Differential Theory on Schemes

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Abstract

These are notes I made while learning about the theory of differential calculus on schemes, as well as abelian and non-abelian Hodge theory. Emphasis is on developing the language, rather than gaining intuition. All intuition should come from the analytic perspective of differential geometry and classical Hodge theory.

Contents

Notations, Conventions, and other Remarks

For the classical viewpoint of algebraic geometry in the field of differential geometry and Hodge theory, the book [GH14] is the gold standard.

Throughout, all rings will be commutative rings with identity.

1 GAGA (and GAGA-esque results)

Serre's GAGA, or *Géometrie Algébrique et Géométrie Analytique*, is one of the most important papers in algebraic geometry. It proves general results comparing objects in analytic (and differential) geometry with algebraic objects, thus paving the way for analytic methods to prove theorems in algebraic geometry.

The main reference is [Har77, Appendix B]. A lot of the properties/theorems about the analytification functors will be taken from [GR02, Exposé XII]. However, [Nee07] is also quite good, especially since it assumes very few prerequisites.

1.1 Analytification of $\mathbb{C}$-schemes

Definition. A *complex analytic variety* is a locally ringed space $(Z, \mathcal{O}_Z)$ where $Z \subset \mathbb{C}^n$ is the vanishing locus of a finite set of holomorphic functions f_i on $\mathbb{C}^n$, and $\mathcal{O}_Z = \mathcal{O}_{\mathbb{C}^n}/\mathcal{I}_Z$ where $\mathcal{I}_Z$ is the ideal sheaf on $\mathbb{C}^n$ generated by the f_i. Here, $\mathcal{O}_{\mathbb{C}^n}$ is the sheaf of all holomorphic functions on $\mathbb{C}^n$.

A *complex analytic space* is a locally ringed space $(X, \mathcal{O}_X)$ over $\mathbb{C}$ such that it is locally isomorphic as ringed spaces to a complex analytic variety.

Morphisms of complex analytic spaces are just morphisms of locally ringed spaces over $\mathbb{C}$.

Remark 1.1. Of course, every complex manifold is a complex-analytic space. In fact, an everywhere smooth complex-analytic space is a complex manifold.

Suppose X is a scheme locally of finite type over $\mathbb{C}$. We try to endow $X(\mathbb{C})$, the set of complex points of X, with a certain topology and with a ring structure so that it becomes a complex analytic space $(X^{an}, \mathcal{O}_{X^{an}})$. We do this by a local argument (we follow the argument given in [Nee07, Chapter 4-5]).

Suppose R is a finitely generated $\mathbb{C}$-algebra; then there exists $n \geq 1$ such that $\operatorname{Spec} R \hookrightarrow \mathbb{A}^n_{\mathbb{C}}$. On $\mathbb{C}$-points, we have a map of sets $(\operatorname{Spec} R)(\mathbb{C}) \hookrightarrow \mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^n$. We endow $\mathbb{C}^n$ with the usual Euclidean topology, and endow $(\operatorname{Spec} R)(\mathbb{C})$ with the subspace topology. Since R is a finitely generated $\mathbb{C}$-algebra and $\operatorname{Spec} R \hookrightarrow \mathbb{A}^n_{\mathbb{C}}$ is induced by a surjection $\mathbb{C}[x_1, ..., x_n] \to R$, it follows that $\operatorname{Spec} R$ is defined by the vanishing of certain polynomials on $\mathbb{C}^n$. We can pick only finitely many such polynomials as $\mathbb{C}[x_1, ..., x_n]$ is Noetherian. Since polynomials are obviously holomorphic, there is an obvious complex analytic variety structure on $(\operatorname{Spec} R)(\mathbb{C})$. Write $(\operatorname{Spec} R)^{an}, \mathcal{O}_{(\operatorname{Spec} R)^{an}}$ for the complex analytic variety defined by $\operatorname{Spec} R$.

One checks that this construction is independent of the closed immersion $\operatorname{Spec} R \hookrightarrow \mathbb{A}^n_{\mathbb{C}}$ and is independent of n itself. Since the set of $\mathbb{C}$-points is the same as the set of closed points of $\operatorname{Spec} R$, it follows that we have a canonical map

$$\lambda_R : (\operatorname{Spec} R)(\mathbb{C}) \hookrightarrow \operatorname{Spec} R.$$

A priori, this is just a map of sets. However, one can check that this map is in fact continuous for the Zariski topology on $\operatorname{Spec} R$, and in fact defines a map of ringed spaces

$$\lambda_R : (\operatorname{Spec} R)^{an} \to \operatorname{Spec} R.$$

One also checks that this is functorial in R, i.e. if $\varphi : \operatorname{Spec} R \to \operatorname{Spec} S$ is a morphism of affine $\mathbb{C}$-schemes of finite type, then the induced map $\varphi(\mathbb{C}) : (\operatorname{Spec} R)(\mathbb{C}) \to (\operatorname{Spec} S)(\mathbb{C})$ of sets yields a natural map of ringed spaces

$$\varphi^{an} : (\operatorname{Spec} R)^{an} \to (\operatorname{Spec} S)^{an}.$$

For an arbitrary $\mathbb{C}$-scheme X locally of finite type, we glue together the above complex analytic rings on an affine open cover. One checks that this is independent of the chosen affine open cover, and so defines a complex analytic space structure on $X(\mathbb{C})$. We write X^{an} for this complex analytic space. The topology on the underlying space $X(\mathbb{C})$ of X^{an} turns out to be the finest topology such that for every open immersion $(\operatorname{Spec} R) \hookrightarrow X$ (for R finitely generated over $\mathbb{C}$), the induced map $(\operatorname{Spec} R)^{an} \to X^{an}$ is continuous. Of course, we have a canonical map $\lambda_X : X^{an} \to X$ of ringed spaces obtained by glueing.

Example 1.2. It is clear that $(\mathbb{A}^n_{\mathbb{C}})^{an} = \mathbb{C}^n$ and $(\mathbb{P}^n_{\mathbb{C}})^{an} = \mathbb{P}^n(\mathbb{C})$. Also, $(\operatorname{Spec} \mathbb{C})^{an}$ is just a point with the sheaf of constant functions.

Theorem 1.3. *The above construction yields a functor $X \mapsto X^{an}$ from the category of $\mathbb{C}$-schemes locally of finite type, to the category of complex analytic spaces. Viewing both of these categories as full subcategories of the category of locally ringed spaces over $\mathbb{C}$, with inclusion functors $I_{\mathbb{C}-sch}$ and $I_{\mathbb{C}-as}$, the above functor $(-)^{an}$ is equipped with a natural transformation*

$$\lambda : I_{\mathbb{C}-as} \circ (-)^{an} \Rightarrow I_{\mathbb{C}-sch}.$$

These objects also satisfiy the following properties.

1. *The functor $(-)^{an}$ preserves fibre products; in fact, it preserves all finite limits.*
2. *Suppose $x \in X(\mathbb{C})$. The map of local rings $\lambda_{X,x} : \mathcal{O}_{X,x} \to \mathcal{O}_{X^{an},x}$ coming from the morphism $\lambda_X : X^{an} \to X$ of ringed spaces induces an isomorphism on completions*

$$\hat{\lambda}_{X,x} : \hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{X^{an},x}.$$

3. *The complex analytic space X^{an} represents the functor from the category of $\mathbb{C}$-analytic spaces to* Set *given*

$$Y \mapsto \mathrm{Hom}(Y, X)$$

   where $\mathrm{Hom}(Y, X)$ is the set of all morphisms of ringed spaces from the complex analytic space Y to the complex scheme X.
4. *Consider the following properties $\mathcal{P}$ that may be enjoyed by $\mathbb{C}$-schemes and by complex analytic spaces:*
   - *regular*
   - *normal*
   - *reduced*
   - *connected*
   - *irreducible*
   - *of dimension n*

   The $\mathbb{C}$-scheme X satisfies property $\mathcal{P}$ if and only if the complex analytic space X^{an} satisfies property $\mathcal{P}$.
5. *Suppose $f : X \to Y$ is a morphism of $\mathbb{C}$-schemes locally of finite type, with corresponding morphism of complex analytic spaces $f^{an} : X^{an} \to Y^{an}$. Then, f satisfies property $\mathcal{P}$ if and only if f^{an} satisfies property $\mathcal{P}$, where $\mathcal{P}$ can be any of the following properties:*
   - *flat*
   - *unramified*
   - *étale*
   - *smooth*
   - *normal*
   - *reduced*
   - *injective*
   - *separated*
   - *an isomorpism*
   - *an open immersion*

   If f is further a morphism of finite type, then $\mathcal{P}$ can be any of the following as well:
   - *surjective*
   - *dominant*
   - *closed immersion*
   - *immersion*
   - *proper*
   - *finite*

Remark 1.4. In the category of complex analytic spaces, a morphism being étale implies that the morphism is a local isomorphism. This is however not true in the category of $\mathbb{C}$-schemes.

