$

3.4 Trace Maps on Prismatic Cohomology for Perfectoids

We analyse the prismatic trace map associated to a (module-)finite locally free extension $R \to S$ where both R and S are perfectoid. Then, $\Delta_R = A_{\mathrm{inf}}(R)$ and $\Delta_S = A_{\mathrm{inf}}(S)$.

The trace map on prismatic cohomology is thus a map

$$\mathrm{tr}^{\Delta}_{S/R} : A_{\mathrm{inf}}(S) \to A_{\mathrm{inf}}(R).$$

On the other hand, by derived Nakayama, we know that the extension $A_{\mathrm{inf}}(R) \to A_{\mathrm{inf}}(S)$ is finite and of the same degree as S/R. By the same argument, it is also clear that $A_{\mathrm{inf}}(S)$ is locally free as an $A_{\mathrm{inf}}(R)$-module as the same is true for S over R. Thus, there is also the usual trace map

$$\mathrm{tr}_{A_{\mathrm{inf}}} : A_{\mathrm{inf}}(S) \to A_{\mathrm{inf}}(R)$$

associated to a finite locally free extension. In this subsection we check that these two maps coincide.

Proposition 3.21. *For any finite locally free extension S/R of p-torsion free integral perfectoids, the two maps $\mathrm{tr}^{\Delta}_{S/R}$ and $\mathrm{tr}_{A_{\mathrm{inf}}}$ described above coincide.*

In other words, Theorem E(5) is true.

Proof. By Zariski descent we assume S is finite free as an R-module. By the results of [BMS19, Section 6.1], Remark 4.1, and multiplicativity of the homotopy-fixed point spectral sequence, the transfer map $\mathrm{TC}^-(S) \to \mathrm{TC}^-(R)$ induces the map of spectral sequences

$$\begin{array}{ccc} S[u][\![t]\!] & \Longrightarrow & A_{\mathrm{inf}}(S)[u,t]/(ut-d) \\ {\scriptstyle \mathrm{tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}^{\Delta}_{S/R}} \\ R[u][\![t]\!] & \Longrightarrow & A_{\mathrm{inf}}(R)[u,t]/(ut-d) \end{array}$$

where d is a generator of $\ker(A_{\mathrm{inf}}(R) \to R)$ (and $\ker(A_{\mathrm{inf}}(S) \to S)$), and the vertical maps send u and t to themselves and act on S (resp. $A_{\mathrm{inf}}(S)$) as the transfer maps $\pi_0\mathrm{tr}^{\mathrm{THH}}$ (resp. $\pi_0\mathrm{tr}^{\mathrm{TC}^-}$). We claim that the map

$$\pi_0\mathrm{tr}^{\mathrm{THH}} : S = \mathrm{THH}_0(S) \to \mathrm{THH}_0(R) = R$$

coincides with the usual trace map $S \to R$. Indeed, recall that $\mathrm{THH}_0 = \mathrm{HH}_0$ in general, and so this claim is equivalent to the claim that

$$\pi_0 \mathrm{tr}^{\mathrm{HH}} : S = \mathrm{HH}_0(S) \to \mathrm{HH}_0(R) = R$$

is the usual trace map. This latter claim follows by a straightforward calculation unravelling the explicit weak equivalence $\mathrm{HH}(-) \simeq \mathrm{HH}(\mathrm{Perf}(-))$ described in [McC94, Section 2].

By functoriality of the transfer with respect to a map of localising invariants, the map $\mathrm{tr}^{\Delta}_{S/R}$ is also Frobenius equivariant. The usual trace map $\mathrm{tr}_{A_{\mathrm{inf}}} : A_{\mathrm{inf}}(S) \to A_{\mathrm{inf}}(R)$ is also Frobenius equivariant, since one can apply Frobenius to the matrix for multiplication by any element $x \in A_{\mathrm{inf}}(S)$. In particular, the difference

$$h : A_{\mathrm{inf}}(S) \to A_{\mathrm{inf}}(R)$$

is a Frobenius equivariant $A_{\mathrm{inf}}(R)$-linear map that is zero modulo d. We claim that any such map h has to be zero. Indeed, since h is zero modulo d, we have $h(A_{\mathrm{inf}}(S)) \subseteq d \cdot A_{\mathrm{inf}}(R)$. Frobenius equivariance then says that $h(A_{\mathrm{inf}}(S)) \subseteq \varphi^n(d) \cdot A_{\mathrm{inf}}(R)$ for all $n \in \mathbb{Z}$. We are done by the following Lemma 3.22. □

The following lemma is probably well-known but the author could not find a reference in the literature.

Lemma 3.22. *Suppose R is any p-torsion free integral perfectoid ring and d a generator of the kernel of Fontaine's map $\theta : A_{\mathrm{inf}}(R) \to R$. Then,*

$$\bigcap_{n \geq 0} \varphi^n(d) A_{\mathrm{inf}}(R) = 0.$$

Proof. Let $R^\flat$ be the tilt of R, so that $A_{\mathrm{inf}}(R) = W(R^\flat)$ and $R^\flat$ is $\overline{d}$-adically complete where $\overline{d}$ is the reduction of d modulo p. Note that $\overline{d}$ is also non-zero divisor in $R^\flat$ by our p-torsion-free assumption on R. Suppose $x \in \bigcap_{n \geq 0} \varphi^n(d) A_{\mathrm{inf}}(R)$. Reducing mod p, one has

$$\bigcap_{n \geq 0} \overline{d}^{p^n} R^\flat = 0$$

by $\overline{d}$-adic completeness of $R^\flat$. In particular, $x = p x_1$ for some $x_1 \in A_{\mathrm{inf}}(R)$. Next, writing $x = \varphi^n(d) b_n$ implies that $\overline{d}^{p^n} \cdot b_n = 0$ modulo p, and since $\overline{d}$ is a non-zero divisor we get $b_n \in p A_{\mathrm{inf}}(R)$. As $A_{\mathrm{inf}}(R)$ is p-torsion free, we thus have $x_1 = \varphi^n(d) \frac{b_n}{p}$ for all n. Thus $\bigcap_{n \geq 0} \varphi^n(d) A_{\mathrm{inf}}(R)$ is p-divisible. Since $A_{\mathrm{inf}}(R)$ is p-complete, it follows that $\bigcap_{n \geq 0} \varphi^n(d) A_{\mathrm{inf}}(R) = 0$. □

3.5 Étale Comparison

In this section, we prove the compatibility of the prismatic trace with étale realisation. For the rest of this section, we assume that $f : X \to Y$ is a regular finite flat map of quasi-syntomic $\mathbb{Z}_p$-flat p-adic formal schemes such that the generic fibres X_η and Y_η are locally Noetherian analytic adic spaces. Then, $f_\eta : X_\eta \to Y_\eta$ is a finite flat map of analytic adic spaces. In this setting, [LRZ24] construct a trace map

$$\mathrm{tr}^{\text{ét}}_f(\mathcal{F}) : f_{\eta *} f_\eta^* \mathcal{F} \to \mathcal{F}$$

for any abelian sheaf $\mathcal{F}$; we recall their result.

Theorem 3.23 ([LRZ24, Theorem 2.5.6])**.** *For any finite flat map $f_\eta : X_\eta \to Y_\eta$ of locally Noetherian analytic adic spaces, there is a unique map*

$$\mathrm{tr}^{\text{ét}}_f(\mathcal{F}) : f_{\eta *} f_\eta^* \mathcal{F} \to \mathcal{F}$$

functorial in $\mathcal{F} \in \mathrm{Ab}(Y_{\eta,\text{ét}})$, compatible with compositions and pull-backs in f_η, and normalized so that the composition $\mathcal{F} \to f_{\eta,*} f_\eta^* \mathcal{F} \to \mathcal{F}$ is locally given by multiplication by $\deg f$.

> **Remark 3.24.** [LRZ24] only construct these étale trace maps for finite flat maps of locally Noetherian analytic adic spaces. Using Theorem 3.12, we may relax the Noetherianity hypothesis at the cost of adding a regularity condition and working with p-complete coefficients.

Here, we note that $f^{\heartsuit}_{\eta,*}$, the underived push-forward, is exact for finite morphisms f_η by [Hub13, Proposition 2.6.3], so that $f_{\eta,*} \simeq f^{\heartsuit}_{\eta,*}$.

We can extend the construction of trace maps for finite flat maps given in [LRZ24] to the derived category $\mathcal{D}^b(Y_{\eta,\text{ét}}, \mathrm{Ab})$ of étale sheaves of bounded complexes of abelian groups. In particular, for any complex of lisse sheaves $\mathcal{L} \in \mathcal{D}^b_{\text{lisse}}(Y_\eta, \mathbb{Z}_p)$, we can look at the étale sheaves $\mathcal{L}/p^n$ for all n, and we have a trace map

$$\mathrm{tr}_f^{\text{ét}}(\mathcal{L}/p^n) : f_{\eta *} f_\eta^*(\mathcal{L}/p^n) \to \mathcal{L}/p^n$$

compatible for the various transition maps. Taking limits, and passing to the pro-étale site for the formation of pushforwards/pullbacks, we get a $\mathbb{Z}_p$-linear trace map

$$\mathrm{tr}_f^{\text{ét}}(\mathcal{L}) : f_{\eta,*} f_\eta^*(\mathcal{L}) \to \mathcal{L}.$$

In particular, for $\mathcal{L} = \underline{\mathbb{Z}_p}_{Y_\eta}$, we have

$$\mathrm{tr}_f^{\text{ét}} := \mathrm{tr}_f^{\text{ét}}(\underline{\mathbb{Z}_p}_{Y_\eta}) : f_{\eta,*}\underline{\mathbb{Z}_p}_{X_\eta} \to \underline{\mathbb{Z}_p}_{Y_\eta}.$$

By the projection formula, if $\mathcal{L}$ is a bounded complex of lisse sheaves, we can identify $\mathrm{tr}_f^{\text{ét}}(\mathcal{L})$ with the map

$$f_{\eta,*} f_\eta^* \mathcal{L} \simeq f_{\eta,*}(\underline{\mathbb{Z}_p}_{X_\eta}) \otimes \mathcal{L} \xrightarrow{\mathrm{tr}_f^{\text{ét}} \otimes \mathrm{id}_{\mathcal{L}}} \mathcal{L}.$$

With this trace map on p-adic lisse sheaves for finite morphisms of rigid spaces in hand, we have the following.

Theorem 3.25. *Suppose $f : X \to Y$ finite locally free p-completely syntomic map of bounded quasi-syntomic p-adic formal schemes flat over $\mathbb{Z}_p$. Suppose also that X_η, Y_η are locally Noetherian. Then, for any prismatic F-gauge $\mathcal{E} \in \mathrm{Perf}(Y^{\mathrm{Syn}})$, the étale realisation of the trace map on F-gauges*

$$f_*^{\mathrm{syn}} f^{\mathrm{syn},*}\mathcal{E} \to \mathcal{E}$$

coincides with the étale trace map

$$\mathrm{tr}_f^{\text{ét}}(T_{\text{ét}}\mathcal{E}) : f_{\eta,*} f_\eta^* \left(T_{\text{ét}}(\mathcal{E})\right) \to T_{\text{ét}}(\mathcal{E})$$

of [LRZ24] described above.

In other words, Theorem E(3) holds.

Remark 3.26. Suppose Y is some fixed bounded qcqs quasi-syntomic p-adic formal scheme flat over $\mathbb{Z}_p$. Étale realisation as defined in [Bha22] is a symmetric monoidal exact functor

$$T_{\text{ét}} : \mathrm{Perf}(Y^{\mathrm{Syn}}) \to \mathcal{D}^b_{\text{lisse}}(Y_\eta, \mathbb{Z}_p).$$

However, for a map $f : X \to Y$ as in Theorem 3.25, the F-gauge $f_*^{\mathrm{syn}} f^{\mathrm{syn},*}\mathcal{E}$ need not lie in $\mathrm{Perf}(Y^{\mathrm{Syn}})$. We thus need to extend the domain of $T_{\text{ét}}$.

We thus define an exact functor

$$T_{\text{ét}} : \mathcal{D}_{\text{geom}}(Y^{\mathrm{Syn}}) \to \mathcal{D}(Y_{\eta,\text{proét}}, \mathbb{Z}_p)$$

where

- the source category is the thick subcategory of $\mathcal{D}_{\text{qcoh}}(Y^{\mathrm{Syn}})$ generated by $\mathrm{Perf}(Y^{\mathrm{Syn}})$ and sheaves of the form $f_*^{\mathrm{syn}} f^{\mathrm{syn},*}\mathcal{E}$ as f runs over all possible quasi-syntomic maps $f : X \to Y$ from X a bounded qcqs quasi-syntomic p-adic formal scheme flat over $\mathbb{Z}_p$ and $\mathcal{E} \in \mathrm{Perf}(Y^{\mathrm{Syn}})$, and
- the target category is the category of $\mathbb{Z}_p$-sheaves on the pro-étale site of Y_η.

One defines, for any $\mathcal{E} \in \mathcal{D}_{\text{geom}}(Y^{\mathrm{Syn}})$, the underlying pro-étale sheaf of $T_{\text{ét}}(\mathcal{E})$ on Y_η to be given on a perfectoid test object $(R[\frac{1}{p}], R)$ (where $\mathrm{Spf}\, R$ maps to Y) by

$$\left(\mathcal{E}(A_{\inf}(R))\left[\frac{1}{d}\right]_p^\wedge\right)^{\varphi=\mathrm{id}}$$

where d generates the kernel of $\theta : A_{\inf}(R) \to R$ and $\mathcal{E}(A_{\inf}(R))$ denotes the pull-back of $\mathcal{E}$ along the syntomification of the map $\mathrm{Spf}\, R \to Y$. This is compatible with the usual étale realisation $\mathrm{Perf}(Y^{\mathrm{Syn}}) \to \mathcal{D}^b_{\text{lisse}}(Y_\eta, \mathbb{Z}_p)$ by construction.

Essentially by definition and [GR24, Theorem 6.1], if $\mathcal{E} = f_*^{\mathrm{syn}}\mathcal{E}'$ for $\mathcal{E}' \in \mathrm{Perf}(X^{\mathrm{Syn}})$ (where $f : X \to Y$ is a quasi-syntomic map), then

$$T_{\text{ét}}(\mathcal{E}) \simeq f_{\eta,*}T_{\text{ét}}(\mathcal{E}')$$

as pro-étale sheaves on Y_η.

Remark 3.27. By following the proof of [GR24, Lemma 7.15], and using Proposition 3.21 and the proof of the étale comparison, one checks rather easily that Theorem E(4) is true.

We now go on to prove Theorem 3.25, for which we record some lemmas.

First, recall that for an adic space X, a geometric point $\overline{x}$ of X is a map $\overline{x} : \mathrm{Spa}(C, C^+) \to X$ where C is a complete algebraically closed nonarchimedean field over $\mathbb{Q}_p$ and C^+ is an open bounded valuation subring of C. For $\mathcal{F}$ a sheaf of abelian groups on the étale site of an adic space X and for $\overline{x} : \mathrm{Spa}(C, C^+) \to X$ a geometric point of X, the stalk $\mathcal{F}_{\overline{x}}$ of $\mathcal{F}$ at $\overline{x}$ is defined to be

$$\mathcal{F}_{\overline{x}} := \mathrm{R}\Gamma(\mathrm{Spa}(C, C^+), \overline{x}^*\mathcal{F}) \in \mathrm{Ab}.$$

The following lemma is [LRZ24, Lemma 2.5.1].

Lemma 3.28. *For $f : X \to Y$ as above, $\overline{y}$ a geometric point of Y_η, and $\mathcal{F}$ a sheaf of abelian groups on $Y_{\text{ét}}$. Then there is a natural isomorphism*

$$\bigoplus_{\overline{x} \in f_\eta^{-1}(\overline{y})} \mathcal{F}_{\overline{y}} \xrightarrow{\simeq} (f_{\eta,*}f_\eta^*\mathcal{F})_{\overline{y}}.$$

Corollary 3.29. *For $f : X \to Y$ as above and $\overline{y}$ a geometric point of Y_η, there is a natural isomorphism*

$$\bigoplus_{\overline{x} \in f_\eta^{-1}(\overline{y})} \underline{\mathbb{Z}_p}_{\overline{y}} \xrightarrow{\simeq} (f_{\eta,*}\underline{\mathbb{Z}_p}_{X_\eta})_{\overline{y}}$$

of sheaves on the pro-étale site of $\overline{y}$.

Proof. Take $\mathcal{F} = \underline{\mathbb{Z}/p^n\mathbb{Z}}_{X_\eta}$ in the lemma and then take the limit over n. ☐

Lemma 3.30. *Suppose $f, g : \mathcal{L} \to \mathcal{L}'$ are two maps of pro-étale discrete (i.e. concentrated in cohomological degree 0) $\mathbb{Z}_p$-sheaves on an analytic adic space X. Then, $f = g$ if and only if for all geometric points $\overline{x}$ of X, we have $f_{\overline{x}} = g_{\overline{x}}$ as maps of stalks $\mathcal{L}_{\overline{x}} \to \mathcal{L}'_{\overline{x}}$.*

Proof. It suffices to prove this for the map $\mathcal{L}/p^n \to \mathcal{L}'/p^n$ of sheaves for all n; in particular, we may reduce to working on the étale site rather than the pro-étale site. In this case, [Hub13, Proposition 2.5.5] implies that the collection of functors $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ as $\overline{x}$ ranges over all geometric points of X is jointly conservative. As this functor is valued in an abelian category (we restricted ourselves to discrete sheaves), this implies that the stalk functors are jointly fully faithful, as required. ☐

The following lemma was used implicitly in the proof of [LRZ24, Theorem 2.5.6], and we will need it ourselves.

Lemma 3.31. *Let $\mathcal{X}$ be a connected adic space with a finite map $f : \mathcal{X} \to \mathrm{Spa}(C, C^+)$, where C^+ is an open integrally closed subring of an algebraically closed extension C of $\mathbb{Q}_p$. Then $\mathcal{X}_{\mathrm{red}} \simeq \mathrm{Spa}(C, C^+)$.*

Proof. Since $\mathrm{Spa}(C, C^+)$ is already reduced, we see that f factors through a finite map $\mathcal{X}_{\mathrm{red}} \to \mathrm{Spa}(C, C^+)$. [LRZ24, Lemma 2.1.3(i)] then implies that $\mathcal{X}_{\mathrm{red}} = \mathrm{Spa}(K, K^+)$ for K a finite extension of C and K^+ the integral closure of C^+ in K. Since C is algebraically closed, we thus have $K = C$ and $K^+ = C^+$ as required. ☐

We finally prove Theorem 3.25. The idea of the proof is based on the proof of uniqueness of the finite flat trace map in [LRZ24, Theorem 2.5.6].

Proof of Theorem 3.25. By the projection formula it suffices to prove this for $\mathcal{E} = \mathcal{O}_{Y^{\mathrm{Syn}}}$, whose étale realisation is the local system $\underline{\mathbb{Z}}_p$ on Y_η. We have two maps

$$f_{\eta,*}\underline{\mathbb{Z}_p}_{X_\eta} \to \underline{\mathbb{Z}_p}_{Y_\eta},$$

one of which comes from the étale trace and one which comes from the étale realization of the prismatic trace. By Lemma 3.30, we reduce to checking the case of a geometric point $Y = \mathrm{Spf}\, C^+$ (so $Y_\eta = \mathrm{Spa}(C, C^+)$) for C

a complete algebraically closed nonarchimedean field over $\mathbb{Q}_p$ and C^+ an open bounded valuation subring (see also the proof of uniqueness of [LRZ24, Theorem 2.5.6]). Recall from Section 2.2 that

$$\mathrm{R}\Gamma_{\text{ét}}(\mathrm{Spa}(R[\tfrac{1}{p}], R), \mathbb{Z}_p) \simeq \mathrm{gr}^0_{\mathrm{mot}} L_{K(1)}\mathrm{TC}(R)$$

for any p-complete ring R, and that this is compatible with the prismatic-étale realisation. It thus suffices to check that the étale trace

$$\mathrm{R}\Gamma_{\text{proét}}(X_\eta, \mathbb{Z}_p) \to \mathbb{Z}_p$$

coincides with the abstract transfer map

$$\mathrm{tr}'_{\text{ét}}(X_\eta) := \mathrm{gr}^0_{\mathrm{mot}} \mathrm{tr}_f^{L_{K(1)}\mathrm{TC}} : \mathrm{R}\Gamma_{\text{proét}}(X_\eta, \mathbb{Z}_p) \simeq \mathrm{gr}^0_{\mathrm{mot}} L_{K(1)}\mathrm{TC}(X) \to \mathrm{gr}^0_{\mathrm{mot}} L_{K(1)}\mathrm{TC}(Y) \simeq \mathbb{Z}_p.$$

Since the motivic filtration on $L_{K(1)}\mathrm{TC}(X) \simeq L_{K(1)}K(X_\eta)$ is actually an invariant of X_η (since it identifies with the Thomason filtration), this abstract transfer map is actually functorial in the analytic adic space X_η. In particular, it follows that for $X_\eta = \bigsqcup \mathcal{X}_i$ a connected component decomposition of the analytic adic space X_η (which need not come from a decomposition of the formal scheme X) we have $\mathrm{tr}'_{\text{ét}}(X_\eta) = \sum_i \mathrm{tr}'_{\text{ét}}(\mathcal{X}_i)$ as maps

$$\mathrm{R}\Gamma_{\text{ét}}(X_\eta, \mathbb{Z}_p) \simeq \bigoplus_i \mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_i, \mathbb{Z}_p) \to \mathbb{Z}_p.$$

Of course, this is also true of the étale trace maps of [LRZ24, Section 2.5].