Remark 1.5. We haven't technically defined any of the above properties $\mathcal{P}$ in the analytic category. For most of these properties, the definition carries over *mutatis mutandis* from the definition for schemes. For the rest, we have the following definitions. Throughout, we let $f : Y \to Y'$ be a map of complex analytic spaces.

- f is *proper* if $f^{-1}(K)$ is compact for all $K \subset Y'$ compact. In particular, being proper over $\mathbb{C}$ is the same as being compact.
- f is finite if every fibre of f is a finite set.
- f is smooth if at any point $x \in Y$ there exist open neighbourhoods U of x in Y and V of $f(x)$ in Y' such that $f(U) \subset V$, U and V are (isomorphic to) complex manifolds, and $f|_U : U \to V$ is a smooth map of complex manifolds.

1.2 Coherent Sheaves, and the Statement of GAGA

Definition. Let Y be a complex analytic space. An *analytic coherent sheaf (of $\mathcal{O}_Y$-modules)* is a sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules such that every $p \in Y$ has a open neighbourhood $U \subset Y$ on which there exists an exact sequence

$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0$$

such that, for any polydisc $V \subset U$, there is an exact sequence

$$\Gamma(V, \mathcal{O}_U^m) \to \Gamma(V, \mathcal{O}_U^n) \to \Gamma(V, \mathcal{F}|_U) \to 0.$$

Analytic coherent sheaves behave as expected. For instance, the full subcategory $\mathrm{Coh}^{an}(Y)$ of coherent sheaves on Y is closed under taking kernels, cokernels, internal Homs, etc.

Suppose now that $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on a $\mathbb{C}$-scheme X where X is locally of finite type. Recall the canonical morphism of ringed spaces $\lambda_X : X^{an} \to X$. We have a functor

$$\mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_{X^{an}}), \quad \mathcal{F} \mapsto \mathcal{F}^{an} := \lambda_X^* \mathcal{F},$$

where recall that a pull-back along a map of ringed spaces is a well-defined functor of sheaves on (locally) ringed spaces. On stalks for instance, we have for any closed point $x \in X(\mathbb{C})$,

$$\mathcal{F}_x^{an} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X^{an},x}.$$

Example 1.6. Clearly, $\mathcal{O}_X^{an} = \mathcal{O}_{X^{an}}$.

Of course, by adjunction, we have a canonical map

$$\lambda_X^* : \mathcal{F} \to \lambda_{X,*} \mathcal{F}^{an}.$$

Proposition 1.7. *Suppose X is a scheme locally of finite type over $\mathbb{C}$. The analytification functor $\mathrm{Coh}(X) \to \mathrm{Coh}^{an}(X^{an}), \mathcal{F} \mapsto \mathcal{F}^{an}$ has the following properties.*

1. *Analytification is an exact, faithful (injective on Hom sets) and conservative functor of abelian categories.*
2. *Analytification commutes with all limits.*
3. *Analytification commutes with the usual constructions on $\mathrm{Coh}(-)$, eg. with direct sums, tensor products, internal homs, etc.*
4. *Analytification sends coherent sheaves to coherent sheaves.*
5. *The map $\lambda_X^*(U) : \Gamma(U, \mathcal{F}) \to \Gamma(U^{an}, \mathcal{F}^{an})$ is injective. If U is affine and $V \subset U^{an}$ is a polydisc, then the composition*
$$\Gamma(U, \mathcal{F}) \xrightarrow{\lambda_X^*(U)} \Gamma(U^{an}, \mathcal{F}^{an}) \to \Gamma(V, \mathcal{F}^{an})$$
has dense image (in the Fréchet topology), and the image moreover generates $\Gamma(V, \mathcal{F}^{an})$ as a modules over $\Gamma(V, \mathcal{O}_X^{an})$.
6. *Suppose $\mathcal{G}$ is a sheaf of $\mathcal{O}_X^{an}$-modules on X^{an}, so that $\lambda_{X,*}\mathcal{G}$ is a sheaf of $\mathcal{O}_X$-modules on X. Suppose $\mathcal{F}$ is a coherent algebraic sheaf on X and $\alpha : \mathcal{F} \to \lambda_{X,*}\mathcal{G}$ is a $\mathcal{O}_X$-linear morphism on X. Then, there is a unique map $A : \mathcal{F}^{an} \to \mathcal{G}$ of analytic $\mathcal{O}_X$-modules on X^{an}, such that α is the composite*
$$\mathcal{F} \xrightarrow{\lambda_X^*} \lambda_{X,*}\mathcal{F}^{an} \xrightarrow{\lambda_{X,*}A} \lambda_{X,*}\mathcal{G}.$$

With the analytification functors out of the way, we can now give the main statements of GAGA. We write down the statements as generalized by Groethnedieck in [GR02, Exposé XII.4]. In order to set up the statement, suppose $f : X \to Y$ is a morphism of $\mathbb{C}$-schemes where X and Y are locally of finite type over $\mathbb{C}$. We have a commutative diagram

$$\begin{array}{ccc} X^{an} & \xrightarrow{\lambda_X} & X \\ {\scriptstyle f^{an}}\downarrow & & \downarrow{\scriptstyle f} \\ Y^{an} & \xrightarrow[\lambda_Y]{} & Y \end{array}$$

Suppose also that $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$. Then, for any integer $p \geq 0$, we have morphisms

$$R^p f_* \mathcal{F} \xrightarrow{i} R^p f_*(\lambda_{X,*}\mathcal{F}^{an}) \xrightarrow{j} R^p(f \circ \lambda_X)_* \mathcal{F}^{an} = R^p(\lambda_Y \circ f^{an})_* \mathcal{F}^{an} \xrightarrow{k} \lambda_{Y,*}(R^p f^{an}_* \mathcal{F}^{an}).$$

Here, i is induced by the canonical morphism $\mathcal{F} \to \lambda_{X,*}\mathcal{F}$, and j and k are enduced by the edge morphisms from the Leray spectral sequence associated to the composition of two derived functors (between derived categories). We thus have a canonical morphism

$$\theta_p : (R^p f_* \mathcal{F})^{an} \to R^p f^{an}_*(\mathcal{F}^{an})$$

of analytic $\mathcal{O}_{Y^{an}}$-modules on Y^{an}.

Theorem 1.8 (Serre's GAGA)**.** *Suppose $X, Y, f, \mathcal{F}$, and θ_p are as above. Suppose additionally that f is* proper *and $\mathcal{F}$ is coherent.*

1. *The above morphism θ_p is an isomorphism for all $p \geq 0$.*
2. *The functor $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{an}), \mathcal{F} \mapsto \mathcal{F}^{an}$, is an equivalence of categories.*

The GAGA theorems yield a lot of powerful corollaries, some of which are as follows.

Corollary 1.8.1. *For X a proper $\mathbb{C}$-scheme and $\mathcal{F} \in \mathrm{Coh}(X)$, we have a canonical isomorphism*

$$H^*(X, \mathcal{F}) \cong H^*(X^{an}, \mathcal{F}^{an}).$$

Corollary 1.8.2. *The functor $X \mapsto X^{an}$ is a fully faithful functor from the category of proper $\mathbb{C}$-schemes to the category of (compact) complex-analytic spaces.*

Corollary 1.8.3. *Suppose X is a proper $\mathbb{C}$-scheme. The functor $X' \mapsto (X')^{an}$ induces an equivalence of categories between the category of finite X-schemes and the category of finite complex analytic spaces over X^{an}.*

Remark 1.9. Here, it should be noted that a finite map of complex analytic spaces need not be covering map. A finite map $X' \to X$ of complex analytic spaces is a *finite covering* if every irreducible subspace of X' maps onto an irreducible subspace of X. However the following

Corollary 1.9.1 (Generalised Riemann Existence Theorem)**.** *Suppose X is a $\mathbb{C}$-scheme locally of finite type. The functor $X' \mapsto (X')^{an}$ induces an equivalence of categories between the category of finite étale coverings of X and the category of finite étale coverings of X^{an}.*

Corollary 1.9.2 (Chow's Theorem)**.** *Suppose Y is a closed complex-analytic subspace of complex projective space (the latter being viewed as a compact complex manifold). Then there exists a projective $\mathbb{C}$-scheme $X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ such that $Y \cong X^{an}$.*

1.3 Kähler and Moishezon Manifolds

(This section is taken from [Har77, Appendix B])

Definition. A compact complex manifold $\mathcal{X}$ is said to be *algebraic* if $\mathcal{X} = X^{an}$ for some proper $\mathbb{C}$-scheme X.

By Corollary 1.8.2, we know that if $\mathcal{X}$ is algebraic then the $\mathbb{C}$-scheme X it defines is necessarily unique. We now have an obvious question: which compact complex manifolds are algebraic? There is no clean answer if we restrict ourselves to schemes. There are some obvious necessary conditions, which follow immediately (by GAGA) from the corresponding property of schemes.

Proposition 1.10. *Suppose $\mathcal{X}$ is a compact complex manifold of dimension n. Then the transcendence degree of the field $K(\mathcal{X})$ of meromorphic functions on $\mathcal{X}$ over $\mathbb{C}$ is at most n. Moreover, equality holds if $\mathcal{X}$ is algebraic.*

Definition. A compact complex manifold is said to be *Moishezon* if the transcendence degree of the field $K(\mathcal{X})$ of meromorphic functions on $\mathcal{X}$ over $\mathbb{C}$ is equal to n.

Thus, being Moishezon is a necessary condition to being algebraic. We have another necessary condition, which is a simple corollary of GAGA and Proposition 2.37.

Proposition 1.11. *Suppose $\mathcal{X}$ is a compact complex manifold. Consider the Hodge-to-de Rham spectral sequence*

$$E_1^{p,q} = H^q(\mathcal{X}, \Omega^p) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{C}),$$

where Ω^p is the space of differential p-forms on $\mathcal{X}$. If $\mathcal{X}$ is algebraic, then this spectral sequence degenerates on the E_1 page.

Moving on to sufficient conditions/characterizations, if we further restrict to projective complex manifolds (i.e. a submanifold of $\mathbb{P}^n(\mathbb{C})$), then we have better results. Recall that any complex manifold admits a Hermitian metric, which is just a differential 2-form h with extra conditions. We have a differential $(1,1)$-form $\omega := \mathrm{Im} \circ h$, often called the fundamental form (associated to h).

Definition. A *Kähler manifold* is a manifold such that the fundamental form ω is closed.

An excellent introduction to the theory of Kähler manifolds is [GH14, Chapter 0]. From loc. cit., we know that every projective complex manifold is canonically a Kähler manifold. In this set up, we have the following two results that give sufficient conditions.

Proposition 1.12. *Suppose $\mathcal{X}$ is a complex manifold. Then, $\mathcal{X}$ is algebraic if and only if it is a compact Kähler manifold with $\omega \in \mathrm{Im}(H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{X}, \mathbb{C}))$.*

Proposition 1.13. *A Kähler Moishezon manifold is projective algebraic.*

Of course, projective algebraic manifolds are both Kähler and Moishezon.

We also have a clean result if we look beyond schemes. It turns out that the analytification functor also continues to work for algebraic spaces. We have the following theorem of Serre.

Theorem 1.14 (Serre)**.** *The category of smooth proper algebraic spaces over $\mathbb{C}$ is equivalent to the category of compact complex Moishezon manifolds.*

2 De Rham Cohomology

The main references for stuff on the de Rham complex and de Rham cohomology is [Sta23, 0FK4]. For stuff on connections, the best reference is [Kat70b]. The expository article [Ill02] is also really good, though its main focus is the theory in characteristic p. The paper [Har75] also contains a lot of information about the homological properties of the de Rham cohomology theory (such as the Künneth formula, Mayer-Vietoris, etc.); however, Hartshorne works only in characteristic 0 in this paper.

Remark 2.1. A word of warning. Most references, including [Ill02] and [Sta23] usually consider arbitrary Y-schemes and X, and restrict to looking at coherent sheaves on X. However, [Kat70b] considers *quasi*-coherent sheaves on *smooth* Y-schemes X. I've tried to be as careful as I can with the assumptions while copying down results, but it's possible that I may have missed some things.

2.1 The de Rham Complex

See [Har77, Section II.8] for details.

Definition. Suppose A is a ring, B an A-algebra, and M a B-module. An *A-derivation of B to M* is an A-linear map $d : B \to M$ such that

$$d(bb') = bdb' + b'db$$

and $da = 0$ for all $a \in A$.

Lemma-Definition. The *module of relative differentials* $\Omega_{B/A}$ is the unique B-module $\Omega_{B/A}$ equipped with an A-derivation $d : B \to \Omega_{B/A}$ that is initial amongst all such B-modules, i.e. for any B-module M with A-derivation $d' : B \to M$, there is a unique B-modules morphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d$.

One can construct the module of relative differentials in an obvious way. Properties of $\Omega_{B/A}$ can be found in [Har77, Section II.8].

Now, recall that if Z is a closed subscheme of X with closed immersion $i : Z \to X$, the *ideal sheaf* $\mathcal{I}_Z$ of Z is the kernel of the canonical morphism

$$\mathcal{O}_X \to i_*\mathcal{O}_Z.$$

It is a quasi-coherent sheaf of ideals of $\mathcal{O}_X$. Moreover, if $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/\mathfrak{a}$, then $\mathcal{I}_Z = \widetilde{\mathfrak{a}}$ is the sheaf of ideals generated by the ideal $\mathfrak{a}$.

Now suppose $f : X \to Y$ is a morphism of schemes. We have the induced diagonal map $\Delta : X \to X \times_Y X$. It is a fact that Δ is a locally closed immersion, i.e. Δ identifies X with a closed subscheme of an open subscheme W of $X \times_Y X$. We can thus consider the ideal sheaf $\mathcal{I}$ of X in W, i.e.

$$\mathcal{I} = \ker(\mathcal{O}_{X \times_Y X}|_W \to i_*\mathcal{O}_X).$$

Definition. The sheaf of relative differentials of $f : X \to Y$ is the sheaf

$$\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2).$$

The natural action of $\mathcal{O}_{\Delta(X)}$ on $\mathcal{I}/\mathcal{I}^2$ induces an action of $\mathcal{O}_X$ on $\Omega_{X/Y}$.

It is known that $\Omega_{X/Y}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules on X. It is coherent if Y is locally Noetherian and f a morphism locally of finite presentation.

Lemma 2.2. *If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ so that the map $f : X \to Y$ is induced by a map $A \to B$, then $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$.*

Lemma 2.3. *Suppose $f : X \to Y$ is a morphism. For any point $x \in X$, we have the equality on stalks*

$$\Omega_{X/Y,x} = \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$$

In particular, by the above lemma, on every open affine $U = \operatorname{Spec} B$ of X with corresponding open affine $V = \operatorname{Spec} A$ of Y, the map $d : B \to \Omega_{B/A}$ induces the map

$$\widetilde{d}_{U,V} : \mathcal{O}_U = \widetilde{B} \to \widetilde{\Omega_{B/A}} = \Omega^1_{V/U}.$$

Glueing these maps together as we range over all open affines, it follows that we have a canonical map

$$d : \mathcal{O}_X \to \Omega_{X/Y}$$

which is a derivation on all fibres. Note that this map d is NOT a morphism of $\mathcal{O}_X$-modules; it is instead a morphism of $f^{-1}\mathcal{O}_Y$-modules, where recall that $f^{-1}\mathcal{O}_Y$ is the sheafification of the presheaf

$$U \mapsto \operatorname*{colim}_{V \text{ open}, f(U) \subset V \subset Y} \mathcal{O}_Y(V).$$

Definition. Suppose $f : X \to Y$, and suppose $\mathcal{M}$ is a $\mathcal{O}_X$-module. A Y-*derivation* is a homomorphism of $f^{-1}\mathcal{O}_Y$-modules $D : \mathcal{O}_X \to \mathcal{M}$ such that on every open subscheme $U \subset X$ the map $D_U : \mathcal{O}_X(U) \to \mathcal{M}(U)$ satisfies Liebniz' rule.

Thus $\Omega_{X/Y}$ is equipped with a canonical Y-derivation $d : \mathcal{O}_X \to \Omega_{X/Y}$.
Suppose now we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

of schemes. We have a morphism of $\mathcal{O}_{X'}$-modules

$$g^*\Omega_{X/Y} := \mathcal{O}_{X'} \otimes_{g^{-1}\mathcal{O}_X} g^{-1}\Omega_{X/Y} \to \Omega_{X'/Y'}, \quad 1 \otimes g^{-1}(d_{X/Y}s) \mapsto d_{X'/Y'}(1 \otimes g^{-1}(s)).$$

Here is a collection of results on the sheaf of relative differentials. Proofs for all of these can be found in either [Har77] or [Ill02].

Proposition 2.4. *Suppose $f : X \to Y$ is a morphism of schemes.*

1. *(*Universal Property*) The functor*

$$\mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Ab}, \quad \mathcal{M} \mapsto \mathrm{Der}_Y(\mathcal{O}_X, \mathcal{M}),$$

   taking a $\mathcal{O}_X$-module $\mathcal{M}$ to the (abelian) group of Y-derivations $\mathcal{O}_X \to \mathcal{M}$, is representable by $\Omega_{X/Y}$.

   In other words, for any $\mathcal{O}_X$-module $\mathcal{M}$ with Y-derivation $d' : \mathcal{O}_X \to \mathcal{M}$, there is a unique morphism $f : \Omega_{X/Y} \to \mathcal{M}$ of $\mathcal{O}_X$-modules such that $d' = f \circ d$.

2. *If $g : Y' \to Y$ is another morphism, the sheaf of relative differentials of the induced map $X' := X \times_Y Y' \to Y'$ is*

$$\Omega_{X'/Y'} = p^*\Omega_{X/Y}$$

   where $p : X' \to X$ is the morphism induced by base-changing g.