We have thus reduced to the case that $\mathcal{X} = \mathrm{Spa}(S, S^+)$ is a connected finite flat analytic adic space over $Y_\eta = \mathrm{Spa}(C, C^+)$, at the cost of losing that the map is the generic fibre of a finite flat map of formal schemes. Lemma 3.31 says that $\mathcal{X}_{\mathrm{red}} = \mathrm{Spa}(C, C^+)$. The topological invariance of the étale site then implies that

$$\mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}, \mathbb{Z}_p) \simeq \mathrm{R}\Gamma_{\text{ét}}(\mathcal{X}_{\mathrm{red}}, \mathbb{Z}_p) \simeq \mathrm{R}\Gamma_{\text{ét}}(\mathrm{Spa}(C, C^+), \mathbb{Z}_p) \xleftarrow{\simeq} \mathbb{Z}_p,$$

where the last arrow is the canonical map $\mathbb{Z}_p \to \mathrm{R}\Gamma_{\text{ét}}(-, \mathbb{Z}_p)$. In particular, the étale trace map of [LRZ24, Theorem 2.5.6] identifies with the map

$$\mathbb{Z}_p \simeq \mathrm{R}\Gamma_{\text{proét}}(\mathcal{X}, \mathbb{Z}_p) \to \mathbb{Z}_p$$

given by multiplication by $d := \mathrm{rank}_C \mathcal{O}(\mathcal{X})$. This is of course also true of the trace map $\mathrm{tr}'_{\text{ét}}(\mathcal{X})$ coming from applying $\mathrm{gr}^0_{\mathrm{mot}}$ to the transfer map $L_{K(1)}K(\mathcal{X}) \to L_{K(1)}K(Y_\eta)$. $\square$

4 Prismatic Trace for the Cyclotomic Tower

In this section we finally prove all of our main theorems except the already established Theorem E.

Throughout this section, we will use the notation $q_n := q^{p^{-n}}$ as well as the notation introduced in Section 2.4.2 without comment. Before we can go on to actually prove Theorem E, we make a few remarks since Theorem 3.12 is not strong enough for our purposes.

Fix $f : X \to Y$ a finite flat regular map. In Theorem 3.12, we have shown that the transfer map $\mathrm{THH}(X) \to \mathrm{THH}(Y)$ upgrades canonically to a map

$$\mathrm{tr}_f^{\mathrm{THH}} : \mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(X) \to \mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(Y)$$

of filtered spectra. However, we will crucially need that $\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}$ takes values in cyclotomic synthetic spectra, and *a priori* Theorem 3.12 does not show construct $\mathrm{tr}_f^{\mathrm{THH}}$ as a map of cyclotomic synthetic spectra. In the following remark, let us explain how the construction of Theorem 3.12 automatically upgrades to a map of cyclotomic synthetic spectra.

Remark 4.1. Recall that a synthetic spectrum is a $\mathbb{S}_{\mathrm{ev}}$-module in $\mathrm{Fil}\,\mathrm{Sp}$, and that a synthetic spectrum with synthetic circle action is a $\mathbb{T}_{\mathrm{ev}}$-module in synthetic spectra. In particular, $\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(Y)$ being an E_∞-ring object in synthetic spectra with synthetic circle action, it is in fact an E_∞-algebra over $\mathbb{T}_{\mathrm{ev}}$ in $\mathrm{Fil}\,\mathrm{Sp}$. By $\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(Y)$-linearity of the filtered transfer map $\mathrm{tr}_f^{\mathrm{THH}}$, the map $\mathrm{tr}_f^{\mathrm{THH}}$ is a $\mathbb{T}_{\mathrm{ev}}$-module map in filtered spectra. Thus $\mathrm{tr}_f^{\mathrm{THH}}$ naturally upgrades to a map of synthetic spectra with synthetic circle action. The same reasoning applies to HH, so that the filtered transfer map

$$\mathrm{tr}_f^{\mathrm{HH}} : \mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{HH}(X) \to \mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{HH}(Y),$$

a priori only a map of filtered spectra, is naturally a map of synthetic spectra with synthetic circle action.

For the cyclotomic Frobenius, note that for any qcqs quasi-syntomic p-adic formal scheme X, one has

$$(\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(X))^{tC_{p,\mathrm{ev}}} \simeq \mathrm{R}\Gamma\left(\mathrm{QSyn}_X, \tau_{\geq 2*}\left(\mathrm{THH}(R)^{tC_p}\right)\right)$$

as a consequence of quasi-syntomic descent to QRSPs and [AR24, Lemma 3.18]. In particular, we can define

$$\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(X)^{tC_p} := (\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(X))^{tC_{p,\mathrm{ev}}},$$

and this promotes $\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}^{tC_p}$ to a sheaf valued in synthetic spectra with synthetic circle action. One checks rather easily that $\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}^{tC_p}$ now satisfies Assumption 3.1 as well as Condition 3.3. In particular, Theorem 3.12 and the previous argument yields a transfer map

$$\mathrm{tr}_f^{\mathrm{THH}^{tC_p}} : \mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(X)^{tC_p} \to \mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(Y)^{tC_p}$$

of synthetic spectra with synthetic circle action natural in f. Moreover, since the synthetic cyclotomic Frobenius is just the evenly filtered cyclotomic Frobenius $\mathrm{THH} \to \mathrm{THH}^{tC_p}$ (at least on affines), the functoriality-in-$\mathcal{F}$ of Theorem 3.12 implies the existence of a commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(X) \arrow[r, "\varphi_X"] \arrow[d, "\mathrm{tr}_f^{\mathrm{THH}}"] & \mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(X)^{tC_p} \arrow[r, phantom, "\simeq"] \arrow[d, "\mathrm{tr}_f^{\mathrm{THH}^{tC_p}}"] & (\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(X))^{tC_{p,\mathrm{ev}}} \arrow[d, "(\mathrm{tr}_f^{\mathrm{THH}})^{tC_{p,\mathrm{ev}}}"] \\
\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(Y) \arrow[r, "\varphi_X"] & \mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(Y)^{tC_p} \arrow[r, phantom, "\simeq"] & (\mathrm{Fil}^\bullet_{\mathrm{mot}}\mathrm{THH}(Y))^{tC_{p,\mathrm{ev}}} .
\end{tikzcd}$$

Here, the identification $\mathrm{tr}_f^{\mathrm{THH}^{tC_p}} = (\mathrm{tr}_f^{\mathrm{THH}})^{tC_{p,\mathrm{ev}}}$ holds by the assertion on functoriality in the filtered sheaf of Theorem 3.12. Hence, $\mathrm{tr}_f^{\mathrm{THH}}$ upgrades to a map of cyclotomic synthetic spectra.

Recall next that Theorem 3.12 does not give a canonical homotopy $\mathrm{tr}^{\mathcal{F}}_{g\circ f} \simeq \mathrm{tr}^{\mathcal{F}}_g \circ \mathrm{tr}^{\mathcal{F}}_f$ where $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two composable finite locally free regular maps.

Remark 4.2. For the rest of this paper we will be dealing with the transfer maps in a tower

$$\cdots \to X_3 \to X_2 \to X_1.$$

We will not have need to deal with the filtered transfer map along $X_{n+2} \to X_n$ directly. Rather, we will only remember the filtered transfer maps $\mathrm{tr}^{\mathcal{F}}_n : \mathrm{Fil}^\bullet\mathcal{F}(X_{n+1}) \to \mathrm{Fil}^\bullet\mathcal{F}(X_n)$ coming from Theorem 3.12, and for $\mathrm{Fil}^\bullet\mathcal{F}(X_{n+k}) \to \mathrm{Fil}^\bullet\mathcal{F}(X_n)$ we will replace $\mathrm{tr}^{\mathcal{F}}_{X_{n+k}\to X_n}$ with the composition $\mathrm{tr}^{\mathcal{F}}_{n+k-1} \circ \cdots \circ \mathrm{tr}^{\mathcal{F}}_n$. This is fine since we only care about computing the limit. Since the above k-fold composition is (non-canonically) homotopic to the true transfer map, the limit will not change up to homotopy.

Notice however that Theorem 3.12 does guarantee that the filtered transfer $\mathrm{tr}^{\mathcal{F}}$ is functorial in pullbacks, so we only actually need to make this choice once and for all for the tower

$$\cdots \to \mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^{n+1}}] \to \mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^n}] \to \cdots .$$

4.1 Preliminary Calculations

In the following, we will need the following standard computation of the homotopy groups of (various S^1-fixed points of) $\mathrm{ku}_{p,n}$.

Lemma 4.3. *For $n \geq 0$, we have*

$$\begin{aligned}
\pi_*\mathrm{ku}_{p,n} &\cong (\mathbb{Z}_p[\![q_n - 1]\!]/(q-1))\,[\beta], \\
\pi_*\mathrm{ku}_{p,n}^{hS^1} &\cong \mathbb{Z}_p[\![q_n - 1]\!][\beta, t]/(\beta t - (q-1)), \\
\pi_*\mathrm{ku}_{p,n}^{tS^1} &\cong \mathbb{Z}_p[\![q_n - 1]\!][t^{\pm 1}] \\
\pi_*\mathrm{ku}_{p,n}^{tC_p} &\cong \mathbb{Z}_p[\zeta_{p^{n+1}}][t^{\pm 1}] \\
\pi_*(\mathrm{ku}_{p,n}^{(-1)})^{hS^1} &\cong \mathbb{Z}_p[\![q_{n+1} - 1]\!][u, t]/(ut - [p]_{q_1}).
\end{aligned}$$

where $|\beta| = |u| = 2, |q| = 0$, and $|t| = -2$. Moreover, the following are true:

1. *The Adams operations corresponding to $g \in (1 + p^{n+1}\mathbb{Z}_p)$ acts on $\mathrm{ku}_{p,n}^{hS^1}$ via*

$$\beta \mapsto g\beta, \quad q \mapsto q^g, \quad \textit{and} \quad t \mapsto \frac{[g]_q}{g} t,$$

where $[g]_q := \frac{q^g - 1}{q-1}$.

2. *The ring map* $\mathrm{ku}_{p,n}^{hS^1} \to (\mathrm{ku}_{p,n}^{(-1)})^{hS^1}$ *is given on homotopy by the graded algebra map*

$$\mathbb{Z}_p[\![q_n - 1]\!][\beta, t]/(\beta t - (q-1)) \to \mathbb{Z}_p[\![q_{n+1} - 1]\!][u, t]/(ut - [p]_{q_1})$$

sending $q_n \mapsto q_{n+1}^p$*,* $t \mapsto t$*, and* $\beta \mapsto (q_1 - 1)u$*.*

3. *The Frobenius map* $\varphi^{hS^1} : (\mathrm{ku}_{p,n}^{(-1)})^{hS^1} \to (\mathrm{ku}_{p,n}^{(-1)})^{tS^1}$ *is given on homotopy by the graded algebra map*

$$\mathbb{Z}_p[\![q_{n+1} - 1]\!][u, t]/(ut - [p]_{q_1}) \to \mathbb{Z}_p[\![q_{n+1} - 1]\!][t^{\pm 1}]$$

sending $q_{n+1} \mapsto q_{n+1}^p$*,* $t \mapsto [p]_q t$*, and* $u \mapsto t^{-1}$*. In particular, the image of the Bott class is 'fixed' by Frobenius; more precisely,* $\varphi^{hS^1}(\beta) = \mathrm{can}(\beta)$*.*

Proof Sketch. A standard result says that $\mathrm{ku}_p^{hS^1} \simeq \mathbb{Z}_p[\![q-1]\!][\beta, t]$, and that

$$\mathrm{ku}_p^{hC_{p^n}} \simeq \mathbb{Z}_p[\![q-1]\!][\beta, t]/([p^n]_q t).$$

This yields the first isomorphism. The second isomorphism follows from evenness (which gives the description as a graded abelian group by degeneration of the homotopy fixed point spectral sequence) and the fact that $\mathrm{ku}_p^{hS^1} \to \mathrm{ku}_{p,n}^{hS^1}$ are ring maps which are injective on homotopy groups in order to identify the multiplicative structure. The third and fourth equivalences are immediate from the second by standard facts about complex orientations. The fifth isomorphism as well as statement (2) follow in the same way as the second isomorphism, though one needs to use the ring map $\mathrm{ku}_{p,n}^{hS^1} \to (\mathrm{ku}_{p,n}^{(-1)})^{hS^1}$ (an injection on homotopy groups) to identify the multiplicative structure. The claim on Adams operations is a standard fact about Adams operations on $\mathrm{ku}_p^{hS^1}$, while the second statement is clear from unravelling identifications.

The final claim comes about by taking S^1-fixed points of the cyclotomic Frobenius map $\varphi^{hS^1} : \mathrm{ku}_{p,n}^{(-1)} \to \mathrm{ku}_{p,n}^{(-1),tC_p}$. By construction of the cyclotomic Frobenius, we see that φ^{hS^1} factors as

$$\mathrm{ku}_{p,n}^{(-1),hS^1} \to (\mathrm{ku}_{p,n}^{tC_p})^{hS^1} \simeq \mathrm{ku}_{p,n}^{tS^1} \to \mathrm{ku}_{p,n}^{(-1),tS^1}.$$

The claim follows by computing the image of q_n and t under the map

$$\frac{\mathbb{Z}_p[\![q_{n+1} - 1]\!][u, t]}{ut - [p]_{q_1}} \cong \pi_* \mathrm{ku}_{p,n}^{(-1),hS^1} \to \pi_* (\mathrm{ku}_{p,n}^{tC_p})^{hS^1} \simeq \pi_* \mathrm{ku}_{p,n}^{tS^1} \cong \mathbb{Z}_p[\![q' - 1]\!][(t')^{\pm 1}],$$

using that q_{n+1} maps to q' and that the Euler class t on the source is sent to the p-fold iteration of the formal group law on t' under the identification $(\mathrm{ku}_{p,n}^{tC_p})^{hS^1} \simeq \mathrm{ku}_{p,n}^{tS^1}$. ☐

Corollary 4.4. *There is a fibre sequence*

$$\widetilde{j}_n \to \mathrm{ku}_{p,n} \xrightarrow{\psi_\circ^{1+p^{n+1}}} \tau_{\geq 2} \mathrm{ku}_{p,n}$$

where $\psi_\circ^{1+p^{n+1}}$ *is the unique map factorizing*

$$\psi^{1+p^{n+1}} - 1 : \mathrm{ku}_{p,n} \to \mathrm{ku}_{p,n}$$

through the connective cover map $\tau_{\geq 2} \mathrm{ku}_{p,n} \to \mathrm{ku}_{p,n}$*.*

Proof. For $\psi_\circ^{1+p^{n+1}}$ to exist, notice that the Adams operation $\psi^{1+p^{n+1}}$ acts trivially on $\pi_0(\mathrm{ku}_{p,n}) = \mathbb{Z}_p[\![q_n - 1]\!]/(q-1)$, since

$$\psi^{1+p^{n+1}}(q_n) = q^{(1+p^{n+1})/p^n} = q_n \cdot q^p \equiv q_n \pmod{q-1}.$$

As $1 + p^{n+1}\mathbb{Z}_p$ is topologically generated by $1 + p^{n+1}$, we see that we have the fibre sequence

$$\mathrm{ku}_{p,n}^{h(1+p^{n+1})\mathbb{Z}_p} \to \mathrm{ku}_{p,n} \xrightarrow{\psi^{1+p^{n+1}} - 1} \mathrm{ku}_{p,n}.$$

Taking connective covers and using the fibre sequence

$$\tau_{\geq 2} \mathrm{ku}_{p,n} \to \mathrm{ku}_{p,n} \to \pi_0 \mathrm{ku}_{p,n}$$

then yields the lemma. ☐

For the remaining corollaries, we introduce the following notation.

Notation. For any $\mathbb{Z}_p^\times$-equivariant module M over ku_p, write

$$M(1)[2] := \tau_{\geq 2}\mathrm{ku}_p \otimes_{\mathrm{ku}_p} M.$$

This notation is reasonable since, ignoring $\mathbb{Z}_p^\times$-actions, there is an equivalence $\tau_{\geq 2}\mathrm{ku}_p \simeq \mathrm{ku}_p[2]$ as a consequence of Bott periodicity. Note also that

$$\tau_{\geq 2}\mathrm{ku}_{p,n} \simeq \mathrm{ku}_{p,n}(1)[2].$$

Corollary 4.5. *There is an equivalence*

$$\tau_{\geq 2}((\tau_{\geq 2}\mathrm{ku}_{p,n})^{tC_p}) \simeq \mathrm{ku}_{p,n}^{(-1)}(1)[2].$$

The homotopy groups of this $\mathrm{ku}_{p,n}^{(-1)}$-module is the (graded) ideal

$$(\zeta_p - 1)u \cdot \mathbb{Z}_p[\zeta_{p^{n+1}}][u]$$

inside $\pi_*\mathrm{ku}_{p,n}^{(-1)} \simeq \mathbb{Z}_p[\![\zeta_{p^{n+1}}[u]$, where the inclusion of the ideal is induced by

$$\tau_{\geq 2}((\tau_{\geq 2}\mathrm{ku}_{p,n})^{tC_p}) \to \tau_{\geq 2}(\mathrm{ku}_{p,n}^{tC_p}) \to \mathrm{ku}_{p,n}^{(-1)}.$$

Similarly, the homotopy groups of the $\mathrm{ku}_{p,n}^{(-1)}$-module

$$(\tau_{\geq 2}((\tau_{\geq 2}\mathrm{ku}_{p,n})^{tC_p})^{hS^1} \simeq \mathrm{ku}_{p,n}^{(-1),hS^1}(1)[2]$$

is the (graded) ideal

$$(q_1 - 1)u \cdot \mathbb{Z}_p[\![q_{n+1} - 1]\!][u,t]/(ut - [p]_{q_1}).$$

Proof. The first equivalence follows from the fact that $(-)^{tC_p}$ is exact and so commutes with shifting. The rest follow from Lemma 4.3 since multiplication by the Bott-class induces the equivalence $\tau_{\geq 2}\mathrm{ku}_{p,n} \simeq \mathrm{ku}_{p,n}[2]$. $\square$

One proves the following corollary similarly to Corollary 4.4.

Corollary 4.6. *For any $g \in 1 + p^{n+1}\mathbb{Z}_p$ a topological generator, there is a fibre sequence*

$$\widetilde{j}_n^{(-1)} \to \mathrm{ku}_{p,n}^{(-1)} \xrightarrow{\psi^g_{\circ\circ}} \mathrm{ku}_{p,n}^{(-1)}(1)[2].$$

where $\psi^g_{\circ\circ}$ is induced by $\psi^g_\circ$.

In particular, using the equivalences of Section 2.4.2, we have a fibre sequence of cyclotomic spectra

$$\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \to \mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!]) \xrightarrow{\psi^g_{\circ\circ}} \mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!])(1)[2]$$

where now g is a topological generator of $1 + p^n\mathbb{Z}_p$.

Remark 4.7. There is a generalisation of this to arbitrary quasi-syntomic $\mathbb{Z}_p[\zeta_{p^n}]$-algebras, given in Proposition 4.26.

The following is a generalisation of [Dev25, Lemma 6.4.10] and may be proven similarly.

Lemma 4.8. *There are pushout squares of cyclotomic E_∞-ring spectra*

$$\begin{tikzcd}
\mathbb{S}_p[Bp^{-n}\mathbb{Z}_p]^{\mathrm{triv}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{S}_p[BC_{p^n}]^{\mathrm{triv}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widetilde{j}_{n-1}^{(-1)} \arrow[d] \\
\mathbb{S}_p^{\mathrm{triv}} \arrow[r] & \mathbb{S}_p[B^2\mathbb{Z}_p]^{\mathrm{triv}} \arrow[r] & \mathrm{ku}_{p,n-1}^{(-1)}
\end{tikzcd}$$

where:

- *the left square is induced by applying $\mathbb{S}_p[-]$ to the cofibre sequence*

$$p^{-n}\mathbb{Z}_p[1] \to C_{p^n}[1] \to \mathbb{Z}_p[2],$$

- *the map* $\mathbb{S}_p[BC_{p^n}]^{\mathrm{triv}} \to \widetilde{j}^{(-1)}_{n-1}$ *is the composition*
$$\mathbb{S}_p[BC_{p^n}]^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{S}_p[C_{p^n}]) \to \mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \simeq \widetilde{j}^{(-1)}_{n-1}$$
where the first map is the assembly map, and
- *the map* $\mathbb{S}_p[B^2\mathbb{Z}_p]^{\mathrm{triv}} \to \mathrm{ku}^{(-1)}_{p,n-1}$ *is adjoint to the map* $\mathbb{Z}_p \to \pi_2(\mathrm{ku}^{(-1),hS^1}_{p,n-1}) \cong \mathbb{Z}_p[\![q_n-1]\!]\cdot u$ *which sends* $1 \in \mathbb{Z}_p$ *to* $\beta = (q_1-1)u$.