3. *Suppose $g : Y \to Z$ is another morphism of schemes. There is an exact sequence*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0,$$

   where the map $f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is the canonical morphism induced by the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ Z & = & Z. \end{array}$$

4. *If $Y' \to Y$ is another morphism, and $X' := X \times_Y Y'$, then*

$$\Omega_{X'/Y} = p^*\Omega_{X/Y} \oplus q^*\Omega_{Y'/Y}$$

   where $p : X' \to X$ and $q : X' \to Y'$ are the canonical projections. Moreover, the canonical map

$$p^*\Omega_{X/Y} \to \Omega_{X'/Y'}$$

   is an isomorphism of $\mathcal{O}_{X'}$-modules.

5. *Suppose Z is a closed subscheme of X with corresponding ideal sheaf $\mathcal{I}_Z$. Then, there is a natural map $\mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z$ fitting into an exact sequence*

$$\mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

6. *For $X = \mathbb{A}^n_Y$ the affine space of dimension n over Y, the $\mathcal{O}_X$-module $\otimes_{X/Y} \cong \mathcal{O}_X^{\oplus n}$ is free (in the obvious way)*

Suppose $f : X \to Y$ a morphism of schemes. Define the $\mathcal{O}_X$-module

$$\Omega^p_{X/Y} := \bigwedge^p \Omega_{X/Y}$$

with $\Omega^0_{X/Y} := \mathcal{O}_X$.

Lemma-Definition. There is a unique family of maps $d : \Omega^p_{X/Y} \to \Omega^{p+1}_{X/Y}$ satisfying

- the degree 0 map $d : \mathcal{O}_X \to \Omega^1_{X/Y}$ is the usual Y-derivation $d : \mathcal{O}_X \to \Omega_{X/Y}$;
- d is a Y-anti-derivation of the exterior algebra $\bigwedge^* \Omega_{X/Y} = \bigoplus_p \Omega^p_{X/Y}$, i.e. d is $f^{-1}(\mathcal{O}_Y)$-linear and

  $$d(ab) = da \wedge b + (-1)^i a \wedge db$$

  for a a local section of $\mathcal{O}_X$ and b a local section of $\Omega^p_{X/Y}$; and
- the family of maps makes

  $$\Omega^\bullet_{X/Y} : \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/Y} \xrightarrow{d} \Omega^2_{X/Y} \to \cdots$$

  into a complex, called the *de Rham complex*.

Remark 2.5. Even though each of the objects $\Omega^p_{X/Y}$ are $\mathcal{O}_X$-modules, the complex $\Omega^\bullet_{X/Y}$ is NOT a complex of $\mathcal{O}_X$-modules. This is because the differential maps $d : \Omega^p_{X/Y} \to \Omega^{p+1}_{X/Y}$ are not $\mathcal{O}_X$-linear but are only $f^{-1}\mathcal{O}_Y$-linear. Hence, $\Omega^\bullet_{X/Y}$ is in fact a complex of $f^{-1}(\mathcal{O}_Y)$-modules.

Now, suppose we have a commuting diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y. \end{array}$$

This square then induces a map

$$\Omega^\bullet_{X/Y} \to g_* \Omega^\bullet_{X'/Y'}$$

of complexes, which is also a morphism of the corresponding differential graded algebra. Of course, if the above square is Cartesian, then this map

$$\Omega^\bullet_{X/Y} \to g_* \Omega^\bullet_{X'/Y'}$$

is an isomorphism.

Finally, let us briefly define the tangent bundle.

Definition. Suppose $f : X \to Y$ is a morphism. The *relative tangent sheaf* $T_{X/Y}$ is the sheaf

$$T_{X/Y} := \mathcal{H}om(\Omega^1_{X/Y}, \mathcal{O}_X).$$

Using the universal property of $\Omega^1_{X/Y}$, we see that for any open subscheme U of X, we have a natural isomorphism

$$T_{X/Y}(U) \cong \mathrm{Der}_Y(\mathcal{O}_U, \mathcal{O}_U).$$

2.2 Smooth Maps

Lemma-Definition. A morphism $f : X \to Y$ of schemes is *flat* if for any open affines $\mathrm{Spec}(A) \subset X$ and $\mathrm{Spec}(B) \subset Y$, the corresponding ring map $B \to A$ is flat. A morphism is *faithfully flat* if it is both surjective and flat.

It is easy to check that flatness is an affine local (in the sense of Vakil) property on the target Y.

Proposition 2.6. *Suppose $f : X \to S$ is a morphism of schemes.*

1. *f is flat if and only if for every $x \in X$ the local ring map $f^\#_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.*
2. *f is flat if and only if for every $S \to S'$, the pull-back functor $\mathrm{QCoh}(S') \to \mathrm{QCoh}(X \times_S S')$ induced by the map $X \times_S S' \to S'$ is an exact functor.*

3. *Composite of flat morphisms is flat.*

4. *Fibre product of two flat (resp. faithfully flat) morphisms is flat (resp. faithfully flat).*

5. *Flatness and faithful flatness are preserved by base change.*

6. *If f is flat, then for every $x \in X$ and every $s \in S$ such that $f(x) \in \overline{\{s\}}$, there exists $x' \in X$ such that $s = f(x')$ and $x \in \overline{\{x'\}}$.*

7. *If f is flat and locally of finite presentation, then it is universally open, i.e. for every $S' \to S$ the induced map $X \times_S S' \to S'$ is an open map.*

8. *If f is quasi-compact and faithfully flat (i.e. fpqc), then $T \subset S$ is open (resp. closed) iff $f^{-1}(T)$ is open (resp. closed).*

Thus, fpqc maps can be thought of as quotient maps.

Definition. A morphism $f : X \to Y$ is *smooth* (resp. *unramified*, resp. *étale*) if f is locally of finite presentation and if the following condition is satisfied: for any commutative diagram

$$\begin{array}{ccccc} & & & & X \\ & & \overset{g_0}{\nearrow} & & \downarrow f \\ T_0 & \xrightarrow[i]{} & T_1 & \longrightarrow & Y \end{array}$$

where i is a closed immersion such that the ideal sheaf $\mathcal{I}_{T_0}$ in $\mathcal{O}_{T_1}$ satisfies $\mathcal{I}_{T_0}^2 = 0$, there exists locally in the Zariski topology on T a (resp. at most one, resp. unique) Y-morphism $g : T \to X$ such that $gi = g_0$.

Remark 2.7. We can remove the phrase ‘locally in the Zariski topology’ from the definition of étale morphism.

We now list basic properties of unramified, smooth, and étale morphisms.

Proposition 2.8. *Suppose $f : X \to Y$ is a morphism of schemes.*

1. *Suppose f is locally of finite presentaiton. Then, f is unramified if and only if any one of the following equivalent conditions hold:*

   (a) *it is locally of finite presentation and for each $x \in X$ and $y = f(x)$, the residue field $k(x)$ is a separable algebraic extension of $k(y)$, and $f_x(\mathfrak{m}_{Y,y})\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ where $f_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.*

   (b) *for any affine opens $\mathrm{Spec}(B) \subset X$ and $\mathrm{Spec}(A) \subset Y$ the induced map $f^{\#} : A \to B$ is formally unramified.*

   (c) *the diagonal map $X \to X \times_Y X$ is an open immersion.*

2. *Composite of unramified morphisms is unramified.*

3. *Base change of an unramified morphism is unramified.*

4. *Open immersions are unramified.*

5. *Unramified-ness is affine local (in the sense of Vakil) on the target Y.*

6. *If $f : X \to Y$ is a morphism of S-schemes where X is unramified over S and Y is locally of finite type over S, then f is unramified.*

7. *Suppose X and Y are S-schemes and $f, g : X \to Y$ morphisms over S. Suppose Y is unramified over S. Let $x \in X$ be such that $f(x) = g(x) =: y$ where the maps $f_x, g_x : k(y) \to k(x)$ induced by f and g are equal. Then, there exists a Zariski open neighbourhood U of x in X such that $f|_U = g|_U$.*

8. *If f is unramified, then $\Omega_{X/Y} = 0$.*

Proposition 2.9. *Suppose $f : X \to S$ is a morphism.*

1. *Suppose f is locally of finite presentation. Then, f is smooth if and only if any one of the following conditions hold:*

   (a) *f is flat and for every S-morphism $\overline{s} : \mathrm{Spec}\, k \hookrightarrow S$ for k algebraically closed, the fibre $X_{\overline{s}} = X \times_S \overline{s}$ is regular.*

(b) f is flat and all fibers $f^{-1}(s)$ are regular and remain so after extension of scalars to some perfect extension of $k(s)$.

2. Composite of smooth morphisms is smooth.

3. Base change of an smooth morphism is smooth.

4. Open immersions are smooth.

5. Smoothness is affine local (in the sense of Vakil) on the target Y.

6. Smooth morphisms is universally open, i.e. f is open and for any base change $Y' \to Y$, the corresponding map $X \times_Y Y' \to Y'$ is also open.

7. If f is smooth, then $\Omega_{X/Y}$ is locally free of finite type. As a consequence, for $f : X \to Y$ smooth we have

$$T^\vee_{X/Y} := \mathcal{H}om(T_{X/Y}, \mathcal{O}_X) \cong \Omega_{X/Y}.$$

8. Suppose $g : Y \to Z$ is another morphism of schemes. If f is smooth, the exact sequence of Proposition 2.4(3) extends to the locally split exact sequence

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

On the other hand, if $g \circ f$ is smooth and if the above sequence is exact and locally split, then f is smooth.

9. Suppose Z is a closed subscheme of X with corresponding ideal sheaf $\mathcal{I}_Z$. If $f|_Z$ is smooth, the exact sequence of Proposition 2.4(4) extends to the locally split exact sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

On the other hand, if f is smooth and the above sequence is exact and locally split, then $f|_Z$ is smooth.

10. (Implicit Function Theorem) Suppose Z is a closed subscheme of X with corresponding ideal sheaf $\mathcal{I}_Z$. Suppose $x \in Z$ is a point such that $f|_Z$ is smooth in some open neighbourhood $U \cap Z$ of x, with $U \subset X$ open. Write $n = \dim(\mathcal{I}_Z/\mathcal{I}_Z^2)_x$ and $n + m = \dim_{k(x)} \Omega_{X/Y,x} \otimes k(x)$. Then, by shrinking U if necessary, there is an étale morphism $U \to \mathbb{A}^{n+m}_Y$ such that

$$U \cap Z = U \times_{\mathbb{A}^{n+m}_Y} \mathbb{A}^n_Y$$

where $\mathbb{A}^n_Y \hookrightarrow \mathbb{A}^{n+m}_Y$ is the inclusion into the first n coordinates.

Now for each point $x \in X$, set

$$\dim_x(f) := \dim_{k(x)} \Omega_{X/Y} \otimes k(x)$$

where $k(x)$ is the residue field of x. If f is smooth, $\Omega_{X/Y}$ is locally free and of finite type, and so $\dim_x(f) : X \to \mathbb{Z}_{\geq 0}$ is a locally constant function of x.

Definition. A smooth function $f : X \to Y$ is *pure of relative dimension r* if $\dim_x(f) = r$ for all $x \in X$.

Lemma 2.10. *If f is smooth and pure of relative dimension r, then $\Omega^p_{X/Y} = 0$ for all $p > r$, and $\Omega^p_{X/Y}$ is a locally free $\mathcal{O}_X$-module of rank $\binom{r}{p}$ for all $0 \leq p \leq r$.*

Proposition 2.11. *Suppose $f : X \to Y$ is a morphism.*

1. f is étale if and only if any one of the following conditions hold:

   (a) f is flat and unramified;

   (b) f is smooth and unramified;

   (c) f is smooth and pure of relative dimension 0;

   (d) f is flat, locally of finite presentation, and every fibre $f^{-1}(y)$ is given by the disjoint union $\bigsqcup_{i \in I} \operatorname{Spec} k_{i,y}$ where each $k_{i,y}$ is a finite separable field extension of the residue field $\kappa(y)$;

   (e) f is smooth and locally quasi-finite;

   (f) f is locally of finite presentation and for any affine opens $\operatorname{Spec}(B) \subset X$ and $\operatorname{Spec}(A) \subset S$ the induced map $f^\# : A \to B$ is formally étale;

(g) *for every $x \in X$ there is an open neighbourhood U of X around x and an open affine $V = \operatorname{Spec} A$ around $f(x)$ with $f(U) \subset V$ such that U is V-isomorphic to an open subscheme of* $\operatorname{Spec}\big(A[t]/\langle f\rangle\big)_{f'}$ *for some monic $f \in A[t]$ (with f' the usual derivative of f).*

2. *Étale morphisms are preserved under composition and base change.*

3. *Being an étale morphism is a local property on both the source and the target.*

4. *Product of a finite family of étale morphisms is étale.*

5. *Suppose $g : Y \to Z$ an unramified map and $f : X \to Y$ a map such that $g \circ f$ is étale. Then, f is étale.*

6. *Any S-morphism between étale S-schemes is étale.*

7. *Étale morphisms are locally quasi-finite.*

8. *Open immersions are étale. Moreover, a morphism is an open immersion if and only if it is étale and universally injective.*

9. *A map $X \to \operatorname{Spec} k$ is étale if and only if X is the disjoint union of* $\operatorname{Spec} k'$ *for k' a finite separable field extension of k.*

10. *Étale morphisms are open.*

2.3 de Rham Cohomology

Suppose $f : X \to Y$ is a morphism of schemes.

Definition. The *de Rham cohomology of X over Y* are the *hyper*-cohomology groups

$$H^p_{dR}(X/Y) := \mathbb{H}^i(X, \Omega^\bullet_{X/Y}) := H^i(R\Gamma(X, \Omega^\bullet_{X/Y})).$$

Remark 2.12. Suppose $Y = \operatorname{Spec} \mathbb{C}$, and suppose X is smooth (just to be safe). By GAGA, the algebraic de Rham cohomology of X coincides with the usual differential-geometric de Rham cohomology of the complex manifold X^{an}.

Remark 2.13. The de Rham cohomology groups are NOT the same as the cohomology of the complex $\Omega^\bullet_{X/Y}$. Of course, the cohomology of the complex $\Omega^\bullet_{X/Y}$ would be a sheaf of $f^{-1}\mathcal{O}_Y$-modules.

These de Rham cohomology groups are naturally modules over $\Gamma(Y, \mathcal{O}_Y)$.

Given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y, \end{array}$$

the canonical maps $\Omega^\bullet_{X/Y} \to g_*\Omega^\bullet_{X'/Y'}$ yields pullback maps

$$g^* : R\Gamma(X, \Omega^\bullet_{X/Y}) \to R\Gamma(X', \Omega_{X'/Y'})$$

and thus maps

$$g^* : H^*_{dR}(X/Y) \to H^*_{dR}(X'/Y').$$

In particular, taking $Y' = Y$, we see that H^q_{dR} defines a contravariant (δ-)functor

$$H^q_{dR} : \mathrm{Sch}_Y \to \mathrm{Mod}\big(\mathcal{O}_Y(Y)\big).$$

Lemma 2.14. *Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, so that the map $f : X \to Y$ is induced by a ring map $A \to B$. Then,*

$$H^*_{dR}(X/Y) \cong H^*(\Omega^\bullet_{B/A})$$

is the usual cohomology of the complex $\Omega^\bullet_{X/Y} = \Omega^\bullet_{B/A}$.

Let us try to compute de Rham cohomology in more generality. Of course, the de Rham cohomology is NOT equal to the usual cohomology $H^*(X, \Omega^q_{X/Y})$ of the bundle $\Omega^q_{X/Y}$ for some specific q, as again we are applying $R\Gamma$ to $\Omega^\bullet_{X/S}$ and then taking cohomology. Thus, in order to compute de Rham cohomology, we need to take an injective resolution $I^{\bullet,\bullet}$ of $\Omega^\bullet_{X/S}$, i.e. $I^{\bullet,\bullet}$ is a double complex of injective objects such that for each p, the complex $I^{p,\bullet}$ is an injective resolution of $\Omega^p_{X/S}$. Writing $I^\bullet_{tot}$ for the total complex of the double complex $I^{\bullet,\bullet}$, our choice of $I^{\bullet,\bullet}$ means that $\Omega^\bullet_{X/S} \to I^\bullet_{tot}$ is a quasi-isomorphism where each term in $I^\bullet_{tot}$ is injective, and so $R\Gamma\Omega^\bullet_{X/S}$ is quasi-isomorphic to $\Gamma(I^\bullet_{tot}) = (\Gamma I)^\bullet_{tot}$. We can compute the cohomology of this total complex by using a spectral sequence on the double complex $\Gamma I^{\bullet,\bullet}$. Since $I^{p,\bullet}$ is an injective resolution of $\Omega^p_{X/Y}$, the cohomology of $\Gamma I^{p,\bullet}$ is precisely $H^q(X, \Omega^p_{X/Y})$. By computing the spectral sequence in two different ways, we get the following result.

Proposition 2.15 (Hodge to de Rham Spectral Sequence)**.** *There is a spectral sequence of* $\Gamma(Y, O_Y)$*-modules*

$$E_1^{pq} = H^{p+q}(X, \Omega^p_{X/Y}) \Rightarrow H^{p+q}_{dR}(X/Y).$$

The differentials on the first page are the maps

$$d_1^{p,q} : H^q(X, \Omega^p_{X/Y}) \to H^q(X, \Omega^{p+1}_{X/Y})$$

induced by the usual differential $d : \Omega^p_{X/Y} \to \Omega^{p+1}_{X/Y}$.