The following result will be crucial in the proof of Theorem A. See Remark 4.10 for a concrete interpretation of these results in prismatic cohomology.

Proposition 4.9. *Fix* $n \geq 1$ *and* g *a topological generator of* $1+p^{n+1}\mathbb{Z}_p$. *For any bounded below* S^1*-equivariant* $\widetilde{j}^{(-1)}_0$*-module* M, *the following commuting square is a fibre square of spectra with* S^1*-action.*

$$\begin{array}{ccc} \left(M \otimes_{\widetilde{j}_0} \mathrm{ku}^{(-1)}_{p,n-1}\right)^{tC_p} & \xrightarrow{\psi^g_{\circ\circ}} & \left(M \otimes_{\widetilde{j}_0} \mathrm{ku}^{(-1)}_{p,n-1}\right)^{tC_p}(1)[2] \\ \downarrow & & \downarrow \\ \left(M \otimes_{\widetilde{j}_0} \mathrm{ku}^{(-1)}_{p,n}\right)^{tC_p} & \xrightarrow{\psi^g_{\circ\circ}} & \left(M \otimes_{\widetilde{j}_0} \mathrm{ku}^{(-1)}_{p,n}\right)^{tC_p}(1)[2]. \end{array}$$

Proof. Let F be the total cofibre of the following commutative square of $\widetilde{j}^{(-1)}_0$-modules in cyclotomic spectra

$$\begin{array}{ccc} \mathrm{ku}^{(-1)}_{p,n-1} & \xrightarrow{\psi^g_{\circ\circ}} & \mathrm{ku}^{(-1)}_{p,n-1}(1)[2] \\ \downarrow & & \downarrow \\ \mathrm{ku}^{(-1)}_{p,n} & \xrightarrow{\psi^g_{\circ\circ}} & \mathrm{ku}^{(-1)}_{p,n}(1)[2]. \end{array} \tag{6}$$

We claim that $F/(p,\beta)$ is S^1-nilpotent in the sense of [DR25, Definition 3.1.2], i.e. it is contained in the thick-tensor ideal of $\mathrm{Mod}_{\mathbb{S}_p^{\mathrm{triv}}}(\mathrm{Sp}^{BS^1})$ generated by $\mathbb{S}_p[S^1]$.

Suppose we know the S^1-nilpotence of $F/(p,\beta)$. Then $X := M \otimes_{\widetilde{j}^{(-1)}_0} F$ is also such that $X/(p,\beta)$ is S^1-nilpotent, which by Lemma 4.11 below implies that
$$(X/(p,\beta))^{tC_p} \simeq X^{tC_p}/(p,\beta) \simeq 0.$$
This implies that $(X^{tC_p})^\wedge_\beta \simeq 0$. Using the same argument as in [DR25, Remark 3.1.6], noting that X is bounded below (since M is), the natural map $X^{tC_p} \to (X^{tC_p})^\wedge_\beta$ is an equivalence, so that $X^{tC_p} \simeq 0$. Since X^{tC_p} is the total cofibre of the square occuring in the statement of Proposition 4.9, we win.

It remains to establish the S^1-nilpotence of $F/(p,\beta)$. By [DR25, Lemma 3.4.2] we see that $F/(p,\beta)$ is an S^1-equivariant $\mathbb{F}_p$-module. In particular, the criterion of [DR25, Proposition 3.1.5] applies, i.e. it suffices to check that t acts nilpotently on $(F/(p,\beta))^{hS^1} \simeq F^{hS^1}/(p,\beta)$. One computes the following:

$$\pi_*\left(\mathrm{ku}^{(-1),hS^1}_{p,n-1}/(p,\beta)\right) = \left(\bigoplus_{i\geq 1} \frac{\mathbb{F}_p[\![q_n-1]\!]}{(q_1-1)}\cdot u^i\right) \oplus \left(\bigoplus_{i\geq 0} \frac{\mathbb{F}_p[\![q_n-1]\!]}{q-1}\cdot t^i\right),$$
$$\pi_*\left(\mathrm{ku}^{(-1),hS^1}_{p,n-1}(1)[2]/(p,\beta)\right) = \left(\bigoplus_{i\geq 2} \frac{(q_1-1)\mathbb{F}_p[\![q_n-1]\!]}{(q_1-1)^2}\cdot u^i\right) \oplus \beta\left(\frac{\mathbb{F}_p[\![q_n-1]\!]}{(q-1)}\right) \oplus \left(\bigoplus_{i\geq 0} \frac{(q-1)\mathbb{F}_p[\![q_n-1]\!]}{(q-1)^2}\cdot t^i\right),$$

and similarly for n replaced by $n+1$. Moreover, the vertical maps are simply the obvious inclusion map.

Recall the identity $ut = [p]_{q_1}$ inside $\mathrm{ku}^{(-1),hS^1}_{p,n}$. Since $[p]_{q_1} = 0$ modulo (p, q_1-1), multiplication by t is zero on $\pi_*(-)$, $* \geq 4$, for every term in the mod (p,β)-reduction of (6). In particular, multiplication by t is zero for $\pi_* F^{hS^1}/(p,\beta)$ for $* \geq 4$. To then see that t acts nilpotently on all of $\pi_* F^{hS^1}/(p,\beta)$ (with uniform exponent), it suffices to show that $\pi_* F^{hS^1}/(p,\beta) = 0$ for $* \leq 1$.

To see this vanishing claim, since the fibres of the horizontal maps in (6) are up to homotopy independent of the choice of g (they are given by $\widetilde{j}^{(-1)}_{n-1}$ and $\widetilde{j}^{(-1)}_n$ respectively), we may choose $g = 1+p^{n+1}$. Using Lemma 4.3, the bottom horizontal map in (6) acts as
$$\psi^{1+p^{n+1}}_{\circ\circ}(q^i_{n+1}t^j) = q^i_{n+1}\left(\frac{q^i[1+p^n]^j_q - 1}{q-1}\right)\cdot(q-1)t^j$$

where the term $q_{n+1}^i\left(\frac{q^i[1+p^n]_q^j-1}{q-1}\right)$ is viewed as living in $\mathbb{F}_p[\![q_{n+1}-1]\!]/(q-1)$. If $p|i$, then this also gives a formula for the top horizontal map in (6). Now, we calculate

$$\frac{[1+p^n]_q-1}{q-1}=\frac{[1+p^n]_q-(1+p^n)}{q-1}=\sum_{k=0}^{p^n}\frac{q^k-1}{q-1}=\sum_{k=0}^{p^n}k=p^n\cdot\frac{p^n+1}{2}=0\in\frac{\mathbb{F}_p[\![q_{n+1}-1]\!]}{(q-1)},$$

where we use the fact that $\frac{q^k-1}{q-1}=[k]_q\equiv k \pmod{q-1}$. This implies

$$q_{n+1}^i\left(\frac{q^i[1+p^n]_q^j-1}{q-1}\right)=q_{n+1}^i[i]_q=iq_{n+1}^i$$

in $\mathbb{F}_p[\![q_{n+1}-1]\!]/(q-1)$, i.e. we have shown that

$$\psi_{\circ\circ}^{1+p^{n+1}}(q_{n+1}^it^j)=iq_{n+1}^i\cdot(q-1)t^j\in q_{n+1}^i\frac{(q-1)\mathbb{F}_p[\![q_n-1]\!]}{(q-1)^2}t^j.$$

Now, taking vertical cofibres in the mod (p,β) reduction of (6) yields the following exact sequence for all $j\geq 0$

$$0\to\pi_{-2j+1}(F^{hS^1}/(p,\beta))\to\bigoplus_{i=1}^{p-1}q_{n+1}^i\frac{\mathbb{F}_p[\![q_n-1]\!]}{q-1}t^j\xrightarrow{\psi_{\circ\circ}^{1+p^{n+1}}}\bigoplus_{i=1}^{p-1}q_{n+1}^i\frac{(q-1)\mathbb{F}_p[\![q_n-1]\!]}{(q-1)^2}t^j\to\pi_{-2j}(F^{hS^1}/(p,\beta))\to 0$$

where we have used the basis decomposition

$$\mathbb{F}_p[\![q_{n+1}-1]\!]\simeq\bigoplus_{i=0}^{p-1}q_{n+1}^i\mathbb{F}_p[\![q_n-1]\!].$$

It thus suffices to show that $\psi_{\circ\circ}^{1+p^{n+1}}$ induces an isomorphism of $\mathbb{F}_p$-modules

$$q_{n+1}^i\frac{\mathbb{F}_p[\![q_n-1]\!]}{q-1}t^j\xrightarrow[\psi_{\circ\circ}^{1+p^{n+1}}]{\simeq}q_{n+1}^i\frac{(q-1)\mathbb{F}_p[\![q_n-1]\!]}{(q-1)^2}t^j$$

for $1\leq i\leq p-1$. The preceding calculation shows that this is simply the map $q_{n+1}^it^j\cdot q_n^h\mapsto q_{n+1}^it^j\cdot(i+ph)q_n^h$, which is clearly invertible since $i+ph$ is a unit as long as $1\leq i\leq p-1$. ☐

Remark 4.10. The preceding proposition has the following concrete consequence in prismatic cohomology.

As a consequence of the results of Section 2.4.2 and Corollary 4.6, we have the following explicit description of prismatic cohomology of $\mathbb{Z}_p[\zeta_{p^n}]$:

$$\mathbb{\Delta}_{\mathbb{Z}_p[\zeta_{p^n}]}\{r\}\simeq\left[\mathbb{Z}_p[\![q_n-1]\!]\{r\}\xrightarrow{\frac{g-1}{q-1}}\mathbb{Z}_p[\![q_n-1]\!]\{r\}\right].$$

Here, g is a topological generator of $1+p^n\mathbb{Z}_p$. However, [Liu25, Theorem 1.0.2] says that

$$\mathbb{\Delta}_{\mathbb{Z}_p[\zeta_{p^n}]}\{r\}\simeq\left[\mathbb{Z}_p[\![q_{n-1}-1]\!]\{r\}\xrightarrow{\frac{g-1}{q-1}}\mathbb{Z}_p[\![q_{n-1}-1]\!]\{r\}\right].$$

The proposition above, in the special case $M=\widetilde{j}_0^{(-1)}$, is simply saying that the two complexes appearing on the right are quasi-isomorphic, as one expects. One can actually check that these are quasi-isomorphic by a direct computation as well (though this direct calculation will essentially boil down to the calculation carried out in the proof of Proposition 4.9).

For arbitrary M, the same remark holds, except one replaces the Breuil-Kisin twists $\mathcal{O}_{\mathbb{Z}_p[\zeta_{p^n}]^{\mathbb{\Delta}}}\{r\}$ with $\pi_*\mathcal{O}_{X_n^{\mathbb{\Delta}}}\{r\}$, where X is a p-adic formal scheme over $\mathbb{Z}_p[\zeta_p]$, $X_n:=X\times_{\operatorname{Spf}\mathbb{Z}_p[\zeta_p]}\operatorname{Spf}\mathbb{Z}_p[\zeta_{p^n}]$, and $\pi: X_n^{\mathrm{Syn}}\to\mathbb{Z}_p[\zeta_{p^n}]^{\mathrm{Syn}}$ is the structure map of syntomifications.

The following lemma was used in the proof of the above proposition.

Lemma 4.11. *Suppose X is an S^1-equivariant spectrum that is S^1-nilpotent in the sense of [DR25, Definition 3.1.2], i.e. it is in the thick tensor ideal of $\operatorname{Mod}_{\mathbb{S}_p^{\mathrm{triv}}}(\mathrm{Sp}^{BS^1})$ generated by $\mathbb{S}_p[S^1]$. Then $X^{tC_p}\simeq 0$.*

Proof. (This proof is due in part to Maxime Ramzi.) By the usual thick subcategory argument, it suffices to show that $(-)^{tC_p}$ kills $X \otimes \mathbb{S}_p[S^1]$ where X is any spectrum with S^1-action. Since $(-)^{tC_p}$ kills the thick tensor ideal generated by spectra with induced C_p-action [NS18, Lemma I.3.8], it suffices to show that $\mathbb{S}_p[S^1]$ is in the thick subcategory generated by $\mathbb{S}_p[C_p]$. To see this, consider the usual C_p-equivariant cell structure on S^1 given by one C_p-orbit of 0-simplices and one C_p-orbit of 1-simplices. This expresses S^1 as the coequalizer in spaces with C_p-action of the two maps $\mathrm{id} : C_p \to C_p$ and $\cdot\sigma : C_p \to C_p$ (where σ is a generator of C_p). Applying $\mathbb{S}_p[-]$ then yields the C_p-equivariant cofibre sequence

$$\mathbb{S}_p[C_p] \xrightarrow{\mathrm{id}-\sigma} \mathbb{S}_p[C_p] \to \mathbb{S}_p[S^1]$$

as required. $\square$

We next study the Frobenius on the cyclotomic spectrum $\mathrm{ku}_{p,n}^{(-1)}$.

Lemma 4.12. *There is a canonical S^1-equivariant factorization of the cyclotomic Frobenius*

$$\varphi : \mathrm{ku}_{p,n}^{(-1)} \to \mathrm{ku}_{p,n}^{(-1),tC_p}$$

through the canonical map $\mathrm{ku}_{p,n}^{(-1)} \to \mathrm{ku}_{p,n+1}^{(-1)}$.

Proof. Unwinding definitions, the cyclotomic Frobenius of $\mathrm{ku}_{p,n}^{(-1)}$ is the composition

$$\mathrm{ku}_{p,n}^{(-1)} \to \mathrm{ku}_{p,n}^{tC_p} \to \left(\tau_{\geq 0}\left((\mathrm{ku}_p^{hC_{p^n}})^{hC_p}\right)\right)^{tC_p} \simeq \left(\tau_{\geq 0}\mathrm{ku}_{p,n}^{hC_p}\right)^{tC_p} \to \left(\mathrm{ku}_{p,n}^{(-1)}\right)^{tC_p}.$$

However, we have $\tau_{\geq 0}(\mathrm{ku}_p^{hC_{p^n}})^{hC_p} \simeq \tau_{\geq 0}\mathrm{ku}_p^{hC_{p^{n+1}}} \simeq \mathrm{ku}_{p,n+1}$. The composition of the first two arrows factors through the connective cover map $\mathrm{ku}_{p,n+1}^{(-1)} \to \mathrm{ku}_{p,n+1}^{tC_p}$, and the induced map $\mathrm{ku}_{p,n}^{(-1)} \to \mathrm{ku}_{p,n+1}^{(-1)}$ is the canonical map. The claim follows. $\square$

> **Notation.** Let $\widetilde{\varphi}_n : \mathrm{ku}_{p,n}^{(-1)} \to \mathrm{ku}_{p,n-1}^{(-1),tC_p}$ be the S^1-equivariant map induced by Lemma 4.12, so that $\varphi : \mathrm{ku}_{p,n-1}^{(-1)} \to \mathrm{ku}_{p,n-1}^{(-1),tC_p}$ factors as the composition of $\mathrm{ku}_{p,n-1}^{(-1)} \to \mathrm{ku}_{p,n}^{(-1)}$ and $\widetilde{\varphi}_n$.

Varying n, the cyclotomic spectra $\mathrm{ku}_{p,n}^{(-1)}$ are related to each other by the following.

Lemma 4.13. *There are $\mathbb{Z}_p^\times$-equivariant cyclotomic E_∞-ring maps $\mathbb{S}_p[\![p^{-n-1}\mathbb{Z}_p]\!]^{\mathrm{Tate}} \to \mathrm{ku}_{p,n}^{(-1)}$ for $n \geq 0$ inducing equivalences of cyclotomic spectra*

$$\mathrm{ku}_{p,n}^{(-1)} \simeq \mathrm{ku}_{p,n-1}^{(-1)} \otimes_{\mathbb{S}_p[\![p^{-n}\mathbb{Z}_p]\!]^{\mathrm{Tate}}} \mathbb{S}_p[\![p^{-n-1}\mathbb{Z}_p]\!]^{\mathrm{Tate}}.$$

Proof. This is obvious from the equivalence

$$\mathrm{ku}_{p,n-1}^{(-1)} \simeq \mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!]) \simeq \mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \otimes_{\mathrm{THH}(\mathbb{S}_p[\![p^{-n}\mathbb{Z}_p]\!])} \mathbb{S}_p[\![p^{-n}\mathbb{Z}_p]\!]^{\mathrm{Tate}}$$

of Corollary 2.32. $\square$

> **Remark 4.14.** On underlying spectra with S^1-action, the map $\mathbb{S}_p[\![p^{-n-1}\mathbb{Z}_p]\!]^{\mathrm{triv}} \to \mathrm{ku}_{p,n}^{(-1)}$ is adjoint to a map of ring spectra $\mathbb{S}_p[\![p^{-n-1}\mathbb{Z}_p]\!] \to \mathrm{ku}_{p,n}^{(-1),hS^1}$, and the data of this latter map is equivalent to the data of a map
>
> $$p^{-n-1}\mathbb{Z}_p \to \pi_0(\mathrm{ku}_{p,n}^{(-1),hS^1}) = \mathbb{Z}_p[\![q^{1/p^{n+1}} - 1]\!].$$
>
> This sends the generator p^{-n-1} of $p^{-n-1}\mathbb{Z}_p$ to q_{n+1}.

> **Construction 4.15.** We construct an $\mathbb{Z}_p^\times \times S^1$-equivariant $\mathrm{ku}_{p,n-1}^{(-1)}$-linear map
>
> $$\rho_n : \mathrm{ku}_{p,n}^{(-1)} \to \mathrm{ku}_{p,n-1}^{(-1)}$$
>
> as follows. Consider the following composition of S^1-equivariant $K(1)$-local spectra
>
> $$\mathrm{KU}_p^{hC_p} \xleftarrow[\mathrm{Nm}_{C_p}]{\simeq} \mathrm{KU}_{p,hC_p} \to \mathrm{KU}_p$$
>
> where KU_p has the trivial circle action, the first equivalence is due to semi-additivity of $K(1)$-local spectra (cf. [CSY22]), and the second map exists due to the trivial circle action on KU_p. This composition is

KU_p-linear. Taking $\tau_{\geq 0}(-)^{hC_{p^{n-1}}}$ then yields an S^1-equivariant $\mathrm{ku}_{p,n-1}$-linear map

$$\mathrm{ku}_{p,n} \to \mathrm{ku}_{p,n-1}.$$

Applying $\tau_{\geq 0}(-)^{tC_p}$ then yields a $\mathrm{ku}_{p,n-1}^{(-1)}$-linear map

$$\mathrm{ku}_{p,n}^{(-1)} \to \mathrm{ku}_{p,n-1}^{(-1)}.$$

Lemma 4.16. *The map ρ_n satisfies the following properties.*

1. *There is a canonical commuting diagram*

$$\begin{tikzcd}
\mathrm{ku}_{p,n}^{(-1)} \arrow[r, "\widetilde{\varphi}_n"] \arrow[d, "\rho_n"] & \mathrm{ku}_{p,n-1}^{(-1),tC_p} \arrow[d, "\rho_{n-1}^{tC_p}"] \\
\mathrm{ku}_{p,n-1}^{(-1)} \arrow[r, "\widetilde{\varphi}_n"] & \mathrm{ku}_{p,n-2}^{(-1),tC_p}.
\end{tikzcd}$$

2. *The map ρ_n canonically identifies with the following composition*

$$\mathrm{ku}_{p,n}^{(-1)} \simeq \mathrm{ku}_{p,n-1}^{(-1)} \otimes_{\mathbb{S}_p[\![q_n-1]\!]} \mathbb{S}_p[\![q_{n+1}-1]\!] \xrightarrow{\mathrm{id}\otimes\mathrm{proj}} \mathrm{ku}_{p,n-1}^{(-1)},$$

where the first equivalence is Lemma 4.13 and $\mathrm{proj} : \mathbb{S}_p[\![q_{n+1}-1]\!] \to \mathbb{S}_p[\![q_n-1]\!]$ *is the $\mathbb{S}_p[\![q_n-1]\!]$-linear map given by $q_{n+1}^i \mapsto 0$ if $p \nmid i$ and $q_{n+1}^{pi} \mapsto q_n^i$.*

Proof. Recall that $\widetilde{\varphi}_n$ is obtained by applying $(-)^{(-1)}$ to

$$\tau_{\geq 0}(\mathrm{ku}_p^{hC_{p^n}}) \simeq \tau_{\geq 0}\left(\tau_{\geq 0}(\mathrm{ku}_p^{hC_{p^{n-1}}})\right)^{hC_p} \xrightarrow{\mathrm{can}} \tau_{\geq 0}\left(\tau_{\geq 0}(\mathrm{ku}_p^{hC_{p^{n-1}}})\right)^{tC_p}$$

followed by the connective cover map $(\mathrm{ku}_{p,n-1}^{(-1)})^{(-1)} \to \mathrm{ku}_{p,n-1}^{(-1),tC_p}$. Since $\tau_{\geq 0}\mathrm{ku}_p^{hC_N} \simeq \tau_{\geq 0}\mathrm{KU}_p^{hC_N}$ for any $N \geq 1$, statement (1) follows by noticing that the following diagram (obtained by functoriality of Nm_{C_p} and $\mathrm{can} : (-)^{hC_p} \to (-)^{tC_p}$) commutes:

$$\begin{tikzcd}
\tau_{\geq 0}(\mathrm{KU}_p^{hC_p})^{hC_{p^{n-1}}} \arrow[r, phantom, "\simeq"] & \tau_{\geq 0}\left(\tau_{\geq 0}\mathrm{KU}_p^{hC_{p^{n-1}}}\right)^{hC_p} \arrow[r, "\mathrm{can}"] & \tau_{\geq 0}\left(\tau_{\geq 0}\mathrm{KU}_p^{hC_{p^{n-1}}}\right)^{tC_p} \\
\tau_{\geq 0}(\mathrm{KU}_{p,hC_p})^{hC_{p^{n-1}}} \arrow[u, "\mathrm{Nm}_{C_p}^{hC_{p^{n-1}}}", "\wr"'] \arrow[d] \arrow[r, phantom, "\simeq"] & \tau_{\geq 0}\left(\tau_{\geq 0}(\mathrm{KU}_{p,hC_p})^{hC_{p^{n-2}}}\right)^{hC_p} \arrow[u, "\mathrm{Nm}_{C_p}", "\wr"'] \arrow[d] \arrow[r, "\mathrm{can}"] & \tau_{\geq 0}\left(\tau_{\geq 0}(\mathrm{KU}_{p,hC_p})^{hC_{p^{n-2}}}\right)^{tC_p} \arrow[u, "\mathrm{Nm}_{C_p}^{hC_{p^{n-2}}}", "\wr"'] \arrow[d] \\
\tau_{\geq 0}\mathrm{KU}_p^{hC_{p^{n-1}}} \arrow[r, phantom, "\simeq"] & \tau_{\geq 0}\left(\tau_{\geq 0}\mathrm{KU}_p^{hC_{p^{n-2}}}\right)^{hC_p} \arrow[r, "\mathrm{can}"] & \tau_{\geq 0}\left(\tau_{\geq 0}\mathrm{KU}_p^{hC_{p^{n-2}}}\right)^{tC_p}
\end{tikzcd}$$

For the second statement, notice first that the composition

$$\mathrm{KU}_p \to \mathrm{KU}_p^{hC_p} \xleftarrow[\mathrm{Nm}_{C_p}]{\simeq} \mathrm{KU}_{p,hC_p} \to \mathrm{KU}_p$$

is the identity map since the $K(1)$-local cardinality of BC_p is 1 (see [CSY21, Proposition 2.2.5]). Applying $\left(\tau_{\geq 0}(-)^{hC_{p^{n-1}}}\right)^{(-1)}$ then shows that the map $\mathrm{ku}_{p,n-1}^{(-1)} \to \mathrm{ku}_{p,n}^{(-1)} \xrightarrow{\rho_n} \mathrm{ku}_{p,n-1}^{(-1)}$ is the identity map, where the first map is the canonical map $\mathrm{ku}_{p,n-1}^{(-1)} \to \mathrm{ku}_{p,n}^{(-1)}$. By $\mathrm{ku}_{p,n-1}^{(-1)}$-linearity of ρ_n and of $\mathrm{id}\otimes\mathrm{proj}$, it suffices to check that the classes $q_{n+1}^i \in \pi_0(\mathrm{ku}_{p,n}^{(-1),hS^1})$ are killed by ρ_n for $1 \leq i \leq p-1$. For this, it actually suffices to show that the composition

$$\mathrm{KU}_p^{hS^1} \simeq (\mathrm{KU}_p^{hC_p})^{hS^1} \xleftarrow[\mathrm{Nm}_{C_p}]{\simeq} (\mathrm{KU}_{p,hC_p})^{hS^1} \to \mathrm{KU}_p^{hS^1}$$

kills q^i for $1 \leq i \leq p-1$. By [Yua24, Example 2.16] and the fact that $q^i \in \pi_0(\mathrm{KU}_p^{hS^1})$ represents the i'th power of the tautological line bundle on $BS^1 \simeq \mathbb{CP}^\infty$, the claim that q^i dies under the above composition follows from the simple observation that the i'th power of the tautological line bundle is not fixed under the p-power map $\mathbb{CP}^\infty \to \mathbb{CP}^\infty$ whenever $p \nmid i$. $\square$

4.2 The Case of a Point

The main result of this subsection is Proposition 4.19. The key idea behind the proof is the following lemma, which allows us to use Schlichtkrull's description of the transfer maps in THH of group algebras (given in Section 2.4.1) to then describe the transfer maps for the cyclotomic extensions of $\mathbb{Z}_p$.

Lemma 4.17. *Suppose A is an E_∞-ring spectrum, and suppose B and C are E_∞-A-algebras such that B is dualizable over A (so that there is a* $\mathrm{THH}(A)$*-linear transfer map* $\mathrm{THH}(B) \to \mathrm{THH}(A)$*). Then, the transfer map* $\mathrm{THH}(C \otimes_A B) \to \mathrm{THH}(C)$ *associated to the dualisable E_∞-C-algebra $C \otimes_A B$ coincides with the map*

$$\mathrm{THH}(C \otimes_A B) \simeq \mathrm{THH}(C) \otimes_{\mathrm{THH}(A)} \mathrm{THH}(B) \to \mathrm{THH}(C)$$

induced by the transfer $\mathrm{THH}(B) \to \mathrm{THH}(A)$*.*

Proof. Notice that the following diagram canonically commutes

$$\begin{tikzcd} \mathrm{Perf}(B) \arrow[r] \arrow[d] & \mathrm{Perf}(A) \arrow[d] \\ \mathrm{Perf}(C \otimes_A B) \arrow[r] & \mathrm{Perf}(C), \end{tikzcd}$$

where both horizontal maps are given by restriction of scalars, and both vertical maps are given by base-change; this is the Beck-Chevalley condition for pull-backs of perfect modules. In more elementary terms, this is simply the statement that for any B-module M, we have an equivalence of C-modules

$$M \otimes_B (B \otimes_A C) \simeq M \otimes_A C.$$

Taking THH, we see that

$$\begin{tikzcd} \mathrm{THH}(B) \arrow[r] \arrow[d] & \mathrm{THH}(A) \arrow[d] \\ \mathrm{THH}(C \otimes_A B) \arrow[r] & \mathrm{THH}(C), \end{tikzcd}$$

where the horizontal maps are transfers and the vertical maps are the canonical ones. The lemma follows since THH is symmetric monoidal, and since the top horizontal map is $\mathrm{THH}(A)$-linear while the bottom horizontal map is $\mathrm{THH}(C)$-linear. ☐

Lemma 4.18. *For $n \geq 1$, there is an equivalence of cyclotomic spectra*

$$\widetilde{j}^{(-1)}_{n-1} \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}_p]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]^{\mathrm{triv}} \simeq \mathrm{fib}(\psi^{g^{p^n}}_{\circ\circ} : \mathrm{ku}^{(-1)}_{p,n-1} \to \mathrm{ku}^{(-1)}_{p,n-1}(1)[2]),$$

where g is a topological generator of $1 + p\mathbb{Z}_p$.

Moreover, the $\widetilde{j}^{(-1)}_{n-1}$-linear map

$$\widetilde{j}^{(-1)}_{n-1} \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}_p]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]^{\mathrm{triv}} \to \widetilde{j}^{(-1)}_{n-1}$$

induced by the stable transfer map $\mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p] \to \mathbb{S}_p[Bp^{-n}\mathbb{Z}_p]$ sits in the following commuting diagram of fibre sequences of cyclotomic spectra.

$$\begin{tikzcd} \widetilde{j}^{(-1)}_{n-1} \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}_p]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]^{\mathrm{triv}} \arrow[r] \arrow[d] & \mathrm{ku}^{(-1)}_{p,n-1} \arrow[r, "\psi^{g^{p^n}}_{\circ\circ}"] \arrow[d, "\sum_{i=0}^{p-1} \psi^{g^{ip^{n-1}}}"] & \mathrm{ku}^{(-1)}_{p,n-1}(1)[2] \arrow[d, equal] \\ \widetilde{j}^{(-1)}_{n-1} \arrow[r] & \mathrm{ku}^{(-1)}_{p,n-1} \arrow[r, "\psi^{g^{p^{n-1}}}_{\circ\circ}"'] & \mathrm{ku}^{(-1)}_{p,n-1}(1)[2] \end{tikzcd}$$

Proof. We need to show that

$$\widetilde{j}^{(-1)}_{n-1} \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}_p]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]^{\mathrm{triv}} \simeq \tau_{\geq 0} \left(\mathrm{ku}^{(-1)}_{p,n-1}\right)^{h(1+p^{n+1}\mathbb{Z}_p)}.$$

There is clearly a map

$$\widetilde{j}^{(-1)}_{n-1} \otimes_{\mathbb{S}_p[BC_{p^n}]^{\mathrm{triv}}} \mathbb{S}_p[BC_{p^{n+1}}]^{\mathrm{triv}} \simeq \widetilde{j}^{(-1)}_{n-1} \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}_p]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]^{\mathrm{triv}} \to \mathrm{ku}^{(-1)}_{p,n-1}.$$

Since $(1+p^{n+1}\mathbb{Z}_p)$ acts trivially on the left hand side, and since the left hand side is connective, the above composition factors as a map

$$\widetilde{j}_{n-1}^{(-1)} \otimes_{\mathbb{S}_p[BC_{p^n}]^{\mathrm{triv}}} \mathbb{S}_p[BC_{p^{n+1}}]^{\mathrm{triv}} \to \tau_{\geq 0}\left(\mathrm{ku}_{p,n-1}^{(-1)}\right)^{h(1+p^{n+1}\mathbb{Z}_p)}$$

of cyclotomic rings. It thus suffices to check that this map is an equivalence on underlying p-complete spectra. By p-completeness, it in turn suffices to check that the map induces an equivalence on mod p homotopy groups. One checks using Corollary 4.6 that there are isomorphisms of graded rings

$$\pi_*\widetilde{j}_{n-1}^{(-1)}/p \cong \mathbb{Z}_p[\zeta_{p^n}]/p[u] \otimes_{\mathbb{F}_p} \bigwedge\nolimits_{\mathbb{F}_p}(\epsilon) \quad \text{and} \quad \pi_*\left(\tau_{\geq 0}\left(\mathrm{ku}_{p,n-1}^{(-1)}\right)^{h(1+p^{n+1}\mathbb{Z}_p)}/p\right) \cong \mathbb{Z}_p[\zeta_{p^n}]/p[u] \otimes_{\mathbb{F}_p} \bigwedge\nolimits_{\mathbb{F}_p}(\epsilon')$$

where u is a class in degree 2, ϵ is a class in degree 1 coming from the fibre sequence of Corollary 4.6, and ϵ' is a class in degree 1 similarly defined. The map $(\mathbb{S}/p)[Bp^{-n}\mathbb{Z}_p] \to \widetilde{j}_{n-1}^{(-1)}/p$ picks out the class ϵ while $(\mathbb{S}/p)[Bp^{-n-1}\mathbb{Z}_p] \to \tau_{\geq 0}\left(\mathrm{ku}_{p,n-1}^{(-1)}\right)^{h(1+p^{n+1}\mathbb{Z}_p)}/p$ picks out the class ϵ'. The claim follows.

It remains to prove the existence of the commutative diagram. The right hand square clearly canonically commutes, and so we just need to show that there exists an S^1-equivariant commutative square

$$\begin{tikzcd}
\widetilde{j}_{n-1}^{(-1)} \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}_p]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]^{\mathrm{triv}} \arrow[r] \arrow[d] & \mathrm{ku}_{p,n-1}^{(-1)} \arrow[d, "\sum_{i=0}^{p-1}\psi^{g^{ip^{n-1}}}"] \\
\widetilde{j}_{n-1}^{(-1)} \arrow[r] & \mathrm{ku}_{p,n-1}^{(-1)}
\end{tikzcd} \tag{7}$$

of cyclotomic spectra. However, notice that we have a commuting diagram of spectra

$$\begin{tikzcd}
\mathbb{S}_p[BC_{p^{n+1}}] \arrow[r] \arrow[d, "\text{gp transfer}"'] & \mathbb{S}_p[B^2\mathbb{Z}_p] \arrow[d, "\sum_{i=0}^{p-1} g^{ip^{n-1}}"] \\
\mathbb{S}_p[BC_{p^n}] \arrow[r] & \mathbb{S}_p[B^2\mathbb{Z}_p]
\end{tikzcd}$$

where $\mathbb{Z}_p^\times$ acts on $B^2\mathbb{Z}_p$ via its multiplicative action on $\mathbb{Z}_p$. Applying $-\otimes_{\mathbb{S}_p[BC_{p^n}]}\widetilde{j}_{n-1}^{(-1)}$ and Lemma 4.8 yields the result; here, we also need to use that the map $\mathbb{S}_p[B^2\mathbb{Z}_p] \to \mathrm{ku}_p$ classifying the Bott element is $\mathbb{Z}_p^\times$-equivariant. $\square$

Proposition 4.19. *Let g be a topological generator of $(1+p\mathbb{Z}_p)$. The transfer map*

$$\mathrm{tr} : \mathrm{THH}(\mathbb{Z}_p[\zeta_{p^{n+1}}]) \to \mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}])$$

sits in the following canonical S^1-equivariant commutative diagram of $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \simeq \widetilde{j}_{n-1}^{(-1)}$-module spectra

$$\begin{tikzcd}
\widetilde{j}_n^{(-1)} \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{ku}_{p,n}^{(-1)} \arrow[r, "\psi_{\circ\circ}^{g^{p^n}}"] \arrow[dd, "\rho_n"] & \mathrm{ku}_{p,n}^{(-1)}(1)[2] \arrow[dd, "\rho_n(1)[2]"] \\
\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^{n+1}}]) \arrow[dd, "\mathrm{tr}"] & & \\
 & \mathrm{ku}_{p,n-1}^{(-1)} \arrow[r, "\psi_{\circ\circ}^{g^{p^n}}"] \arrow[dd, "\sum_{i=0}^{p-1}\psi^{g^{ip^{n-1}}}"] & \mathrm{ku}_{p,n-1}^{(-1)}(1)[2] \arrow[dd, equal, "\mathrm{id}"] \\
\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \arrow[d, phantom, "\wr"] & & \\
\widetilde{j}_{n-1}^{(-1)} \arrow[r] & \mathrm{ku}_{p,n-1}^{(-1)} \arrow[r, "\psi_{\circ\circ}^{g^{p^{n-1}}}"] & \mathrm{ku}_{p,n-1}^{(-1)}(1)[2]
\end{tikzcd}$$

where the top and bottom rows are fibre sequences and the maps ρ_n are from Construction 4.15.

Remark 4.20. This is NOT a diagram of cyclotomic spectra. The problem is that the map

$$\mathrm{proj} : \mathbb{S}_p[\![p^{-n}\mathbb{Z}_p]\!]^{\mathrm{triv}} \to \mathbb{S}_p[\![p^{1-n}\mathbb{Z}_p]\!]^{\mathrm{triv}},$$

while a map of S^1-equivariant rings, does not upgrade to a map of cyclotomic spectra

$$\mathrm{proj} : \mathbb{S}_p[\![p^{-n}\mathbb{Z}_p]\!]^{\mathrm{Tate}} \to \mathbb{S}_p[\![p^{1-n}\mathbb{Z}_p]\!]^{\mathrm{Tate}}.$$

The problem is that this map does not commute with Tate valued Frobenius, as can be checked rather easily by a computation in π_0.

Proof. By Lemma 4.17, the transfer map $\mathrm{tr} : \widetilde{j}_n^{(-1)} \to \widetilde{j}_{n-1}^{(-1)}$ of cyclotomic spectra is the base-change along $\mathrm{THH}(\mathbb{S}_p[p^{-n}\mathbb{Z}]) \to \widetilde{j}_{n-1}^{(-1)}$ of the free loop transfer map

$$\mathbb{S}_p[p^{-n-1}\mathbb{Z} \times Bp^{-n-1}\mathbb{Z}] \simeq \mathrm{THH}(\mathbb{S}_p[p^{-n-1}\mathbb{Z}]) \to \mathrm{THH}(\mathbb{S}_p[p^{-n}\mathbb{Z}]) \simeq \mathbb{S}_p[p^{-n}\mathbb{Z} \times Bp^{-n}\mathbb{Z}].$$

where the S^1-action on the suspension spectra in the source and target is the twisted one. By Proposition 2.27, the free loop transfer is the S^1-equivariant composition

$$\begin{aligned}\mathrm{THH}(\mathbb{S}_p[p^{-n-1}\mathbb{Z}]) \simeq & \bigoplus_{\gamma \in p^{-n-1}\mathbb{Z}/p^{-n}\mathbb{Z}} \mathrm{THH}(\mathbb{S}_p[p^{-n}\mathbb{Z}]) \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}]_{(\gamma)} \\ & \xrightarrow{\mathrm{proj}_{\gamma=0}} \mathrm{THH}(\mathbb{S}_p[p^{-n}\mathbb{Z}]) \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}]^{\mathrm{triv}} \xrightarrow{\mathrm{id}\otimes\mathrm{tr}_{st}} \mathrm{THH}(\mathbb{S}_p[p^{-n}\mathbb{Z}])\end{aligned}$$

where tr_{st} is the stable transfer. Tensoring up yields the composition

$$\begin{aligned}\widetilde{j}_n^{(-1)} \simeq & \bigoplus_{\gamma \in p^{-n-1}\mathbb{Z}/p^{-n}\mathbb{Z}} \widetilde{j}_{n-1}^{(-1)} \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}]_{(\gamma)} \\ & \xrightarrow{\mathrm{proj}_{\gamma=0}} \widetilde{j}_{n-1}^{(-1)} \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}]^{\mathrm{triv}} \xrightarrow{\mathrm{id}\otimes\mathrm{tr}_{st}} \widetilde{j}_{n-1}^{(-1)}.\end{aligned}$$

By Lemma 4.18, it remains to check the existence of a 2-simplex witnessing commutativity of the following diagram

$$\begin{tikzcd}
\widetilde{j}_n^{(-1)} \arrow[rr] \arrow[d, phantom, "\wr"] & & \mathrm{ku}_{p,n}^{(-1)} \arrow[d, phantom, "\wr"] \\
\widetilde{j}_{n-1}^{(-1)} \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}]^{\mathrm{triv}}} \bigoplus_\gamma \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}]_{(\gamma)} \arrow[d, "\mathrm{proj}_{\gamma=0}"] & & \mathrm{ku}_{p,n-1}^{(-1)} \otimes_{\mathbb{S}_p[p^{-n}\mathbb{Z}]} \mathbb{S}_p[p^{-n-1}\mathbb{Z}] \arrow[d, "\mathrm{proj}_{p^{-n-1}\mathbb{Z}/p^{-n}\mathbb{Z}}"] \\
\widetilde{j}_{n-1}^{(-1)} \otimes_{\mathbb{S}_p[Bp^{-n}\mathbb{Z}]^{\mathrm{triv}}} \mathbb{S}_p[Bp^{-n-1}\mathbb{Z}]^{\mathrm{triv}} \arrow[rr] & & \mathrm{ku}_{p,n-1}^{(-1)}
\end{tikzcd}$$

of S^1-equivariant $\widetilde{j}_{n-1}^{(-1)}$-module spectra compatible in n, where the horizontal maps are the usual ring maps. Here, we have used Lemma 4.16 (and the fact that $- \otimes_{\mathbb{S}_p[\![p^{-n}\mathbb{Z}_p]\!]} \mathbb{S}_p[\![p^{-n-1}\mathbb{Z}_p]\!] \simeq - \otimes_{\mathbb{S}_p[p^{-n}\mathbb{Z}]} \mathbb{S}_p[p^{-n-1}\mathbb{Z}]$) to identify ρ_n with $\mathrm{id} \otimes \mathrm{proj}$. That this commutative diagram exists is in fact clear by Remark 4.14 and by the fact that the map

$$\bigoplus_{\gamma \in p^{-n}\mathbb{Z}} \mathbb{S}_p[Bp^{-n}\mathbb{Z}]_{(\gamma)} \simeq \mathbb{S}_p[p^{-n}\mathbb{Z} \times Bp^{-n}\mathbb{Z}] \simeq \mathrm{THH}(\mathbb{S}_p[p^{-n}\mathbb{Z}]) \to \mathbb{S}_p[p^{-n}\mathbb{Z}]$$

sends the γ-summand to $\mathbb{S}_p \cdot q^\gamma$. $\square$

Remark 4.21. Compare this theorem with the following description of the usual ring map $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \to$

$\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^{n+1}}])$:

$$\begin{tikzcd}
\widetilde{j}^{(-1)}_{n-1} \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{ku}^{(-1)}_{p,n-1} \arrow[r, "\psi^{g^{p^{n-1}}}_{\circ\circ}"] \arrow[d, "\mathrm{can}"] & \mathrm{ku}^{(-1)}_{p,n-1}(1)[2] \arrow[d, "\mathrm{can}"] \\
\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]) \arrow[d, "\mathrm{tr}"'] & \mathrm{ku}^{(-1)}_{p,n} \arrow[r, "\psi^{g^{p^{n-1}}}_{\circ\circ}"] \arrow[dd, equal] & \mathrm{ku}^{(-1)}_{p,n}(1)[2] \arrow[dd, "\sum_{i=0}^{p-1}\psi^{g^{ip^{n-1}}}"] \\
\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^{n+1}}]) \arrow[d, phantom, "\wr"] & & \\
\widetilde{j}^{(-1)}_{n} \arrow[r] & \mathrm{ku}^{(-1)}_{p,n} \arrow[r, "\psi^{g^{p^{n}}}_{\circ\circ}"] & \mathrm{ku}^{(-1)}_{p,n}(1)[2]
\end{tikzcd}$$

where $\mathrm{can} : \mathrm{ku}^{(-1)}_{p,n-1} \to \mathrm{ku}^{(-1)}_{p,n}$ is the usual ring map.