Corollary 2.15.1. $H^0_{dR}(X/Y) = \ker\left(d : \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \Omega_{X/Y})\right)$

Definition. The $\Gamma(Y, \mathcal{O}_Y)$-modules $H^{p,q}_{Hodge}(X/Y) := H^q(X, \Omega^p_{X/Y})$ are called the *Hodge cohomology groups of* X *over* Y.

If for example $Y = \operatorname{Spec} k$ and $f : X \to Y$ is proper, then the Hodge cohomology groups are known to be finite dimensional k-vector spaces, and thus the spectral sequence implies that the de Rham cohomology groups are finite dimensional k-vector spaces.

There is another spectral sequence that computes de Rham cohomology (cf. [Sta23, 0FM6]). This second spectral sequence relates the cohomology of the cohomology sheaves of the complex $\Omega^\bullet_{X/Y}$ to de Rham cohomology.

Proposition 2.16 (Conjugate Spectral Sequence for de Rham Cohomology)**.** *There is a spectral sequence*

$$E_2^{p,q} = H^q(X, H^p(\Omega^\bullet_{X/Y})) \Rightarrow H^{p+q}_{dR}(X/Y),$$

where $H^p(\Omega^\bullet_{X/Y}) \in \operatorname{QCoh}(X)$ *is the cohomology sheaf of the de Rham complex.*

2.4 Connections

We mostly follow [Con23], generalizing to arbitary schemes. This reference also has a lot of computations in local coordinates, linking back to the classical theory.

2.4.1 Definition

Let $p : X \to Y$ be a morphism of schemes. Suppose $\mathcal{E}$ is a quasi-coherent sheaf on X (for example, $\mathcal{E}$ could be a locally free coherent sheaf aka a vector bundle).

Definition. A *connection on* $\mathcal{E}$ *relative to* Y (or a Y*-connection on* $\mathcal{E}$) is a map of abelian sheaves

$$\nabla : \mathcal{E} \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$$

that satisfies the Liebniz rule

$$\nabla(fs) = df \otimes s + f \cdot \nabla s$$

for $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{E}(U)$, for any open subscheme $U \subset X$.

Notice that ∇ is $p^{-1}\mathcal{O}_Y$-linear. Moreover, if ∇ and ∇' are connections, then $\nabla - \nabla' : \mathcal{E} \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$ is easily seen to be $\mathcal{O}_X$-linear. Thus, the space of connections on $\mathcal{E}$ relative to Y is a principal homogeneous space under the group

$$\operatorname{Hom}(\mathcal{E}, \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}).$$

Example 2.17. The canonical differential $d : \mathcal{O}_X \to \Omega_{X/Y} = \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_X$ is a connection on $\mathcal{O}_X$ relative to Y.

Example 2.18. Suppose Λ is a locally free $p^{-1}\mathcal{O}_Y$-module with finite rank. Set $\mathcal{E} := \mathcal{O}_X \otimes_{p^{-1}\mathcal{O}_Y} \Lambda$. Then, the map

$$\nabla := d \otimes 1 : \mathcal{E} = \mathcal{O}_X \otimes_{p^{-1}\mathcal{O}_Y} \Lambda \to \Omega_{X/Y} \otimes_{p^{-1}\mathcal{O}_Y} \Lambda = \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$$

satisfies Liebniz rule (because d does), and hence is a connection. Moreover, one sees that under the canonical map $\Lambda \to \mathcal{E}$, the image of Λ is killed by ∇. It turns out that ∇ is the unique connection on $\mathcal{E}$ killing Λ. If p is smooth then it turns out that $\ker \nabla = \Lambda$.

If $Y = \operatorname{Spec} k$, then by definition Λ is a local system (i.e. locally constant sheaf) of k-vector spaces, and the above characterisation of ∇ is a key ingredient of the Riemann-Hilbert correspondence.

If we take $\Lambda = p^{-1}\mathcal{O}_Y$, the above construction recovers the previous example.

Example 2.19. There is a bijective correspondence between Y-connections ∇ on $\mathcal{O}_X$ and global sections of $\Omega_{X/Y}$. Namely, to a Y-connection ∇ on $\mathcal{O}_S$ is associated the global section $\omega = \nabla(1) \in \Omega_{X/Y}$. Conversely, given a global section $\omega \in \Omega_{X/Y}(X)$ we have the connection $\nabla_\omega : \mathcal{O}_X \to \Omega_{X/Y}$ given by

$$\nabla_\omega(f) = df + f\omega.$$

Under this correspondence, the standard Y-connection d on $\mathcal{O}_X$ corresponds to the zero section.

Classically, connections are a mechanism to differentiate sections of a vector bundle on a manifold along a vector field. We can recover this construction in this more general case.

Definition. A *vector field of X relative to Y* is a Y-derivation of $\mathcal{O}_X$, i.e. it is an element of $\operatorname{Der}_Y(\mathcal{O}_X, \mathcal{O}_X)$.

By the universal property of the sheaf of relative differentials, a vector field of X relative to Y can also be viewed as a $\mathcal{O}_X$-linear morphism $\Omega_{X/Y} \to \mathcal{O}_X$.

Recall the relative tangent bundle $T_{X/Y}$. We see immediately that for any open subscheme $U \hookrightarrow X$, the space of sections $T_{X/Y}(U)$ is the space of all vector fields of U relative to Y.

Suppose v is a vector field of X relative to Y, which we view as a $\mathcal{O}_X$-linear morphism $v : \Omega_{X/Y} \to \mathcal{O}_X$. If $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y}$ is a connection on a coherent sheaf $\mathcal{E}$, then we have the composite

$$\nabla_v : \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y} \xrightarrow{1 \otimes v} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{E}.$$

A similar statement holds for any open $U \subset X$. Hence, a connection defines (and is defined by) a $\mathcal{O}_X$-linear map

$$T_{X/Y} \to \mathcal{H}om(\mathcal{E}, \mathcal{E}) =: \mathcal{E}nd(\mathcal{E}), \quad v \in T_{X/Y}(U) \mapsto (1 \otimes v) \circ \nabla \in \operatorname{End}(\mathcal{E}|_U).$$

Since ∇ satisfies the Liebniz rule and v is $\mathcal{O}_X$-linear, it follows that ∇_v also satisfies the Liebniz rule

$$\nabla_v(f \cdot s) = v(f) \cdot s + f \cdot \nabla_v(s)$$

for all $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{E}(U)$, for all open $U \subset X$. Here, notice that we are now viewing v as a $p^{-1}\mathcal{O}_Y$-module morphism $v : \mathcal{O}_X \to \mathcal{O}_X$.

2.4.2 Another Perspective on Connections

There is another perspective on connections that is more algebraic-geometric. Consider the diagonal morphism $\Delta : X \to X \times_Y X$, and let $\mathcal{I}_\Delta \subset \mathcal{O}_{X \times_Y X}$ denote the ideal sheaf of this locally closed immersion. Set

$$\mathcal{P}_{X/Y} := \mathcal{O}_{X \times_Y X} / \mathcal{I}_\Delta^2.$$

This is the structure sheaf of the first infinitesimal thickening of the diagonal (see [Ill02, Section 1.1-1.2]). In particular, it is naturally a sheaf on X. Now, the two projections $p_1 : X \times_Y X \to X$ and $p_2 : X \times_Y X \to X$ induce maps

$$\mathcal{O}_X \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \Delta^* \mathcal{O}_{X \times_Y X} \twoheadrightarrow \mathcal{P}_{X/Y},$$

denoted by j_1 and j_2 respectively. These morphisms $j_i : \mathcal{O}_X \to \mathcal{P}_{X/Y}$ induce two different $\mathcal{O}_X$-algebra structures on $\mathcal{P}_{X/Y}$. By definition, $\Omega_{X/Y} = \mathcal{I}_\Delta / \mathcal{I}_\Delta^2$, and so we have an injection $\Omega_{X/Y} \hookrightarrow \mathcal{P}_{X/Y}$. One checks that we have an exact sequence

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathcal{P}_{X/Y} \underset{\underset{j_2}{\hookleftarrow}}{\overset{\overset{j_1}{\hookleftarrow}}{\longrightarrow}} \mathcal{O}_X \longrightarrow 0$$

split by both j_1 and j_2. Here, the map $\mathcal{P}_{X/Y} \to \mathcal{O}_X$ is the map induced by $\Delta^* : \mathcal{O}_{X \times_Y X} \to \mathcal{O}_X$, noting that the kernel of Δ is by definition $\mathcal{I}_\Delta$. Moreover, one checks that $j_2 - j_1 : \mathcal{O}_X \to \Omega_{X/Y}$ coincides with the canonical differential $d : \mathcal{O}_X \to \Omega_{X/Y}$ (that $j_2 - j_1$ lands in the subsheaf $\Omega_{X/Y}$ follows by exactness and the fact that j_1 and j_2 split the exact sequence.)

Proposition 2.20. *The data of a connection on a coherent $\mathcal{O}_X$-module $\mathcal{E}$ is the same as the data of a $\mathcal{P}_{X/Y}$-linear isomorphism*

$$\mathcal{P}_{X/Y} \otimes_{j_2,\mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X,j_1} \mathcal{P}_{X/Y}$$

lifting the identity on $\mathcal{E}$.