Remark 4.22. At the level of homotopy groups, one computes rather easily that the transfer map on π_0 is given by the usual trace map

$$\mathrm{Tr}_{\mathbb{Q}_p(\zeta_{p^{n+1}})/\mathbb{Q}_p(\zeta_{p^n})} := \sum_{g\in \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^{n+1}})/\mathbb{Q}_p(\zeta_{p^n}))} g : \mathbb{Z}_p[\zeta_{p^{n+1}}] \to \mathbb{Z}_p[\zeta_{p^n}],$$

while the transfer map on π_{2i-1} ($i \geq 1$) (which are the only other non-zero homotopy groups of $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}])$) is given by

$$\frac{\mathbb{Z}_p[\zeta_{p^{n+1}}]}{ip^n\frac{p}{\zeta_p-1}} \xrightarrow{\mathrm{Tr}^{\mathrm{Tate}}_n} \frac{\mathbb{Z}_p[\zeta_{p^n}]}{ip^{n-1}\frac{p}{\zeta_p-1}},$$

where

$$\mathrm{Tr}^{\mathrm{Tate}}_n := \frac{1}{p}\mathrm{Tr}_{\mathbb{Q}_p(\zeta_{p^{n+1}})/\mathbb{Q}_p(\zeta_{p^n})} : \mathbb{Z}_p[\zeta_{p^{n+1}}] \to \mathbb{Z}_p[\zeta_{p^n}]$$

is Tate's normalized trace for the perfectoid tower $(\mathbb{Q}_p(\zeta_{p^n}))_n$.

4.3 General Case

We now upgrade Proposition 4.19 to arbitrary bounded quasi-syntomic p-adic formal schemes X over $\mathbb{Z}_p[\zeta_p]$, while also keeping track of motivic filtrations. In the following, we endow $F^{\geq *}_{\mathrm{ev}}\mathrm{ku}^{(-1)}_{p,n}$ with the cyclotomic synthetic spectrum structure coming from Example 2.22. Recall the following definition from [HRW22].

Definition. A ring map of E_∞-rings $A \to B$ is said to be p-completely evenly faithfully flat (abbreviated eff) if for every p-complete E_∞-A-algebra E that is even and with bounded p-power-torsion, the p-completed tensor product $E \otimes_A B$ is also even, and the map

$$\pi_{2*}E \to \pi_{2*}(E \otimes_A B)$$

is p-completely faithfully flat.

Remark 4.23. The property of being p-completely evenly faithfully flat is usual abbreviated to p-completely eff. Since we will not have need to consider non-p-complete objects, we abusively drop the 'p-completely' and simply write eff.

Lemma 4.24. *For all $m \geq n \geq 0$, the maps $\widetilde{j}^{(-1)}_n \to \mathrm{ku}^{(-1)}_{p,m}$ and $\widetilde{j}^{(-1),tC_p}_n \to \mathrm{ku}^{(-1),tC_p}_{p,m}$ are p-completely eff.*

Proof. Notice that $\mathrm{ku}^{(-1)}_{p,m}$ is a finite free $\mathrm{ku}^{(-1)}_{p,n}$-algebra (for $m > n$) by Lemma 4.13. In particular, we reduce to $m = n$. Since

$$\widetilde{j}^{(-1)}_n \otimes_{\mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]} \mathbb{S}_p \simeq \mathrm{ku}^{(-1)}_{p,n} \quad \text{and} \quad \widetilde{j}^{(-1),tC_p}_n \otimes_{\mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]} \mathbb{S}_p \simeq \left(\widetilde{j}^{(-1)}_n \otimes_{\mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]^{\mathrm{triv}}} \mathbb{S}^{\mathrm{triv}}_p\right)^{tC_p} \simeq \mathrm{ku}^{(-1),tC_p}_{p,n}$$

by Lemma 4.8 (we also use the projection formula for $(-)^{tC_p}$ to pull $-\otimes_{\mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]}\mathbb{S}_p$ out), it suffices to check whether $\mathbb{S}_p[Bp^{-n-1}\mathbb{Z}_p]\simeq\mathbb{S}_p[S^1]\to\mathbb{S}_p$ is p-completely eff. This is proved in the same way as [AR24, Corollary 2.36]. ☐

Corollary 4.25. *Suppose $m\geq n\geq 1$. If $R\in\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$ is a QRSP, then $\mathrm{THH}(R\otimes_{\mathbb{Z}_p[\zeta_{p^n}]}\mathbb{Z}_p[\zeta_{p^m}]/\mathbb{S}_p[\![q_m-1]\!])$ is even and the motivic filtration is the double-speed Postnikov filtration.*

Proof. This follows from the lemma, the fact that

$$\mathrm{THH}(R\otimes_{\mathbb{Z}_p[\zeta_{p^n}]}\mathbb{Z}_p[\zeta_{p^m}]/\mathbb{S}_p[\![q_m-1]\!])\simeq\mathrm{THH}(R)\otimes_{\widetilde{j}^{(-1)}_{n-1}}\mathrm{ku}^{(-1)}_{p,m-1},$$

and since $\mathrm{THH}(R)$ is even by [BMS19, Theorem 7.1]. The assertion about the motivic filtration follows from its identification with the even filtration. ☐

The following is a generalisation of Corollary 4.6.

Proposition 4.26. *For any $n\geq 1$ and any choice of topological generator $\gamma_n\in 1+p^n\mathbb{Z}_p$, there is a fibre sequence of sheaves (in fact, a fibre sequence at the level of pre-sheaves) of cyclotomic synthetic spectra on $\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$*

$$\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(-)\to\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])\xrightarrow{\psi^{\gamma_n}_{\circ\circ}}\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])(1)[2].$$

> **Remark 4.27.** Upon taking $(-)^{t\mathbb{T}_{\mathrm{ev}}}$ and then taking associated graded, this gives a computation of absolute (Nygaard-completed) prismatic cohomology in terms of (Nygaard-completed) prismatic cohomology relative to $\mathbb{Z}_p[\![q_n-1]\!]$. A Nygaard-decompleted generalisation of this latter result to coefficients is [Liu25, Theorem 1.0.2]. On the other hand, *loc. cit.* does not consider the Nygaard filtration.

Proof. Note that both $\mathrm{THH}(-)$ and $\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])$ are sheaves of cyclotomic spectra on $\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$, and that

$$\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])\simeq\mathrm{THH}(-)\otimes_{\widetilde{j}^{(-1)}_{n-1}}\mathrm{ku}^{(-1)}_{p,n-1}$$

as pre-sheaves, and thus as sheaves where the right hand side denotes the point-wise tensor product. Since $\mathrm{ku}^{(-1)}_{p,n-1}(1)[2]$ is a perfect $\mathrm{ku}^{(-1)}_{p,n-1}$-module, it also follows that

$$\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])(1)[2]\simeq\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])\otimes_{\mathrm{ku}^{(-1)}_{p,n-1}}\mathrm{ku}^{(-1)}_{p,n-1}(1)[2]$$

is a sheaf of cyclotomic spectra on $\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$. Applying $\mathrm{THH}(-)\otimes_{\widetilde{j}^{(-1)}_{n-1}}-$ to the fibre sequence in Corollary 4.6 thus yields the fibre sequence of sheaves of cyclotomic spectra

$$\mathrm{THH}(-)\to\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])\xrightarrow{\psi^{\gamma_n}_{\circ\circ}}\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])(1)[2].$$

We need to upgrade this to motivic filtrations. The motivic filtrations are defined via quasi-syntomic descent from QRSPs, and so it suffices to consider a QRSP R. In this case, $\mathrm{THH}(R)$ and $\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])$ are even. We thus get a fibre sequence

$$\tau_{\geq 2*}\mathrm{THH}(R)\to\tau_{\geq 2*}\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])\xrightarrow{\psi^{\gamma_n}_{\circ\circ}}\tau_{\geq 2*}\left(\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])(1)[2]\right)$$

of filtered (in fact synthetic) cyclotomic spectra. Since

$$\tau_{\geq 2*}\left(\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])(1)[2]\right)\simeq\left(\tau_{\geq 2*-2}\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])\right)(1)[2],$$

unfolding the preceding fibre sequence yields the result. ☐

Corollary 4.28. *For R a QRSP $\mathbb{Z}_p[\zeta_p]$-algebra, one has an equivalence*

$$\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R\otimes_{\mathbb{Z}_p[\zeta_p]}\mathbb{Z}_p[\zeta_{p^n}])\simeq\tau_{\geq 2*-1}\mathrm{THH}(R\otimes_{\mathbb{Z}_p[\zeta_p]}\mathbb{Z}_p[\zeta_{p^n}]).$$

Proof. Set $R_n:=R\otimes_{\mathbb{Z}_p[\zeta_p]}\mathbb{Z}_p[\zeta_{p^n}]$. By Corollary 4.25, $\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])$ is even and its motivic filtration is the double-speed Postnikov filtration. The corollary follows from taking connective covers in the fibre sequence

$$\mathrm{THH}(R_n)\to\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])\to\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])(1)[2]$$

from the proposition. ☐

The following generalises Lemma 4.13.

Lemma 4.29. *For each $r \in \mathbb{Z}$, there is a natural $\mathbb{Z}_p^\times$-equivariant $\mathbb{S}_p[\![q_n - 1]\!]^{\mathrm{triv}}$-module structure on the sheaf $\mathrm{Fil}^{\geq r}_{\mathrm{mot}}\mathrm{THH}(-/\mathbb{S}_p[\![q_n - 1]\!])$.*

Moreover, there is a $\mathbb{Z}_p^\times$-equivariant equivalence of sheaves of synthetic spectra with synthetic circle action

$$\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!]) \simeq \Big(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(-/\mathbb{S}_p[\![q_1 - 1]\!])\Big) \otimes_{\mathbb{S}_p[\![q_1-1]\!]^{\mathrm{triv}}} \mathbb{S}_p[\![q_n - 1]\!]^{\mathrm{triv}}.$$

Proof. The first follows from the fact that

$$\tau_{\geq 2r}\mathrm{ku}^{(-1)}_{p,n-1} \simeq \mathrm{Fil}^{\geq r}_{\mathrm{mot}}\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!])$$

has an obvious $\mathbb{S}_p[\![q_n - 1]\!]^{\mathrm{triv}}$ module structure. For the second statement, there is an obvious map from the right to the left, which is an equivalence for QRSPs as a consequence of Lemma 4.13 (in which case the motivic filtration is the double-speed Postnikov filtration) and thus for everything by quasi-syntomic descent. ☐

Lemma 4.30. *Suppose $R \in \mathrm{QSyn}_{\mathbb{Z}_p[\zeta_p]}$. Then,*

$$\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!]) \simeq \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R/\mathbb{S}_p[\![q_1 - 1]\!]) \otimes_{F^{\geq *}_{\mathrm{ev}}\mathrm{ku}_p^{(-1)}} F^{\geq *}_{\mathrm{ev}}\mathrm{ku}^{(-1)}_{p,n-1}$$

as cyclotomic synthetic spectra, functorially in R.

Proof. By definition and Corollary 2.32, this equivalence holds on underlying objects. Since the motivic filtration is the even filtration, and the even filtration is lax-symmetric monoidal, we get a canonical map from the right to the left. By quasi-syntomic descent, we may assume R is QRSP, in which case $\mathrm{THH}(R/\mathbb{S}_p[\![q_n - 1]\!])$ is even and all filtrations at play are the double-speed Postnikov filtration, by Corollary 4.25. To prove the filtered equivalence, it suffices to consiedr this equivalence on associated graded, i.e. we need to check that the canonical map

$$\pi_{2*}\mathrm{ku}^{(-1)}_{p,n-1} \otimes_{\pi_{2*}\mathrm{ku}_p^{(-1)}} \pi_{2*}\mathrm{THH}(R/\mathbb{S}_p[\![q_1 - 1]\!]) \to \pi_{2*}\mathrm{THH}(R \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!])$$

is an isomorphism. Since $\mathrm{ku}_p^{(-1)} \to \mathrm{ku}^{(-1)}_{p,n-1}$ is p-completely eff (in fact $\mathrm{ku}^{(-1)}_{p,n-1}$ is a finite free even $\mathrm{ku}_p^{(-1)}$-module), and since $\mathrm{THH}(R/\mathbb{S}_p[\![q_1 - 1]\!])$ is even, this isomorphism is [HRW22, Proposition 2.2.6]. ☐

> **Definition.** Let ρ_n denote the map of sheaves of $\mathbb{Z}_p^\times$-equivariant synthetic spectra with synthetic circle action obtained by the composition
>
> $$\begin{aligned}\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^{n+1}}]/\mathbb{S}_p[\![q_{n+1} - 1]\!]) &\simeq \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(-/\mathbb{S}_p[\![q_1 - 1]\!]) \otimes_{F^{\geq *}_{\mathrm{ev}}\mathrm{ku}_p^{(-1)}} F^{\geq *}_{\mathrm{ev}}\mathrm{ku}^{(-1)}_{p,n} \\ &\xrightarrow{\mathrm{id}\otimes F^{\geq *}_{\mathrm{ev}}\rho_n} \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(-/\mathbb{S}_p[\![q_1 - 1]\!]) \otimes_{F^{\geq *}_{\mathrm{ev}}\mathrm{ku}_p^{(-1)}} F^{\geq *}_{\mathrm{ev}}\mathrm{ku}^{(-1)}_{p,n-1} \\ &\simeq \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!])\end{aligned}$$
>
> where the equivalences are from Lemma 4.30 and the map ρ_n in the middle is the map from Construction 4.15.

The following is immediate from Lemma 4.16.

Lemma 4.31. *The map of sheaves ρ_n defined above identifies with the composition*

$$\begin{aligned}\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^{n+1}}]/\mathbb{S}_p[\![q_{n+1} - 1]\!]) &\simeq \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(-/\mathbb{S}_p[\![q_1 - 1]\!]) \otimes_{\mathbb{S}_p[\![q_1-1]\!]^{\mathrm{triv}}} \mathbb{S}_p[\![q_{n+1} - 1]\!]^{\mathrm{triv}} \\ &\xrightarrow{\mathrm{id}\otimes\mathrm{proj}_n} \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(-/\mathbb{S}_p[\![q_1 - 1]\!]) \otimes_{\mathbb{S}_p[\![q_1-1]\!]^{\mathrm{triv}}} \mathbb{S}_p[\![q_n - 1]\!]^{\mathrm{triv}} \\ &\simeq \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!])\end{aligned}$$

where the equivalences are from Lemma 4.29 and proj_n is the map $\mathbb{S}_p[\![q_{n+1} - 1]\!] \to \mathbb{S}_p[\![q_n - 1]\!]$ given by

$$q_{n+1}^i \mapsto \begin{cases} 0 & p \nmid i, \\ q_n^{i/p} & p | i. \end{cases}$$

We can finally prove the main result of this subsection.

Theorem 4.32. *Let g be a topological generator of $(1+p\mathbb{Z}_p)$, and $R \in \mathrm{QSyn}_{\mathbb{Z}_p[\zeta_p]}$. Set $R_n := R \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]$. The transfer map*

$$\mathrm{tr} : \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_{n+1}) \to \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n)$$

sits in the following functorial-in-R commutative diagram of $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n)$-modules in synthetic spectra with synthetic S^1-action:

$$\begin{tikzcd}
\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_{n+1}) \arrow[r] \arrow[dd, "\mathrm{tr}"'] & \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!]) \arrow[r, "\psi^{g^{p^n}}_{\circ\circ}"] \arrow[d, "\rho_n"'] & \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!])(1)[2] \arrow[d, "\rho_n(1)[2]"'] \\
 & \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!]) \arrow[r, "\psi^{g^{p^n}}_{\circ\circ}"] \arrow[d, "\sum_{i=0}^{p-1}\psi^{g^{ip^{n-1}}}"] & \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])(1)[2] \arrow[d, equal, "\mathrm{id}"] \\
\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n) \arrow[r] & \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!]) \arrow[r, "\psi^{g^{p^{n-1}}}_{\circ\circ}"] & \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])(1)[2]
\end{tikzcd} \tag{8}$$

where the top and bottom rows are the fibre sequences of Proposition 4.26.

Proof. Hitting Proposition 4.19 with $\mathrm{THH}(R) \otimes_{\mathrm{THH}(\mathbb{Z}_p[\zeta_p])} -$, and using Lemma 4.17, yields the result at the unfiltered level. For the motivic filtrations, by quasi-syntomic descent it suffices to consider R a QRSP, in which case $\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])$ is even with the motivic filtration the double-speed Postnikov filtration, and

$$\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n) \simeq \tau_{\geq 2*-1}\mathrm{THH}(R_n)$$

by Corollary 4.28. We can thus take connective covers in the unfiltered version of (8) to obtain the filtered diagram (8). ☐

Proof of Proposition F. Take associated graded of the motivic filtration of $(-)^{h\mathbb{T}_{\mathrm{ev}}}$ (resp. $(-)^{t\mathbb{T}_{\mathrm{ev}}}$) in Theorem 4.32. ☐

4.4 Limits Over the Cyclotomic Tower

We now compute

$$\varprojlim_n \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}])$$

as a sheaf of $\mathbb{Z}_p^\times$-equivariant cyclotomic synthetic spectra.

We thank Tomer Schlank for suggesting the following lemma. Recall that for a connective E_∞-ring R, a bounded below R-module spectrum M is said to be *almost perfect* if $\tau_{\leq n}M$ is a perfect R-module for all n. If R is a classical ring, this notion of almost perfect is equivalent to the notion of being pseudo-coherent in the sense of [SP, Tag064Q].

Lemma 4.33. *Suppose R is a connective E_∞-ring, N an almost perfect connective R-module, and $\{M_i\}_i$ any sequential diagram of connective R-modules with limit $M := \lim_i M_i$. Then, the natural map*

$$M \otimes_R N \to \lim_i (M_i \otimes_R N)$$

is an equivalence.

> **Remark 4.34.** If the connective E_∞-ring R is coherent (in the sense of [Lur, Definition 7.2.4.16]), then an R-module M is almost perfect if and only if M is bounded below and $\pi_i M$ is finitely presented as a $\pi_0 R$-module [Lur, Proposition 7.2.4.17]. In particular, if R is Noetherian (i.e. $\pi_0 R$ is Noetherian and $\pi_i R$ for $i > 0$ is a finitely generated $\pi_0 R$-module), then R is coherent and the above criterion applies.

Proof. It suffices to show that

$$\tau_{\leq n}(M \otimes_R N) \to \tau_{\leq n} \lim_i (M_i \otimes_R N)$$

is an equivalence for all $n \geq 0$. So fix $n \geq 0$. Since N is almost perfect over R, there exists a perfect R-module N' and a map $N' \to N$ that is an equivalence on $\tau_{\leq n+1}$. Consider the diagram

$$\begin{tikzcd}
M \otimes_R N' \arrow[r, "\simeq"] \arrow[d] & \lim_i (M_i \otimes_R N') \arrow[d] \\
M \otimes_R N \arrow[r] & \lim_i (M_i \otimes_R N).
\end{tikzcd} \tag{9}$$

where the top arrow is an equivalence as N' is a perfect R-module. Since M_i is connective and $\operatorname{cofib}(N' \to N) \in \mathrm{Sp}_{>n+1}$, we have $M_i \otimes \operatorname{cofib}(N' \to N) \in \mathrm{Sp}_{>n+1}$ and so

$$\tau_{\leq n+1}(M_i \otimes_R N') \xrightarrow{\simeq} \tau_{\leq n+1}(M_i \otimes_R N).$$

By the Milnor exact sequence, $\tau_{\leq n} \lim_i(M_i \otimes_R N)$ depends only on $\tau_{\leq n+1}(M_i \otimes_R N)$ and similarly for $\lim_i(M_i \otimes_R N')$. In particular, the preceding connectivity estimate implies that the right vertical arrow in (9) is an equivalence on $\tau_{\leq n}$. Again by the Milnor sequence and the connectivity of M_i, we have $M \in \mathrm{Sp}_{\geq -1}$ so that $M \otimes \operatorname{cofib}(N' \to N) \in \mathrm{Sp}_{>n}$ and thus

$$\tau_{\leq n}(M \otimes_R N') \xrightarrow{\simeq} \tau_{\leq n}(M \otimes_R N).$$

Hence, in (9), both vertical arrows and the top horizontal arrow are equivalences on $\tau_{\leq n}$. It then follows that the bottom horizontal arrow is also an equivalence on $\tau_{\leq n}$ as required. $\square$

Lemma 4.35. *For any $R \in \mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$, the $1 + p^n\mathbb{Z}_p$-action on*

$$\mathrm{gr}^*_{\mathrm{mot}} \Phi \mathrm{THH}_{\Delta}(R/\mathbb{S}_p[\![q_n - 1]\!]) \quad \textit{and} \quad \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(R/\mathbb{S}_p[\![q_n - 1]\!])$$

is canonically trivial modulo $\zeta_p - 1$.