Remark 2.21. By coherence of $\mathcal{E}$, any $\mathcal{P}_{X/Y}$-linear map

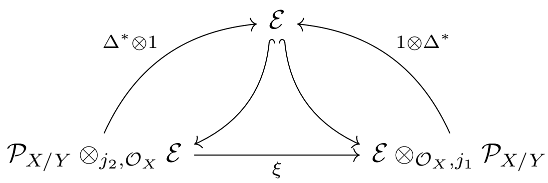
$$\mathcal{P}_{X/Y} \otimes_{j_2,\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X,j_1} \mathcal{P}_{X/Y}$$

lifting the identity on $\mathcal{E}$ is automatically an isomorphism.

Proof. Let

$$\xi : \mathcal{P}_{X/Y} \otimes_{j_2,\mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X,j_1} \mathcal{P}_{X/Y}$$

be any morphism of abelian sheaves. It suffices to specify ξ at the level of the corresponding pre-sheaves, since sheafification is an exact functor. Consider the comutative diagram



where $\Delta^* : \mathcal{P}_{X/Y} \to \mathcal{O}_X$. Since $\Delta^* \circ j_1 = \Delta^* \circ j_2 = id_{\mathcal{O}_X}$, it follows that ξ lifts the identity on $\mathcal{E}$ if and only if for any $s \in \mathcal{E}(U)$, the section

$$\xi(1 \otimes s) - s \otimes 1$$

of $\mathcal{E} \otimes_{\mathcal{O}_X,j_1} \mathcal{P}_{X/Y}$ lies in the kernel of

$$1 \otimes \Delta : \mathcal{E} \otimes_{\mathcal{O}_X,j_1} \mathcal{P}_{X/Y} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{E},$$

which is precisely $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y}$. Thus, the morphism ξ lifts the identity on $\mathcal{E}$ if and only if we have a well-defined a map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y},$$

the correspondence given by $\xi(1 \otimes s) = s \otimes 1 + \nabla s$.

Now, ξ is furthermore $\mathcal{P}_{X/Y}$-linear if and only if $j_2(f) \cdot \xi(1 \otimes s) = \xi(1 \otimes fs)$ for $f \in \mathcal{O}_X(U)$, where $j_2 : \mathcal{O}_X \to \mathcal{P}_{X/Y}$. This equation is equivalent to

$$s \otimes j_2(f) + f \cdot \nabla s = s \otimes j_1(f) + \nabla(fs),$$

which by the formula $d = j_2 - j_1$ is precisely equivalent to ∇ satisfying the Liebniz rule. □

Remark 2.22. This proposition can be viewed as saying that a connection is a first-order descent data on the coherent sheaf $\mathcal{E}$. See [BO15, Chapter 2].

2.5 Curvature, Integrable Connections, and Riemann-Hilbert

Recall that we could extend $d : \mathcal{O}_X \to \Omega_{X/Y}$ to form the differentials of a complex $\Omega^\bullet_{X/Y}$. We can do a similar thing for arbitrary connections. As always, fix a Y-scheme X.

Theorem 2.23. *Suppose $(\mathcal{E}, \nabla)$ is a coherent sheaf on X with a connection ∇ relative to Y. Then, for all $p \geq 0$, there is a unique abelian sheaf map*

$$\nabla^p : \Omega^p_{X/Y} \otimes \mathcal{E} \to \Omega^{p+1}_{X/Y} \otimes \mathcal{E}$$

such that $\nabla^0 = \nabla$ satisfying

$$\nabla^{p+q}((\omega_p \wedge \omega_q) \otimes s) = \nabla^p(\omega_p \otimes s) \wedge \omega_q + (-1)^p \omega_p \wedge \nabla^q(\omega_q \otimes s)$$

for local sections s of $\mathcal{E}$, ω_p of $\Omega^p_{X/Y}$, and ω_q of $\Omega^q_{X/Y}$.

The definition of ∇^p is essentially as follows. Suppose $U \subset X$ open, and suppose $s \in \mathcal{E}(U)$ and $\omega \in \Omega^p_{X/S}(U)$ are local sections. Then, we set

$$\nabla^p(\omega \otimes s) := d\omega \otimes s + (-1)^p \omega \wedge \nabla s.$$

Here and throughout, when we write $\omega \wedge \rho$ for a section of $\Omega^i_{X/Y} \otimes_{\mathcal{O}_X} (\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$ we really mean the image of $\omega \otimes \rho \in \Omega^i_{X/Y} \otimes_{\mathcal{O}_X} (\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$ under the canonical map

$$\Omega^i_{X/Y} \otimes_{\mathcal{O}_X} (\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E} \to \Omega^{i+1}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}, \quad \omega \otimes \tau \otimes e \mapsto (\omega \wedge \tau) \otimes e.$$

Some tedious computation checks that ∇^p is well-defined (for instance, that $\nabla^p((f\omega) \otimes s) = \nabla^p(\omega \otimes (fs))$ for any $f \in \mathcal{O}_X(U)$) and satisfies the required equation. By linearity it can then be extended to a well-defined abelian sheaf map

$$\nabla^p : \Omega^p_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \Omega^{p+1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}.$$

Uniqueness is obvious from the given characterising equation. The remaining statements are further tedious calculations.

Some computation yields the following.

Lemma 2.24. *For any ω a local section of $\Omega^i_{X/Y}$ and any s a section of $\mathcal{E}$, one has*

$$(\nabla^{p+1} \circ \nabla^p)(\omega \otimes e) = \omega \wedge (\nabla^1 \circ \nabla^0)(e).$$

Corollary 2.24.1. *$\Omega^\bullet_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$ is a complex if and only if $\nabla^1 \circ \nabla^0 = 0$.*

This motivates the following definition.

Definition. The *curvature* K_∇ of a connection ∇ is the map $K_\nabla := \nabla^1 \circ \nabla^0 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/Y}$.

A connection is *integrable* (or *flat*) if its curvature is identically zero. The resulting complex $\Omega^\bullet_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$ with differentials ∇^p is called the *de Rham complex of* $(\mathcal{E}, \nabla)$.

Another corollary of the previous computation is the following suprising (at least at first glance) fact.

Corollary 2.24.2. *The curvature map $K_\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/Y}$ is a $\mathcal{O}_X$-linear map of $\mathcal{O}_X$-modules.*

Remark 2.25. Most authors abuse notation by writing ∇ for any of the ∇^p. This abuse of notation is similar in spirit to writing $d : A^p \to A^{p+1}$ for the differentials of a complex $A^\bullet$, for all p.

Example 2.26. The connection d on $\mathcal{O}_X$ is integrable. More generally, for a locally free $\pi^{-1}\mathcal{O}_Y$-module Λ of finite rank (with $\pi : X \to Y$ the structure map), the resulting connection $\nabla_\Lambda = d \otimes 1$ on $\mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \Lambda$ is integrable.

Example 2.27. Recall the bijective correspondence $\omega \leftrightarrow \nabla_\omega$ between global sections of $\Omega_{X/Y}$ and Y-connections on $\mathcal{O}_X$. Then,

$$K_{\nabla_\omega} : \mathcal{O}_X \to \Omega^2_{X/Y} \quad \text{is given by} \quad f \mapsto f \cdot d\omega.$$

In particular, ∇_ω is integrable if and only if ω is closed.

Remark 2.28. One can translate the condition of integrability in terms of the perspective given in Section 2.4.2, and this is done in [BO15, Chapter 2]. Here, they show that a connection is integrable if and only if it can be extended to a certain 'stratification' of the sheaf $\mathcal{E}$.

Lemma 2.29. *If $X \to Y$ is smooth, then ∇ is integrable if and only if $\nabla_{[v,w]} = [\nabla_v, \nabla_w]$ for relative vector fields v, w (i.e. local sections of $T_{X/Y}$).*

Recall that a connection ∇ on $\mathcal{E}$ induces a morphism of sheaves

$$\widetilde{\nabla} : T_{X/Y} \to \mathcal{E}nd(\mathcal{E}).$$

Notice also that both $T_{X/Y}$ and $\mathcal{E}nd(\mathcal{E})$ have $\pi^{-1}\mathcal{O}_Y$-Lie algebra structures on them via the standard commutator pairing. The previous lemma can thus be recast as the following.

Lemma 2.30. *A connection ∇ is integrable if and only if $\widetilde{\nabla}$ is a Lie algebra morphism.*

The importance of integrability is due to the Riemann-Hilbert correspondence.

Theorem 2.31. *Suppose $X \to \operatorname{Spec} \mathbb{C}$ is a smooth map of finite type, and suppose $(\mathcal{E}, \nabla)$ is a coherent sheaf on X with integrable connection. The sheaf $\ker \nabla \subset \mathcal{E}$ is a locally constant sheaf with $\underline{\mathbb{C}}$-rank equal to the $\mathcal{O}_X$-rank of $\mathcal{E}$, and the natural map*

$$\mathcal{O}_X \otimes_{\underline{\mathbb{C}}} \ker \nabla \to \mathcal{E}$$

is an isomorphism identifying ∇ with $\nabla_{\ker \nabla} = d \otimes 1$.