Proof. By [Liu25, Theorem 1.0.6], evaluation at the prism $(\mathbb{Z}_p[\![q_{n-1} - 1]\!], [p]_q)$ induces a fully faithful functor from $\mathcal{D}_{\mathrm{qcoh}}(\mathbb{Z}_p[\zeta_{p^n}]^{\Delta})$ into the ∞-category of $\mathbb{Z}_p[\![q_{n-1} - 1]\!]$-modules with semi-linear $1 + p^n\mathbb{Z}_p$ action with a trivialisation of the action modulo $q - 1$. Passing to the Hodge-Tate locus (see also the proof of [Liu25, Theorem 3.4.6]), this embeds $\mathcal{D}_{\mathrm{qcoh}}(\mathbb{Z}_p[\zeta_{p^n}]^{\mathrm{HT}})$ fully faithfully into the ∞-category of $\mathbb{Z}_p[\zeta_{p^n}]$-modules with linear $1 + p^n\mathbb{Z}_p$-action equipped with a trivialisation modulo

$$\frac{\mathbb{Z}_p[\![q_{n-1} - 1]\!]}{([p]_q, q - 1)} \simeq \frac{\mathbb{Z}_p[\zeta_{p^n}]}{\zeta_p - 1}.$$

In particular, this holds for the Breuil-Kisin twists and the object $\mathrm{R}f_*\mathcal{O}$ where $f : R^{\mathrm{HT}} \to \mathbb{Z}_p[\zeta_{p^n}]^{\mathrm{HT}}$ is the canonical map of stacks (for more on the stacky approach see [BL22a; BL22b]). The evaluation of the Hodge-Tate crystal $\mathrm{R}f_*\mathcal{O}$ at $\mathbb{Z}_p[\![q_{n-1} - 1]\!]$ is precisely $\overline{\Delta}_{R/\mathbb{Z}_p[\![q_{n-1}-1]\!]}$. Thus, the natural $1 + p^n\mathbb{Z}_p$-action on $\overline{\Delta}_{R/\mathbb{Z}_p[\![q_{n-1}-1]\!]}\{*\}$ is canonically trivialised modulo $\zeta_p - 1$.

Now suppose R is a large quasi-syntomic $\mathbb{Z}_p[\zeta_{p^n}]$-algebra (see [BS22, Definition 15.1] for the definition of large); such R form a basis for the quasi-syntomic topology on $\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$. As a consequence of [BS22, Theorem 15.2(2)], we see that $\mathrm{gr}^*_{\mathrm{Nyg}} \Delta^{(1)}_{R/\mathbb{Z}_p[\![q_{n-1}-1]\!]}\{*\}$, being functorially identified as a subset of $\overline{\Delta}_{R/\mathbb{Z}_p[\![q_{n-1}-1]\!]}\{*\}$ via Frobenius, has the trivial $1 + p^n\mathbb{Z}_p$ action modulo $\zeta_p - 1$. However,

$$\mathrm{gr}^*_{\mathrm{Nyg}} \Delta^{(1)}_{R/\mathbb{Z}_p[\![q_{n-1}-1]\!]}\{*\} \simeq \mathrm{gr}^*_{\mathrm{mot}} \Phi \mathrm{THH}_{\Delta}(R/\mathbb{S}_p[\![q_n - 1]\!])[-2*].$$

The lemma follows by quasi-syntomic descent. $\square$

Finally, we give some lemmas that will be used to study the cyclotomic Frobenius on $\varprojlim_n \mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}])$.

Construction 4.36. Define the map of sheaves on $\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_p]}$ valued in $\mathbb{Z}_p^\times$-equivariant synthetic spectra with synthetic circle action

$$\widetilde{\varphi}_n : \mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^{n+1}}]/\mathbb{S}_p[\![q_{n+1} - 1]\!]) \to \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!])\right)^{tC_{p,\mathrm{ev}}}$$

by the composition

$$\begin{aligned}
\mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^{n+1}}]/\mathbb{S}_p[\![q_{n+1} - 1]\!]) &\simeq \mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(-/\mathbb{S}_p[\![q_1 - 1]\!]) \otimes_{F^{\geq *}_{\mathrm{ev}} \mathrm{ku}_p^{(-1)}} F^{\geq *}_{\mathrm{ev}} \mathrm{ku}_{p,n}^{(-1)} \\
&\xrightarrow{\varphi_{\mathrm{THH}(-/\mathbb{S}_p[\![q_1-1]\!])} \otimes F^{\geq *}_{\mathrm{ev}} \widetilde{\varphi}_n} (\mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(-/\mathbb{S}_p[\![q_1 - 1]\!]))^{tC_{p,\mathrm{ev}}} \otimes_{(F^{\geq *}_{\mathrm{ev}} \mathrm{ku}_p^{(-1)})^{tC_{p,\mathrm{ev}}}} (F^{\geq *}_{\mathrm{ev}} \mathrm{ku}_{p,n}^{(-1)})^{tC_{p,\mathrm{ev}}} \\
&\longrightarrow \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(-/\mathbb{S}_p[\![q_1 - 1]\!]) \otimes_{F^{\geq *}_{\mathrm{ev}} \mathrm{ku}_p^{(-1)}} F^{\geq *}_{\mathrm{ev}} \mathrm{ku}_{p,n}^{(-1)}\right)^{tC_{p,\mathrm{ev}}} \\
&\simeq \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n - 1]\!])\right)^{tC_{p,\mathrm{ev}}}
\end{aligned}$$

where the first and last equivalences are Lemma 4.30, the second arrow is the tensor product of the cyclotomic Frobenius on $\mathrm{THH}(-/\mathbb{S}_p[\![q_1 - 1]\!])$ with the evenly filtered map $\widetilde{\varphi}_n : \mathrm{ku}_{p,n}^{(-1)} \to \mathrm{ku}_{p,n-1}^{(-1),tC_p}$, and the third arrow comes from the lax symmetric monoidality of Tate fixed points. Here, we are using that $F_{\mathrm{ev}}^{\geq *}\mathrm{ku}_{p,n-1}^{(-1),tC_p} \simeq \left(F_{\mathrm{ev}}^{\geq *}\mathrm{ku}_{p,n-1}^{(-1)}\right)^{tC_{p,\mathrm{ev}}}$ by evenness of $\mathrm{ku}_{p,n-1}$ [AR24, Lemma 3.18].

These maps $\widetilde{\varphi}_n$ are moreover compatible for varying n as in the following lemma.

Lemma 4.37. *There is a commutative diagram of sheaves on $\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_p]}$ valued in synthetic spectra with synthetic circle action*

$$\begin{tikzcd} \mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(-\otimes_{\mathbb{Z}_p[\zeta_p]}\mathbb{Z}_p[\zeta_{p^{n+1}}]/\mathbb{S}_p[\![q_{n+1}-1]\!]) \arrow[r, "\widetilde{\varphi}_n"] \arrow[d, "\rho_n"] & \left(\mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(-\otimes_{\mathbb{Z}_p[\zeta_p]}\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n-1]\!])\right)^{tC_{p,\mathrm{ev}}} \arrow[d, "\rho_{n-1}^{tC_{p,\mathrm{ev}}}"] \\ \mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(-\otimes_{\mathbb{Z}_p[\zeta_p]}\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n-1]\!]) \arrow[r, "\widetilde{\varphi}_{n-1}"] & \left(\mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(-\otimes_{\mathbb{Z}_p[\zeta_p]}\mathbb{Z}_p[\zeta_{p^{n-1}}]/\mathbb{S}_p[\![q_{n-1}-1]\!])\right)^{tC_{p,\mathrm{ev}}} \end{tikzcd}$$

where the vertical maps are the projection maps.

Proof. By Lemma 4.30 and the definition of $\widetilde{\varphi}_n$ and ρ_n, it suffices to show that the following diagram of S^1-equivariant spectra commutes

$$\begin{tikzcd} F_{\mathrm{ev}}^{\geq *}\mathrm{ku}_{p,n}^{(-1)} \arrow[r, "\widetilde{\varphi}_n"] \arrow[d, "\rho_n"] & F_{\mathrm{ev}}^{\geq *}\mathrm{ku}_{p,n-1}^{(-1),tC_p} \arrow[d, "\rho_{n-1}^{tC_p}"] \\ F_{\mathrm{ev}}^{\geq *}\mathrm{ku}_{p,n-1}^{(-1)} \arrow[r, "\widetilde{\varphi}_{n-1}"] & F_{\mathrm{ev}}^{\geq *}\mathrm{ku}_{p,n-2}^{(-1),tC_p}. \end{tikzcd}$$

This is Lemma 4.16. □

Lemma 4.38. *Fix a topological generator g of $1+p^{n+1}\mathbb{Z}_p$. Define the sheaf $F^{\geq *}\mathcal{F}$ of cyclotomic synthetic spectra on $\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$ to be the mapping fibre of*

$$\mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!]) \xrightarrow{\psi_{\circ\circ}^g} \mathrm{Fil}_{\mathrm{mot}}^{\geq *-1}\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])(1)[2].$$

Then there is the following commutative diagram of cyclotomic synthetic spectra functorial in $R \in \mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$

$$\begin{tikzcd} F^{\geq *}\mathcal{F}(R) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!]) \arrow[r, "\psi_{\circ\circ}^g"] \arrow[d, "\mathrm{can}"] & \mathrm{Fil}_{\mathrm{mot}}^{\geq *-1}\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])(1)[2] \arrow[d, "\mathrm{can}"] \\ \mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(R_{n+1}) \arrow[r] \arrow[d, "\alpha'"] & \mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!]) \arrow[r, "\psi_{\circ\circ}^g"] \arrow[d, "\rho_n"] & \mathrm{Fil}_{\mathrm{mot}}^{\geq *-1}\mathrm{THH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!])(1)[2] \arrow[d, "\rho_n"] \\ F^{\geq *}\mathcal{F}(R) \arrow[r] & \mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!]) \arrow[r, "\psi_{\circ\circ}^g"] & \mathrm{Fil}_{\mathrm{mot}}^{\geq *-1}\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])(1)[2] \end{tikzcd}$$

where the vertical compositions are the identity, all rows are fibre sequences, and $R_{n+1} := R \otimes_{\mathbb{Z}_p[\zeta_{p^n}]} \mathbb{Z}_p[\zeta_{p^{n+1}}]$. Moreover, $\alpha^{tC_{p,\mathrm{ev}}}$ induces an equivalence

$$(F^{\geq *}\mathcal{F}_n(-))^{tC_{p,\mathrm{ev}}} \simeq \left(\mathrm{Fil}_{\mathrm{mot}}^{\geq *}\mathrm{THH}(-\otimes_{\mathbb{Z}_p[\zeta_{p^n}]}\mathbb{Z}_p[\zeta_{p^{n+1}}])\right)^{tC_{p,\mathrm{ev}}}$$

of sheaves of synthetic spectra with synthetic circle action on $\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$, with inverse $(\alpha')^{tC_{p,\mathrm{ev}}}$.

Proof. That the diagram exists is obvious from Proposition 4.26, the fact that

$$\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!]) \simeq \mathrm{THH}(R) \otimes_{\widetilde{j}_0^{(-1)}} \mathrm{ku}_{p,n-1}^{(-1)}$$

(and similarly for n replaced with $n+1$), and the definition of ρ_n. Here, the maps α and α' are defined to be the fibres of the corresponding middle and right vertical maps.

Thus, the real claim to check is that $\alpha^{tC_{p,\mathrm{ev}}}$ is an equivalence. We first assume that R is a large quasi-syntomic algebra over $\mathbb{Z}_p[\zeta_{p^n}] := A/I$ in the sense of [BS22, Definition 15.1], where (A, I) is the oriented prism $(\mathbb{Z}_p[\![q_{n-1}-1]\!], [p]_q)$. Using Proposition 4.9, we obtain a fibre square

$$\begin{tikzcd} \mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])^{tC_p} \arrow[r, "\psi_{\circ\circ}^g"] \arrow[d, "\mathrm{can}"] & \mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])^{tC_p}(1)[2] \arrow[d, "\mathrm{can}"] \\ \mathrm{THH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!])^{tC_p} \arrow[r, "\psi_{\circ\circ}^g"] & \mathrm{THH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!])^{tC_p}(1)[2]. \end{tikzcd} \tag{10}$$

We claim that both $\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])$ and $\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])^{tC_p}$ are even. By [BS22, Theorem 15.2], $\mathbbl{\Delta}_{R/A}$ is a discrete (p,I)-faithfully flat A-algebra. In particular, both $\mathbbl{\Delta}_{R/A}$ and $\widehat{\mathbbl{\Delta}}_{R/A}$ are discrete and I-torsion free. As Frobenius on A is faithfully flat, this is also true for $\mathbbl{\Delta}^{(1)}_{R/A}$ and $\widehat{\mathbbl{\Delta}}^{(1)}_{R/A}$. Moreover, by Theorem 15.2 of *loc. cit.*, the Nygaard filtrations are honest filtrations. In particular, it follows that all of

$$\mathrm{TC}^-(R/\mathbb{S}_p[\![q_n-1]\!]), \mathrm{TP}(R/\mathbb{S}_p[\![q_n-1]\!]), \mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])$$

are even, which dispenses with the first claim. Moreover, since $\mathrm{ku}_p^{(-1)}$ is complex-oriented and both $\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])$ and $\Phi\mathrm{THH}_{\mathbbl{\Delta}}(R/\mathbb{S}_p[\![q_n-1]\!])$ are $\mathrm{ku}_p^{(-1)}$-algebras, we have

$$\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])^{tC_p} \simeq \mathrm{TP}(R/\mathbb{S}_p[\![q_n-1]\!])/[p]_q t.$$

Since $\widehat{\mathbbl{\Delta}}^{(1)}_{R/A}$ is $[p]_q$-torsion free, this implies that $\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])^{tC_p}$ is even.

Evenness of $\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])$ and [AR24, Lemma 3.18] then imply that

$$\left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!]\right)^{tC_{p,\mathrm{ev}}} \simeq \tau_{\geq 2*}\left(\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])^{tC_p}\right).$$

Similarly for n replaced with $n+1$ since $\mathrm{ku}^{(-1)}_{p,n-1} \to \mathrm{ku}^{(-1)}_{p,n}$ is finite free and thus p-completely eff. Hence, applying $\tau_{\geq 2*}$ to (10) yields a fibre square of filtered spectra

$$\begin{tikzcd}
\left(\mathrm{Fil}^{\geq 2*}_{\mathrm{mot}}\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])\right)^{tC_{p,\mathrm{ev}}} \arrow[r, "\psi^g_{\circ\circ}"] \arrow[d, "\mathrm{can}"] & \left(\mathrm{Fil}^{\geq 2*}_{\mathrm{mot}}\mathrm{THH}(R/\mathbb{S}_p[\![q_n-1]\!])\right)^{tC_{p,\mathrm{ev}}}(1)[2] \arrow[d, "\mathrm{can}"] \\
\left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!])\right)^{tC_{p,\mathrm{ev}}} \arrow[r, "\psi^g_{\circ\circ}"] & \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!])\right)^{tC_{p,\mathrm{ev}}}(1)[2].
\end{tikzcd}$$

Taking horizontal fibres and using Proposition 4.26 to identify the bottom horizontal fibre, we immediately get that

$$\alpha^{tC_{p,\mathrm{ev}}}(R) : (F^{\geq *}\mathcal{F}_n(R))^{tC_{p,\mathrm{ev}}} \simeq \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n)\right)^{tC_{p,\mathrm{ev}}}$$

is an equivalence. Quasi-syntomic descent then gives the lemma in general. $\square$

Proposition 4.39. *Fix a choice of topological generator g of $1+p\mathbb{Z}_p$. For all $n \geq 2$, the following is a commutative diagram of synthetic spectra with synthetic circle action functorial in $R \in \mathrm{QSyn}_{\mathbb{Z}_p[\zeta_p]}$, where $R_n := R \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]$. To fit the diagram on the page, we abbreviate $\mathbb{S}_p[\![q_n-1]\!] =: \mathbb{S}_n^{\mathrm{qdR}}$ and drop all references to $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}$ (with shifts in filtration denoted by $\langle - \rangle$ from Section 2.3).*

$$\begin{tikzcd}
 & \mathrm{THH}(R_{n+1}/\mathbb{S}^{\mathrm{qdR}}_{n+1})(1)\langle 1\rangle[1] \arrow[dl, "\rho_n(1)\langle 1\rangle[1]"'] \arrow[dr, "\widetilde{\varphi}_n"] \arrow[ddddd, "\rho_n(1)\langle 1\rangle[1]"] & \\
\mathrm{THH}(R_n/\mathbb{S}^{\mathrm{qdR}}_n)(1)\langle 1\rangle[1] \arrow[d, "\partial_n"'] \arrow[ddd, bend right=60, "\rho_{n-1}(1)\langle 1\rangle[1]"'] & & \mathrm{THH}(R_n/\mathbb{S}^{\mathrm{qdR}}_n)^{tC_{p,\mathrm{ev}}}(1)\langle 1\rangle[1] \arrow[d, "\partial_n^{tC_{p,\mathrm{ev}}}"] \arrow[ddd, bend left=60, "\rho_{n-1}^{tC_{p,\mathrm{ev}}}(1)\langle 1\rangle[1]"] \\
\mathrm{THH}(R_n) \arrow[rr, "\varphi_{\mathrm{THH}(R_n)}"'] \arrow[d, "\mathrm{tr}"] & & \mathrm{THH}(R_n)^{tC_{p,\mathrm{ev}}} \arrow[d, "\mathrm{tr}^{tC_{p,\mathrm{ev}}}"'] \\
\mathrm{THH}(R_{n-1}) \arrow[rr, "\varphi_{\mathrm{THH}(R_{n-1})}"'] & & \mathrm{THH}(R_{n-1})^{tC_{p,\mathrm{ev}}} \\
\mathrm{THH}(R_{n-1}/\mathbb{S}^{\mathrm{qdR}}_{n-1})(1)\langle 1\rangle[1] \arrow[u, "\partial_{n-1}"] & & \mathrm{THH}(R_{n-1}/\mathbb{S}^{\mathrm{qdR}}_{n-1})^{tC_{p,\mathrm{ev}}}(1)\langle 1\rangle[1] \arrow[u, "\partial_{n-1}^{tC_{p,\mathrm{ev}}}"'] \\
 & \mathrm{THH}(R_n/\mathbb{S}^{\mathrm{qdR}}_n)(1)\langle 1\rangle[1] \arrow[ul, "\rho_{n-1}(1)\langle 1\rangle[1]"] \arrow[ur, "\widetilde{\varphi}_{n-1}"'] &
\end{tikzcd}$$

Here $\mathrm{tr} : \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n) \to \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_{n-1})$ *denotes the transfer map and*

$$\partial_n : \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])(1)[1] \to \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n)$$

is the connecting map in the fibre sequence of Proposition 4.26 (where we choose the topological generator $g^{p^{n-1}}$ for $1+p^n\mathbb{Z}_p$ for all n).

Proof. The left-most and right-most commutative squares are obtained by rotating the diagram in Theorem 4.32. The middle rectangle commutes since the transfer maps on THH are maps of cyclotomic spectra. The big trapezium (with a curved short side) on the left commutes trivially, while the big right trapezium (with a curved short side) commutes by Lemma 4.37. Thus, the only thing left to check is that the following pentagon commutes for all n.

$$\begin{tikzcd}
& \mathrm{Fil}^{\geq r-1}_{\mathrm{mot}}\mathrm{THH}(R_{n+1}/\mathbb{S}^{\mathrm{qdR}}_{n+1})(1)[1] \arrow[dl, "\rho_n(1)[1]"'] \arrow[dr, "\widetilde{\varphi}_n"] & \\
\mathrm{Fil}^{\geq r-1}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}^{\mathrm{qdR}}_n)(1)[1] \arrow[d, "\partial_n"'] & & \mathrm{Fil}^{\geq r-1}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}^{\mathrm{qdR}}_n)^{tC_{p,\mathrm{ev}}}(1)[1] \arrow[d, "\partial_n"] \\
\mathrm{Fil}^{\geq r}_{\mathrm{mot}}\mathrm{THH}(R_n) \arrow[rr, "\varphi_{\mathrm{THH}(R_n)}"'] & & \mathrm{Fil}^{\geq r}_{\mathrm{mot}}\mathrm{THH}(R_n)^{tC_{p,\mathrm{ev}}}
\end{tikzcd} \tag{11}$$

Define the sheaf

$$F^{\geq *}\mathcal{F}_n := \mathrm{fib}\left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n-1]\!]) \xrightarrow{\psi^{g_n}_{\circ\circ}} \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(- \otimes_{\mathbb{Z}_p[\zeta_p]} \mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n-1]\!])(1)[2]\right)$$

where g_n is a topological generator $1+p^{n+1}\mathbb{Z}_p$ chosen so that $g_{n+1}=g_n^p$; there is an equivalence $\alpha^{tC_{p,\mathrm{ev}}}$: $F^{\geq *}\mathcal{F}_n(R) \xrightarrow{\simeq} \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_{n+1})\right)^{tC_{p,\mathrm{ev}}}$ from Lemma 4.38. Rotating the diagram of fibre sequences in Lemma 4.38 yields the following two diagrams

$$\begin{tikzcd}
\left(\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(R_{n-1}/\mathbb{S}_p[\![q_{n-1}-1]\!])\right)^{tC_{p,\mathrm{ev}}}(1)[1] \arrow[r, "\widetilde{\partial}^{tC_{p,\mathrm{ev}}}_n"] \arrow[d, "\iota^{tC_{p,\mathrm{ev}}}"] & (F^{\geq *}\mathcal{F}_{n-1}(R))^{tC_{p,\mathrm{ev}}} \arrow[d, "\alpha^{tC_{p,\mathrm{ev}}}"', "\wr"] \\
\left(\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])\right)^{tC_{p,\mathrm{ev}}}(1)[1] \arrow[r, "\partial^{tC_{p,\mathrm{ev}}}_n"'] & \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n)\right)^{tC_{p,\mathrm{ev}}}
\end{tikzcd} \tag{12}$$

and

$$\begin{tikzcd}
\left(\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])\right)^{tC_{p,\mathrm{ev}}}(1)[1] \arrow[r, "\partial^{tC_{p,\mathrm{ev}}}_n"'] \arrow[d, "\rho^{tC_{p,\mathrm{ev}}}_{n-1}(1)[1]"'] & \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n)\right)^{tC_{p,\mathrm{ev}}} \arrow[d, "(\alpha^{tC_{p,\mathrm{ev}}})^{-1}", "\wr"'] \\
\left(\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(R_{n-1}/\mathbb{S}_p[\![q_{n-1}-1]\!])\right)^{tC_{p,\mathrm{ev}}}(1)[1] \arrow[r, "\widetilde{\partial}^{tC_{p,\mathrm{ev}}}_n"] & (F^{\geq *}\mathcal{F}_{n-1}(R))^{tC_{p,\mathrm{ev}}}
\end{tikzcd} \tag{13}$$

where ι is induced by the canonical map $\mathrm{ku}^{(-1)}_{p,n-1} \to \mathrm{ku}^{(-1)}_{p,n}$ and $\widetilde{\partial}_n$: $\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])(1)[1] \to F^{\geq *}\mathcal{F}_n(R)$ is the connecting map coming from the definition of $\mathcal{F}_n$. On the other hand, since the fibre sequence in Proposition 4.26 is of cyclotomic synthetic spectra, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])(1)[1] \arrow[r, "\partial_n"] \arrow[d, "\varphi_{n,\mathrm{qdR}}"] & \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n) \arrow[d, "\varphi_n"] \\
\left(\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])\right)^{tC_{p,\mathrm{ev}}}(1)[1] \arrow[r, "\partial^{tC_{p,\mathrm{ev}}}_n"] & \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n)\right)^{tC_{p,\mathrm{ev}}}
\end{tikzcd} \tag{14}$$

where for notational simplicity we have denoted the cyclotomic Frobenius for $\mathrm{THH}(R_n)$ and for $\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])(1)[1]$ by φ_n and $\varphi_{n,\mathrm{qdR}}$. However, as a consequence of the definition of $\widetilde{\varphi}_{n-1}$ the map $\varphi_{n,\mathrm{qdR}}$ factors through $\widetilde{\varphi}_{n-1}$ (see Lemma 4.12), and so we can glue diagrams (12), (13), and the one from Lemma 4.37 onto diagram (14) to get the following large diagram. As before, in order to fit the diagram onto the page, we have abbreviated $\mathbb{S}_p[\![q_n-1]\!] := \mathbb{S}^{\mathrm{qdR}}_n$ and have dropped the notation $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}$ (using $\langle - \rangle$ to denote shifts in filtration instead).

$$
\begin{tikzcd}
\mathrm{THH}(R_{n+1}/\mathbb{S}_{n+1}^{\mathrm{qdR}})(1)\left\langle 1\right\rangle[1] \arrow[r, "\rho_n(1)\langle1\rangle[1]"'] \arrow[dddd, bend right, "\widetilde{\varphi}_n"'] \arrow[dd, phantom, "\text{Lemma 4.37}"] & \mathrm{THH}(R_n/\mathbb{S}_n^{\mathrm{qdR}})(1)\left\langle 1\right\rangle[1] \arrow[dr, "\partial_n"] \arrow[dd, "\varphi_{n,\mathrm{qdR}}"] \arrow[dddl, "\widetilde{\varphi}_{n-1}"'] \arrow[ddr, phantom, "(14)"] & \\
 & & \mathrm{THH}(R_n) \arrow[dd, "\varphi_n"] \\
 & \mathrm{THH}(R_n/\mathbb{S}_n^{\mathrm{qdR}})^{tC_{p,\mathrm{ev}}}(1)\left\langle 1\right\rangle[1] \arrow[dr, "\partial_n^{tC_{p,\mathrm{ev}}}"] \arrow[d, phantom, "(12)"] & \\
\mathrm{THH}(R_{n-1}/\mathbb{S}_{n-1}^{\mathrm{qdR}})^{tC_{p,\mathrm{ev}}}(1)\left\langle 1\right\rangle[1] \arrow[ur, "\iota^{tC_{p,\mathrm{ev}}}"'] \arrow[r, "\widetilde{\partial}_n^{tC_{p,\mathrm{ev}}}"] & (F^{\geq *}\mathcal{F}_n(R))^{tC_{p,\mathrm{ev}}} \arrow[r, "\simeq", "\alpha^{tC_{p,\mathrm{ev}}}"'] & \mathrm{THH}(R_n)^{tC_{p,\mathrm{ev}}} \\
\mathrm{THH}(R_n/\mathbb{S}_n^{\mathrm{qdR}})^{tC_{p,\mathrm{ev}}}(1)\left\langle 1\right\rangle[1] \arrow[u, "\rho_{n-1}^{tC_{p,\mathrm{ev}}}(1)\langle1\rangle[1]"] \arrow[urr, bend right=10, "\partial_n^{tC_{p,\mathrm{ev}}}"'] \arrow[ur, phantom, "(13)"] & &
\end{tikzcd}
$$

The outermost pentagon is the one we are after. ☐

Lemma 4.40. *For any bounded qcqs quasi-syntomic p-adic formal scheme X over $\mathbb{Z}_p[\zeta_p]$, write*

$$X_n := X \times_{\mathrm{Spf}\,\mathbb{Z}_p[\zeta_p]} \mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^n}] \quad \text{and} \quad X_\infty := X \times_{\mathrm{Spf}\,\mathbb{Z}_p[\zeta_p]} \mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}.$$

There are functorial-in-X equivalences of synthetic spectra with synthetic circle action

$$\left(\varprojlim_n \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(X_n)\right)^{tC_{p,\mathrm{ev}}} \simeq \varprojlim_n \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(X_n)\right)^{tC_{p,\mathrm{ev}}},$$

$$\left(\varprojlim_n \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(X_n/\mathbb{S}_p[\![q_n-1]\!])\right)^{tC_{p,\mathrm{ev}}} \simeq \varprojlim_n \left(\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(X_n/\mathbb{S}_p[\![q_n-1]\!])\right)^{tC_{p,\mathrm{ev}}},$$

where the transition maps in the first two limits are the transfer maps and in the last two limits are the maps ρ_n.

Proof. We may first restrict to $X = \mathrm{Spf}\,R$ affine, since the totalisation for the Čech nerve of a finite Zariski cover by affines is a finite limit, by the qcqs condition on X. In the affine case, the motivic filtrations in question are all the even filtrations on connective E_∞-ring spectra, which by [AR24, Remark 2.13] are all connective in the Postnikov t-structure on synthetic spectra. The lemma follows after an application of [AR24, Lemma 2.80]. ☐

We can now prove a stronger form of Theorem A, the main result of this paper.

Theorem 4.41. *For any bounded qcqs quasi-syntomic p-adic formal X over $\mathbb{Z}_p[\zeta_p]$, write*

$$X_n := X \times_{\mathrm{Spf}\,\mathbb{Z}_p[\zeta_p]} \mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^n}] \quad \text{and} \quad X_\infty := X \times_{\mathrm{Spf}\,\mathbb{Z}_p[\zeta_p]} \mathrm{Spf}\,\mathbb{Z}_p^{\mathrm{cyc}}.$$

There is a functorial-in-X $\mathbb{Z}_p^\times$-equivariant equivalence of $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(X_\infty)$ modules in cyclotomic synthetic spectra:

$$\varprojlim_n \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(X_n) \simeq \varprojlim_n \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(X_n/\mathbb{S}_p[\![q_n-1]\!])(1)[1],$$

where the transition maps on the right hand side are the maps ρ_n (resp. $\Phi\rho_n$), and where the cyclotomic synthetic Frobenius on the right hand side is given by the following composition

$$\begin{aligned}
\varprojlim_{n\geq 1} \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(X_n/\mathbb{S}_p[\![q_n-1]\!])(1)[1] &\xrightarrow[\text{shift}]{\simeq} \varprojlim_{n\geq 1} \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(X_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!])(1)[1] \\
&\xrightarrow{(\widetilde{\varphi}_n)} \varprojlim_{n\geq 1} \left(\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(X_n/\mathbb{S}_p[\![q_n-1]\!])(1)[1]\right)^{tC_{p,\mathrm{ev}}}, \\
&\simeq \left(\varprojlim_{n\geq 1} \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{THH}(X_n/\mathbb{S}_p[\![q_n-1]\!])(1)[1]\right)^{tC_{p,\mathrm{ev}}},
\end{aligned}$$

where we use Lemma 4.37 to see that $\widetilde{\varphi}_n$ commutes with the transition maps in the limit and the last equivalence is due to Lemma 4.40.

Proof. We will implicitly use Lemma 4.40 to commute the Tate construction past limits over transfers.

First, taking the limit in Theorem 4.32, we get a fibre sequence of S^1-equivariant spectra

$$\varprojlim_n \mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(R_n) \to \varprojlim_n \mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(R_n/\mathbb{S}_p[\![q_n - 1]\!]) \to \varprojlim_n \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}} \mathrm{THH}(R_n/\mathbb{S}_p[\![q_n - 1]\!])(1)[2],$$

where the transition maps in the middle term are given by

$$\begin{aligned}\mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1} - 1]\!]) &\simeq \mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(R_n/\mathbb{S}_p[\![q_n - 1]\!]) \otimes_{\mathbb{S}_p[\![p^{-n}\mathbb{Z}_p]\!]} \mathbb{S}_p[\![p^{-n-1}\mathbb{Z}_p]\!] \\ &\xrightarrow{\mathrm{id}\otimes\mathrm{proj}} \mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(R_n/\mathbb{S}_p[\![q_n - 1]\!]) \\ &\xrightarrow{\sum_{i=0}^{p-1} \psi^{g^{ip^{n-1}}}} \mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(R_n/\mathbb{S}_p[\![q_n - 1]\!]),\end{aligned}$$

and the transition maps in the third limit are the maps ρ_n. Here, we have fixed a topological generator g of $1 + p\mathbb{Z}_p$.

We first claim that the middle term in the above fibre sequence is zero. To do this, it suffices to show that the underlying spectrum is zero, so we may forget the circle action. Moreover, the motivic filtration being complete and exhaustive, it suffices to show the associated graded vanishes. Working non-S^1-equivariantly (or, more accurately, non-$\mathrm{gr}^*\mathbb{T}_{\mathrm{ev}}$-equivariantly), the middle term is a $\mathbb{Z}_p[\zeta_p]$-algebra, and since we are working in the p-complete category (equivalently, the $(\zeta_p - 1)$-complete category) we may work mod $\zeta_p - 1$. In this case, it suffices to check that the transition maps

$$\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1} - 1]\!])/(\zeta_p - 1) \to \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(R_n/\mathbb{S}_p[\![q_n - 1]\!])/(\zeta_p - 1)$$

of sheaves are zero, as the limit over zero maps is zero. By definition, this transition map factors through the map

$$\sum_{i=0}^{p-1} \psi^{g^{ip^{n-1}}} : \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(R_n/\mathbb{S}_p[\![q_n - 1]\!])/(\zeta_p - 1) \to \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(R_n/\mathbb{S}_p[\![q_n - 1]\!])/(\zeta_p - 1).$$

Lemma 4.35 identifies this map with the multiplication by p map, which is zero mod $\zeta_p - 1$ as claimed.

It remains to study the cyclotomic synthetic Frobenius. For this, we take limits in n of the diagram (11) in Proposition 4.39 to obtain the following commutative pentagon. Again, in order to fit the diagram onto the page, we have abbreviated $\mathbb{S}_p[\![q_n - 1]\!] := \mathbb{S}^{\mathrm{qdR}}_n$ and have dropped the notation $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}$ (using $\langle - \rangle$ to denote shifts in filtration instead).

$$\begin{tikzcd}
& \varprojlim_n \mathrm{THH}(R_{n+1}/\mathbb{S}^{\mathrm{qdR}}_{n+1})(1)\langle 1\rangle[1] \arrow[dl, "(\rho_n(1)\langle 1\rangle[1])_n"'] \arrow[dr, "(\widetilde{\varphi}_n)_n"] & \\
\varprojlim_n \mathrm{THH}(R_n/\mathbb{S}^{\mathrm{qdR}}_n)(1)\langle 1\rangle[1] \arrow[d, "(\partial_n)_n"'] & & \varprojlim_n \mathrm{THH}(R_n/\mathbb{S}^{\mathrm{qdR}}_n)^{tC_{p,\mathrm{ev}}}(1)[1] \arrow[d, "(\partial_n^{tC_{p,\mathrm{ev}}})_n"] \\
\varprojlim_n \mathrm{THH}(R_n) \arrow[rr, "(\varphi_{\mathrm{THH}(R_n)})_n"'] & & \varprojlim_n \mathrm{THH}(R_n)^{tC_{p,\mathrm{ev}}}
\end{tikzcd}$$

Now, we have established that the maps $(\partial_n)_n$ and $(\partial_n^{tC_{p,\mathrm{ev}}})_n$ are both equivalences. Moreover, since the limit is over $\rho_n(1)\langle 1\rangle[1])$ anyway, the map $(\rho_n(1)\langle 1\rangle[1])_n$ is the shift down map with inverse the shift up map appearing in the statement of Theorem 4.41. The claim follows. $\square$

Remark 4.42. We elaborate on Remark 1.3. Under the isomorphism

$$\mathbb{Z}_p[\![q_1 - 1]\!]\{1\} \cong \pi_2 \mathrm{TP}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q_1 - 1]\!]) \simeq \pi_2 \mathrm{ku}_p^{(-1),tS^1} \cong \mathbb{Z}_p[\![q_1 - 1]\!]t^{-1},$$

a generator of the Breuil-Kisin twist goes to the class t^{-1}. On the other hand, by [BL22a, Proposition 2.6.1], a generator of the Breuil-Kisin twist is

$$\frac{1}{q-1}\log_{\Delta}(q^p)$$

where $\log_{\Delta}$ is the prismatic logarithm. Thus, up to a unit $\alpha \in \mathbb{Z}_p[\![q_1 - 1]\!]^\times$, we have

$$t^{-1} = \alpha \cdot \frac{1}{q-1}\log_{\Delta}(q^p)$$

and so $\beta = (q-1)t^{-1} = \alpha \cdot \log_{\Delta}(q^p)$ (c.f. [BM23, Construction 2.7]).

Since the previous isomorphisms were $\mathbb{Z}_p^\times$-equivariant, the above equality of elements in $\mathbb{Z}_p[\![q-1]\!]\{1\}$ must also be $\mathbb{Z}_p^\times$-equivariant. We claim that $\mathbb{Z}_p^\times$ acts by multiplication on $\log_{\Delta}(q^p)$: by definition of the prismatic logarithm ([BL22a, Section 2.2]) it suffices to check that

$$q^{\gamma p^i} - 1 \equiv \gamma(q^{p^i} - 1) \pmod{[p^i]_q^2},$$

which is an easy calculation. Hence, $\mathbb{Z}_p^\times$ acts trivially on α, so that $\alpha \in \mathbb{Z}_p^\times$. In fact, this unit α should be 1, though we will not need it here. A related discussion is given in [BM23, Section 2.1].

Construction 4.43. We endow

$$\mathscr{M}_{\mathrm{Iw}} := \varprojlim_n \mathbb{S}_p[\![q_n - 1]\!],$$

the limit over the $\mathbb{S}_p[\![q_1-1]\!]$-linear projection maps $\mathrm{proj}_n : \mathbb{S}_p[\![q_{n+1}-1]\!] \to \mathbb{S}_p[\![q_n-1]\!]$, with a cyclotomic structure.

The circle action is the trivial one. The cyclotomic Frobenius is given by the composition

$$\varprojlim_{n\geq 1} \mathbb{S}_p[\![q_n-1]\!] \xrightarrow{\text{shift}} \varprojlim_{n\geq 1} \mathbb{S}_p[\![q_{n+1}-1]\!] \to \varprojlim_{n\geq 1} \mathbb{S}_p[\![q_n-1]\!] \simeq \varprojlim_{n\geq 1} \mathbb{S}_p[\![q_n-1]\!]^{tC_p} \simeq \left(\varprojlim_{n\geq 1} \mathbb{S}_p[\![q_n-1]\!]\right)^{tC_p}$$

where the second arrow is the ring map

$$\widetilde{\varphi} : \mathbb{S}_p[\![q_{n+1}-1]\!] \xrightarrow{\simeq} \mathbb{S}_p[\![q_n-1]\!], \quad q_{n+1} \mapsto q_n.$$

One checks that the tautological $\mathbb{S}_p[\![q_1-1]\!]$-module structure on $\varprojlim_n \mathbb{S}_p[\![q_n-1]\!]$ in Sp extends to a $\mathbb{S}_p[\![q_1-1]\!]^{\mathrm{Tate}}$-module structure on $\varprojlim_n \mathbb{S}_p[\![q_n-1]\!]$ in CycSp. Moreover, this cyclotomic spectrum $\mathscr{M}_{\mathrm{Iw}}$ is the same as that in the introduction, as one can check that the following diagram canonically commutes

$$\begin{tikzcd}
\mathbb{S}_p[\![q_{n+1}-1]\!] \arrow[r, "\widetilde{\varphi}^n"] \arrow[d, "\mathrm{proj}"] & \mathbb{S}_p[\![q_1-1]\!] \arrow[d, "\tau"] \\
\mathbb{S}_p[\![q_n-1]\!] \arrow[r, "\widetilde{\varphi}^{n-1}"] & \mathbb{S}_p[\![q_1-1]\!]
\end{tikzcd}$$

where both horizontal maps are equivalences, and $\tau : \mathbb{S}_p[\![q_1-1]\!] \to \mathbb{S}_p[\![q_1-1]\!]$ is given by

$$q_1^k \mapsto \begin{cases} 0 & p \nmid k, \\ q_1^{k/p} & p|k. \end{cases}$$

Remark 4.44. One may have noticed a difference in the construction of $\mathscr{M}_{\mathrm{Iw}}$ given here as compared to the one given in Theorem A. This is purely a matter of convention. See Remark 2.33 for more.

Corollary 4.45. *Continue with notation as in Theorem 4.41. If X is such that for each i, $\mathrm{gr}^i_{\mathrm{Nyg}} \widehat{\Delta}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}$ is almost perfect over $\mathbb{Z}_p[\![q_1-1]\!]$, then there is an equivalence of cyclotomic synthetic spectra*

$$\varprojlim_n \mathrm{Fil}^{\geq *}_{\mathrm{mot}} \mathrm{THH}(X_n) \simeq \left(\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}} \mathrm{THH}(X/\mathbb{S}_p[\![q_1-1]\!])\right)(1)[1] \otimes_{\mathbb{S}_p[\![q_1-1]\!]^{\mathrm{Tate}}} \varprojlim_n \mathbb{S}_p[\![q_n-1]\!].$$

Proof. By the theorem, and using Lemma 4.29 and Lemma 4.31, it suffices to show that the canonical map of spectra

$$\varprojlim_n \left(\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}} \mathrm{THH}(X/\mathbb{S}_p[\![q_1-1]\!])(1)[1] \otimes_{\mathbb{S}_p[\![q_1-1]\!]} \mathbb{S}_p[\![q_n-1]\!]\right)$$
$$\to \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}} \mathrm{THH}(X/\mathbb{S}_p[\![q_1-1]\!])(1)[2] \otimes_{\mathbb{S}_p[\![q_1-1]\!]} \varprojlim_n \mathbb{S}_p[\![q_n-1]\!]$$

is an equivalence under the extra condition on X. The motivic filtration being complete, we may pass to associated graded. Notice that $\mathbb{S}_p[\![p^{-1}\mathbb{Z}_p]\!]$ is a Noetherian E_∞-ring: $\pi_0 \mathbb{S}_p[\![p^{-1}\mathbb{Z}_p]\!] \cong \mathbb{Z}_p[\![T]\!]$ is Noetherian and for $i > 0$,

$$\pi_i \mathbb{S}_p[\![p^{-1}\mathbb{Z}_p]\!] \simeq (\pi_i \mathbb{S}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![p^{-1}\mathbb{Z}_p]\!]$$

is finitely generated over $\mathbb{Z}_p[\![p^{-1}\mathbb{Z}_p]\!]$. Moreover, the condition on X combined with Remark 4.34 guarantees that $\mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(X/\mathbb{S}_p[\![q_1-1]\!])$ is almost perfect over $\mathbb{S}_p[\![q_1-1]\!]$ for each $*$. We may then conclude by Lemma 4.33. $\square$

The following is just the associated graded version of Corollary 4.45.

Corollary 4.46. *Fix $r \geq 0$. Let X and X_n be as in Theorem 4.41. If $\mathrm{gr}^i_{\mathrm{Nyg}}\widehat{\Delta}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}$ is almost perfect (i.e. pseudo-coherent) as a $\mathbb{Z}_p[\![q-1]\!]$-module for all i, then*

$$\varprojlim_n (\mathrm{Fil}^{\geq r}_{\mathrm{Nyg}}\widehat{\Delta}_{X_n})\{r\} \simeq (\mathrm{Fil}^{\geq r-1}_{\mathrm{Nyg}}\widehat{\Delta}^{(1)}_{X_1/\mathbb{Z}_p[\![q-1]\!]})\{r-1\}\widehat{\otimes}_{\mathbb{Z}_p[\![q_1-1]\!]}\varprojlim_{n\geq 1}\mathbb{Z}_p[\![q_n-1]\!](1)[-1]$$

$$\varprojlim_n \widehat{\Delta}_{X_n}\{r\} \simeq \widehat{\Delta}^{(1)}_{X_1/\mathbb{Z}_p[\![q-1]\!]}\{r-1\}\widehat{\otimes}_{\mathbb{Z}_p[\![q_1-1]\!]}\varprojlim_{n\geq 1}\mathbb{Z}_p[\![q_n-1]\!](1)[-1]$$

where the right hand side is $(p, q-1)$-adically completed.

Corollary 4.47. *Suppose X is a qcqs proper regular bounded p-adic formal scheme over $\mathbb{Z}_p[\zeta_p]$. Write $X_n := X \times_{\mathrm{Spf}\,\mathbb{Z}_p[\zeta_p]} \mathrm{Spf}\,\mathbb{Z}_p[\zeta_{p^n}]$. Then, for any $r \in \mathbb{Z}$, there is a $\mathbb{Z}_p^\times$-equivariant equivalence of objects in $\mathcal{D}(\mathbb{Z}_p)^\wedge_p$*

$$\varprojlim_n \mathbb{Z}_p(r)^{\mathrm{syn}}(X_n) \simeq \mathrm{fib}\left(\mathrm{Fil}^{r-1}_{\mathrm{Nyg}}\widehat{\Delta}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}\{r-1\}\otimes_{\mathbb{Z}_p[\![q_1-1]\!]}\varprojlim_\tau \mathbb{Z}_p[\![q_1-1]\!] \xrightarrow{\varphi_X\{r-1\}\otimes\mathrm{shift}-\mathrm{id}} \widehat{\Delta}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]}\{r-1\}\otimes_{\mathbb{Z}_p[\![q_1-1]\!]}\varprojlim_\tau \mathbb{Z}_p[\![q_1-1]\!]\right)(1)[-1]$$

where the transition maps on the limit on the left hand side are given by the transfer maps in prismatic cohomology and where (1) is the Tate twist.

Proof. This follows from the previous corollary and the definition of syntomic cohomology as the fibre of $\mathrm{can}-\varphi$, once we verify that $\mathrm{gr}^i_{\mathrm{Nyg}}\widehat{\Delta}^{(1)}_{X_1/\mathbb{Z}_p[\![q-1]\!]}$ is perfect over $\mathbb{Z}_p[\![q_1-1]\!]$ for X a proper regular Noetherian p-adic formal scheme over $\mathbb{Z}_p[\zeta_p]$.

By [BL22a, Remark 5.1.2], we have

$$\mathrm{gr}^i_{\mathrm{Nyg}}\widehat{\Delta}^{(1)}_{X/\mathbb{Z}_p[\![q-1]\!]} \simeq \mathrm{Fil}^{\mathrm{conj}}_i\overline{\Delta}_{X/\mathbb{Z}_p[\![q-1]\!]}\{i\}$$

where the right hand side is the (increasing) conjugate filtration on relative Hodge-Tate cohomology as defined in [BS22, Construction 7.6]. Now,

$$\mathrm{gr}^{\mathrm{conj}}_j\overline{\Delta}_{X/\mathbb{Z}_p[\![q-1]\!]} \simeq (\bigwedge_X^j \mathbb{L}_{X/\mathbb{Z}_p[\zeta_p]})^\wedge_p[-j]\{j\}.$$

However, as X is regular over $\mathbb{Z}_p[\zeta_p]$, it follows that $(\mathbb{L}_{-/\mathbb{Z}_p[\zeta_p]})^\wedge_p$ is a perfect quasi-coherent sheaf on X. Properness of X then implies that

$$(\mathbb{L}_{X/\mathbb{Z}_p[\zeta_p]})^\wedge_p := \mathrm{R}\Gamma(X, (\mathbb{L}_{-/\mathbb{Z}_p[\zeta_p]})^\wedge_p)$$

is a perfect $\mathbb{Z}_p[\zeta_p]$-module. All exterior powers are then also perfect, i.e. $(\bigwedge_X^j \mathbb{L}_{X/\mathbb{Z}_p[\zeta_p]})^\wedge_p[-j]\{j\}$ is a perfect $\mathbb{Z}_p[\zeta_p]$-module. Since $\mathbb{Z}_p[\zeta_p]$ is a perfect $\mathbb{Z}_p[\![q-1]\!]$-module, it follows that $\mathrm{gr}^{\mathrm{conj}}_j\overline{\Delta}_{X/\mathbb{Z}_p[\![q-1]\!]}$ is a perfect $\mathbb{Z}_p[\![q-1]\!]$-module. Since $\mathrm{Fil}^{\mathrm{conj}}_{-1}\overline{\Delta}_{X/\mathbb{Z}_p[\![q-1]\!]} \simeq 0$, it follows by a finite induction that $\mathrm{Fil}^{\mathrm{conj}}_i\overline{\Delta}_{X/\mathbb{Z}_p[\![q-1]\!]}\{i\}$ is itself a perfect $\mathbb{Z}_p[\![q-1]\!]$-module, as required. □

4.5 de Rham Cohomology

In this section we prove Corollary D in Corollary 4.54 below. Forgetting motivic filtrations, one just needs to apply $-\otimes_{\mathrm{THH}(\mathbb{Z}_p)}\mathbb{Z}_p$ to Theorem 4.41 since $\mathrm{HH} \simeq \mathrm{THH}\otimes_{\mathrm{THH}(\mathbb{Z}_p)}\mathbb{Z}_p$. However, one needs to keep track of motivic filtrations, which is made difficult by the fact that none of $\mathrm{THH}(\mathbb{Z}_p) \to \mathbb{Z}_p$, $\mathrm{THH}(-) \to \mathrm{HH}(-)$, or $\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!]) \to \mathrm{HH}(-/\mathbb{Z}_p[\![q_n-1]\!])$ are p-completely eff. Thus, we retrace our steps.

Lemma 4.48. *There are equivalences of sheaves of S^1-equivariant spectra on $\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$:*

$$\begin{aligned}\mathrm{HH}(-/\mathbb{Z}_p[\![q_n-1]\!]) &\simeq \mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])\otimes_{\mathrm{THH}(\mathbb{Z}_p)}\mathbb{Z}_p^{\mathrm{triv}}\\ &\simeq \mathrm{HH}(-)\otimes_{\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}])}\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n-1]\!])\end{aligned}$$

where the right hand side denotes the point-wise tensor product.

Proof. It suffices to show that the equivalence holds at the level of pre-sheaves. We then compute:

$$\begin{aligned}\mathrm{THH}(-/\mathbb{S}_p[\![q_n-1]\!])\otimes_{\mathrm{THH}(\mathbb{Z}_p)}\mathbb{Z}_p^{\mathrm{triv}} &\simeq \mathbb{S}_p[\![q_n-1]\!]^{\mathrm{triv}}\otimes_{\mathrm{THH}(\mathbb{S}_p[\![q_n-1]\!])}\mathrm{THH}(-)\otimes_{\mathrm{THH}(\mathbb{Z}_p)}\mathbb{Z}_p^{\mathrm{triv}}\\ &\simeq \mathbb{S}_p[\![q_n-1]\!]^{\mathrm{triv}}\otimes_{\mathrm{THH}(\mathbb{S}_p[\![q_n-1]\!])}\mathrm{HH}(-)\\ &\simeq \mathbb{Z}_p[\![q_n-1]\!]^{\mathrm{triv}}\otimes_{\mathrm{THH}(\mathbb{S}_p[\![q_n-1]\!])\otimes\mathbb{Z}_p}\mathrm{HH}(-)\\ &\simeq \mathrm{HH}(-/\mathbb{Z}_p[\![q_n-1]\!]).\end{aligned}$$

This is the first equivalence. The second equivalence follows since

$$\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n-1]\!])\simeq \mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}])\otimes_{\mathrm{THH}(\mathbb{S}_p[\![q_n-1]\!])}\mathbb{S}_p[\![q_n-1]\!]^{\mathrm{triv}}.$$

□

Corollary 4.49. *For R a QRSP, $\mathrm{HH}(R)$ is even. If R is furthermore a $\mathbb{Z}_p[\zeta_{p^n}]$-algebra, then $\mathrm{HH}(R/\mathbb{Z}_p[\![q_n-1]\!])$ is also even. The motivic filtrations are moreover the double-speed Postnikov filtration.*

Proof. The first is [BMS19, Lemma 5.14]. The second follows from the lemma using that $\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}])\to\mathrm{THH}(\mathbb{Z}_p[\zeta_{p^n}]/\mathbb{S}_p[\![q_n-1]\!])$ is p-completely eff by Lemma 4.24. The final claim is immediate from evenness and the fact that the motivic filtration is the even filtration. □

The following is a version of Proposition 4.26, and is proved similarly.

Lemma 4.50. *For any $n\geq 1$ and any choice of topological generator $\gamma_n\in 1+p^n\mathbb{Z}_p$, there is a fibre sequence of sheaves (in fact, a fibre sequence at the level of pre-sheaves) of synthetic spectra with synthetic circle action on $\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_{p^n}]}$*

$$\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(-)\to \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(-/\mathbb{S}_p[\![q_n-1]\!])\xrightarrow{\psi^{\gamma_n}_{\circ\circ}}\mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{HH}(-/\mathbb{S}_p[\![q_n-1]\!])(1)[2].$$

Proof. On underlying sheaves of spectra, one can just apply $-\otimes_{\mathrm{THH}(\mathbb{Z}_p)}\mathbb{Z}_p$ to the underlying fibre sequence of Proposition 4.26. For motivic filtrations, by quasi-syntomic descent to QRSP $\mathbb{Z}_p[\zeta_{p^n}]$-algebras R, we can just apply $\tau_{\geq 2*}$ to the fibre sequence on underlying S^1-spectra for R. □

The following is proved in the exact same way as Corollary 4.28.

Corollary 4.51. *For R a QRSP $\mathbb{Z}_p[\zeta_p]$-algebra, one has an equivalence*

$$\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R\otimes_{\mathbb{Z}_p[\zeta_p]}\mathbb{Z}_p[\zeta_{p^n}])\simeq\tau_{\geq 2*-1}\mathrm{HH}(R\otimes_{\mathbb{Z}_p[\zeta_p]}\mathbb{Z}_p[\zeta_{p^n}]).$$

Proposition 4.52. *Let g be a topological generator of $(1+p\mathbb{Z}_p)$, and $R\in\mathrm{QSyn}_{\mathbb{Z}_p[\zeta_p]}$. Set $R_n:=R\otimes_{\mathbb{Z}_p[\zeta_p]}\mathbb{Z}_p[\zeta_{p^n}]$. The transfer map*

$$\mathrm{tr}:\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_{n+1})\to\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n)$$

sits in the following functorial-in-R commutative diagram of $\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n)$-modules in synthetic spectra with synthetic S^1-action:

$$\begin{tikzcd}
\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_{n+1}) \arrow[r] \arrow[dd, "\mathrm{tr}"'] & \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!]) \arrow[r, "\psi^{g^{p^n}}_{\circ\circ}"] \arrow[d, "\rho_n"'] & \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{HH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!])(1)[2] \arrow[d, "\rho_n(1)[2]"'] \\
 & \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n/\mathbb{S}_p[\![q_n-1]\!]) \arrow[r, "\psi^{g^{p^n}}_{\circ\circ}"] \arrow[d, "\sum_{i=0}^{p-1}\psi^{g^{ip^{n-1}}}"] & \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{HH}(R_n/\mathbb{S}_p[\![q_n-1]\!])(1)[2] \arrow[d, equal, "\mathrm{id}"] \\
\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n) \arrow[r] & \mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n/\mathbb{S}_p[\![q_n-1]\!]) \arrow[r, "\psi^{g^{p^{n-1}}}_{\circ\circ}"] & \mathrm{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{HH}(R_n/\mathbb{S}_p[\![q_n-1]\!])(1)[2]
\end{tikzcd}$$

where the top and bottom rows are the fibre sequences of Lemma 4.50.

Proof. Forgetting filtrations, we can just apply $-\otimes_{\mathrm{THH}(\mathbb{Z}_p)}\mathbb{Z}_p$ to the underlying unfiltered diagram in Theorem 4.32. For motivic filtrations, just as in the proof of Theorem 4.32, we can take connective covers in the underlying unfiltered diagram for R a QRSP, using Corollary 4.49 to identify

$$\tau_{\geq 2*}\mathrm{HH}(R_n/\mathbb{Z}_p[\![q_n-1]\!])\simeq\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n/\mathbb{Z}_p[\![q_n-1]\!])$$

as well as Corollary 4.51 to identify $\tau_{\geq 2*-1}\mathrm{HH}(R_n)\simeq\mathrm{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n)$. □

Proposition 4.53. *For any bounded qcqs quasi-syntomic p-adic formal scheme X over $\mathbb{Z}_p[\zeta_p]$, write*

$$X_n := X \times_{\operatorname{Spf}\mathbb{Z}_p[\zeta_p]} \operatorname{Spf}\mathbb{Z}_p[\zeta_{p^n}].$$

There is a functorial-in-X $\mathbb{Z}_p^\times$-equivariant equivalence of synthetic spectra with synthetic circle action:

$$\varprojlim_n \operatorname{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(X_n) \simeq \varprojlim_n \operatorname{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{HH}(X_n/\mathbb{Z}_p[\![q_n-1]\!])(1)[1]$$

where the transition maps on the right hand side are given by

$$\begin{aligned}\operatorname{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{HH}(X_n/\mathbb{Z}_p[\![q_n-1]\!]) &\simeq \left(\operatorname{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{HH}(X_1/\mathbb{Z}_p[\![q_1-1]\!])\right)\otimes_{\mathbb{Z}_p[\![q_1-1]\!]}\mathbb{Z}_p[\![q_n-1]\!]\\ &\xrightarrow{\mathrm{id}\otimes\mathrm{proj}} \left(\operatorname{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{HH}(X_1/\mathbb{Z}_p[\![q_1-1]\!])\right)\otimes_{\mathbb{Z}_p[\![q_1-1]\!]}\mathbb{Z}_p[\![q_{n-1}-1]\!]\\ &\simeq \operatorname{Fil}^{\geq *-1}_{\mathrm{mot}}\mathrm{HH}(X_{n-1}/\mathbb{Z}_p[\![q_{n-1}-1]\!]).\end{aligned}$$

Proof. As in the proof of Theorem 4.41, as a consequence of Proposition 4.52 it suffices to show that

$$\varprojlim_n \operatorname{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])$$

is zero where the transition maps in the middle term are given by

$$\begin{aligned}\operatorname{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_{n+1}/\mathbb{S}_p[\![q_{n+1}-1]\!]) &\simeq \operatorname{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n/\mathbb{S}_p[\![q_n-1]\!])\otimes_{\mathbb{S}_p[\![p^{-n}\mathbb{Z}_p]\!]}\mathbb{S}_p[\![p^{-n-1}\mathbb{Z}_p]\!]\\ &\xrightarrow{\mathrm{id}\otimes\mathrm{proj}} \operatorname{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n/\mathbb{S}_p[\![q_n-1]\!])\\ &\xrightarrow{\sum_{i=0}^{p-1}\psi^{g^{ip^{n-1}}}} \operatorname{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n/\mathbb{S}_p[\![q_n-1]\!]).\end{aligned}$$

Since $\varprojlim_n \operatorname{Fil}^{\geq *}_{\mathrm{mot}}\mathrm{HH}(R_n/\mathbb{S}_p[\![q_n-1]\!])$ is a quasi-syntomic sheaf in R, it suffices by quasi-syntomic descent to show the vanishing when R is a QRSP. Using Corollary 4.49, we need to show that

$$\varprojlim_n \tau_{\geq 2*}\mathrm{HH}(R_n/\mathbb{S}_p[\![q_n-1]\!])$$

vanishes. By the Milnor sequence, it suffices to show that $\varprojlim_n \mathrm{HH}(R_n/\mathbb{S}_p[\![q_n-1]\!])=0$. This follows from

$$\varprojlim_n \mathrm{THH}(R_n/\mathbb{S}_p[\![q_n-1]\!])\simeq 0$$

established in the proof of Theorem 4.41 by applying $-\otimes_{\mathrm{THH}(\mathbb{Z}_p)}\mathbb{Z}_p$, noting that this commutes with the limit as a consequence of Lemma 4.33. ☐

The same argument as in Corollary 4.46 then yields the following.

Corollary 4.54. *Suppose X is a qcqs bounded p-adic formal scheme over $\mathbb{Z}_p[\zeta_p]$ such that $\mathbb{L}_{X/\mathbb{Z}_p[\![q_1-1]\!]}$ is pseudo-coherent as a $\mathbb{Z}_p[\![q_1-1]\!]$-module, then there are equivalences*

$$\begin{aligned}\varprojlim_n \widehat{\mathrm{dR}}^{\geq r}_{X_n} &\simeq \widehat{\mathrm{dR}}^{\geq r-1}_{X/\mathbb{Z}_p[\![q_1-1]\!]}\widehat{\otimes}_{\mathbb{Z}_p[\![q_1-1]\!]}\varprojlim_n \mathbb{Z}_p[\![q_n-1]\!](1)[-1],\\ \varprojlim_n \widehat{\mathrm{dR}}_{X_n} &\simeq \widehat{\mathrm{dR}}_{X/\mathbb{Z}_p[\![q_1-1]\!]}\widehat{\otimes}_{\mathbb{Z}_p[\![q_1-1]\!]}\varprojlim_n \mathbb{Z}_p[\![q_n-1]\!](1)[-1],\\ \varprojlim_n \widehat{\mathrm{dR}}^{\leq r}_{X_n} &\simeq \widehat{\mathrm{dR}}^{\leq r-1}_{X/\mathbb{Z}_p[\![q_1-1]\!]}\widehat{\otimes}_{\mathbb{Z}_p[\![q_1-1]\!]}\varprojlim_n \mathbb{Z}_p[\![q_n-1]\!](1)[-1],\end{aligned}$$

where $X_n := X\times_{\operatorname{Spf}\mathbb{Z}_p[\zeta_p]}\operatorname{Spf}\mathbb{Z}_p[\zeta_{p^n}]$, and the right hand side of all three equivalences are (p,q_1-1)-adically completed.

> **Remark 4.55.** The usual transitivity triangle for the cotangent complex, and the standard computation $\mathbb{L}_{\mathbb{Z}_p[\zeta_p]/\mathbb{Z}_p[\![q_1-1]\!]}\simeq\mathbb{Z}_p[\zeta_p]\{1\}[1]$ yield a fibre sequence
>
> $$\mathrm{R}\Gamma(X,\mathcal{O}_X)[1]\to\mathbb{L}_{X/\mathbb{Z}_p[\![q_1-1]\!]}\to\mathbb{L}_{X/\mathbb{Z}_p[\zeta_p]}.$$
>
> Here, $\mathrm{R}\Gamma(X,\mathcal{O}_X)$ is the coherent cohomology of the structure sheaf of X. Since $\mathbb{Z}_p[\zeta_p]$ is itself a pseudo-coherent $\mathbb{Z}_p[\![q_1-1]\!]$-algebra, [SP, Tag064Z] implies that $\mathbb{L}_{X/\mathbb{Z}_p[\![q_1-1]\!]}$ is a pseudo-coherent $\mathbb{Z}_p[\![q_1-1]\!]$-module if $\mathrm{R}\Gamma(X,\mathcal{O}_X)$ and $\mathbb{L}_{X/\mathbb{Z}_p[\zeta_p]}$ are pseudo-coherent $\mathbb{Z}_p[\zeta_p]$-modules.
>
> If X is proper, then $\mathrm{R}\Gamma(X,\mathcal{O}_X)$ is a perfect $\mathbb{Z}_p[\zeta_p]$-module. We also saw in the proof of Corollary 4.47 that $\mathbb{L}_{X/\mathbb{Z}_p[\zeta_p]}$ is a perfect quasi-coherent sheaf whenever X is proper regular over $\mathbb{Z}_p[\zeta_p]$. Thus, Corol-

lary 4.54 holds for proper regular X over $\mathbb{Z}_p[\zeta_p]$.

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