Moreover, the functors $(\mathcal{E}, \nabla) \mapsto \ker \nabla$ and $\Lambda \mapsto (\mathcal{O}_X \otimes_{\underline{\mathbb{C}}} \Lambda, \nabla_\Lambda)$ are inverse equivalences of categories between the category of flat coherent sheaves and the category of local systems of finite dimensional $\mathbb{C}$-vector spaces.

We can relativize this correspondence as follows (cf. [Del70, Théorème 2.23]).

Definition. Suppose $f : X \to Y$ is smooth map of schemes/analytic spaces. A *local relative system* on X is a sheaf of $f^{-1}\mathcal{O}_Y$-modules that is locally isomorphic to the sheaf-theoretic inverse image of a coherent sheaf on Y.

Theorem 2.32. *Suppose $f : X \to Y$ is a smooth map of complex analytic spaces X and Y, and suppose $(\mathcal{E}, \nabla)$ is a coherent sheaf on X with integrable Y-connection. The sheaf $\ker \nabla \subset \mathcal{E}$ is a local relative system with $f^{-1}\mathcal{O}_Y$-rank equal to the $\mathcal{O}_X$-rank of $\mathcal{E}$, and the natural map*

$$\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \ker \nabla \to \mathcal{E}$$

is an isomorphism identifying ∇ with $\nabla_{\ker \nabla} = d \otimes 1$.

Moreover, the functors $(\mathcal{E}, \nabla) \mapsto \ker \nabla$ and $\Lambda \mapsto (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} \Lambda, \nabla_\Lambda)$ are inverse equivalences of categories between the category of coherent sheaves on X with a flat Y-connection, and the category of local relative systems on S.

Bhatt and Lurie also proved a version of Riemann Hilbert in positive characteristic, though they work with étale sheaves rather than Zariski sheaves as above.

An integrable connection imposes non-trivial conditions on the geometry of the underlying sheaf. For instance, we have the following proposition.

Proposition 2.33 ([Kat70b, Proposition 8.9])**.** *Suppose X is smooth over a field of characteristic 0, and let $\mathcal{E}$ be a coherent sheaf with flat connection. Then, $\mathcal{E}$ is a locally free sheaf on X.*

2.6 The Category $MC(X/Y)$ of Sheaves with Connections

Throughout, we fix $p : X \to Y$.

Definition. Suppose $(\mathcal{E}, \nabla)$ and $(\mathcal{F}, \nabla')$ are quasi-coherent $\mathcal{O}_X$-modules with Y-connections. An $\mathcal{O}_X$-linear mapping $\Phi : \mathcal{E} \to \mathcal{F}$ is *horizontal* if

$$\Phi|_U \circ \nabla_v = \nabla'_v \circ \Phi|_U$$

for all local sections $v \in T_{X/Y}(U)$, for all open $U \subset X$.

Let $MC(X/Y)$ denote the abelian category whose objects are $(\mathcal{E}, \nabla)$ where $\mathcal{E}$ are quasi-coherent $\mathcal{O}_X$-modules with connection ∇, and whose morphisms are horizontal $\mathcal{O}_X$-linear maps.

Let $MC_{int}(X/Y)$ be the full subcategory of $MC(X/Y)$ consisting of sheaves with flat connection.

Of course, in order that $MC(X/Y)$ is an abelian category, we need kernels and cokernels. Suppose $\Phi : (\mathcal{E}, \nabla) \to (\mathcal{F}, \nabla')$ is a horizontal morphism. The kernel of Φ in $\operatorname{QCoh}(X)$ is simply the sheaf $U \mapsto \ker \Phi_U$. Since Φ is horizontal, one checks that the image of

$$\nabla|_{\ker \Phi} : \ker \Phi \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$$

is in fact contained in $\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \ker \Phi$. It is clear that $(\ker \Phi, \nabla|_{\ker \Phi})$ acts as the kernel of Φ in the category $MC(X/Y)$. One can do a similar thing for cokernels, though the sheafification required in the definition of a cokernel of a map of quasicoherent sheaves makes things slightly more annoying.

This abelian category $MC(X/Y)$ has further operations defined on it, such as a direct sum, tensor product, etc.

Definition. Given two quasi-coherent sheaves $\mathcal{E}, \mathcal{E}'$ on X with connections ∇, ∇' respectively, the *direct sum* $\nabla \oplus \nabla'$ is the connection

$$\mathcal{E} \oplus \mathcal{E}' \xrightarrow{\nabla \oplus \nabla'} (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y}) \oplus (\mathcal{E}' \otimes_{\mathcal{O}_X} \Omega_{X/Y}) \cong (\mathcal{E} \oplus \mathcal{E}') \otimes_{\mathcal{O}_X} \Omega_{X/Y}.$$

Lemma-Definition. Given two quasi-coherent sheaves $\mathcal{E}, \mathcal{E}'$ on X with connections ∇, ∇' respectively, the *tensor product* $\nabla \otimes \nabla'$ is the connection

$$\mathcal{E} \otimes \mathcal{E}' \to (\mathcal{E} \otimes \mathcal{E}') \otimes_{\mathcal{O}_X} \Omega_{X/Y}$$

given by

$$s \otimes s' \mapsto \nabla(s) \otimes s' + s \otimes \nabla'(s').$$

Recall that for a quasi-coherent sheaf $\mathcal{E}$, the dual sheaf is

$$\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X), \quad \text{i.e. given by} \quad \mathcal{E}^\vee(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{E}|_U, \Omega_{X/Y}|_U).$$

Lemma-Definition. Given a quasi-coherent sheaf $\mathcal{E}$ on X with connection ∇, the *dual connection* $\nabla^\vee$ is the connection

$$\mathcal{E}^\vee \to \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \Omega_{X/Y}$$

given by

$$\nabla^\vee(\ell) := d \circ \ell - (1 \otimes \ell) \circ \nabla \in \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{E}|_U, \Omega_{X/Y}|_U) = \Gamma(U, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \Omega_{X/Y})$$

for any $\ell \in \mathcal{E}^\vee(U)$.

Notice that, for any $U \subset X$ open and any $\ell \in \mathcal{E}^\vee(U)$ and $s \in \mathcal{E}(U)$, we have the section $\ell(s) \in \mathcal{O}_X$. Then, the dual connection $\nabla^\vee$ is constructed so that

$$d\big(\ell(s)\big) = (\nabla^\vee \ell)(s) + (1 \otimes \ell)(\nabla s).$$

One can check that $(\mathcal{E}^\vee)^\vee \cong \mathcal{E}$ and $\nabla^{\vee\vee} = \nabla$.

One also has a internal Hom functor, though certain extra conditions are required.

Lemma-Definition. Suppose given two quasi-coherent sheaves $\mathcal{E}_1, \mathcal{E}_2$ on X with connections ∇_1, ∇_2 respectively. Suppose also that $\mathcal{E}_1$ is locally of finite presentation. We can define a connection ∇ on $\mathcal{E} = \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2) = \mathcal{E}_2 \otimes \mathcal{E}_1^\vee$ via

$$\nabla = \nabla_2 \otimes \nabla_1^\vee.$$

One checks that for a local section $\varphi : \mathcal{E}_1|_U \to \mathcal{E}_2|_U$ of $\mathcal{E}$, we have

$$\nabla\varphi = \nabla_2 \circ \varphi - (1 \otimes \varphi) \circ \nabla_1 \in \Gamma(U, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y}) = \mathrm{Hom}_U\big(\mathcal{E}_1|_U, (\mathcal{E}_2 \otimes_{\mathcal{O}_X} \Omega_{X/Y})|_U\big).$$

One can check that all of the above constructions are compatible for each other. One can also check that all of the above constructions preserve the category $MC_{int}(X/Y)$.

The next construction gives a sort of functoriality between different $MC(-)$ categories.

Lemma-Definition. Suppose we have a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{h'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{h} & Y \end{array}$$

and a quasi-coherent sheaf $\mathcal{E}$ on X with a connection ∇ relative to Y. The *pullback connection* ∇' on the quasi-coherent sheaf $h'^*\mathcal{E}$ on X' relative to Y' is the map

$$\nabla' : h'^*\mathcal{E} := \mathcal{O}_{X'} \otimes_{h'^{-1}\mathcal{O}_X} h'^{-1}\mathcal{E} \to \Omega_{X'/Y'} \otimes_{\mathcal{O}_{X'}} h'^*\mathcal{E}$$

given by

$$\nabla'(f' \otimes h'^* s) := df' \otimes h'^{-1} s + f' \cdot (\eta \otimes 1)\big(h'^*(\nabla s)\big)$$

where $\eta : h'^*\Omega_{X/Y} \to \Omega_{X'/Y'}$ is the canonical map.

The above construction yields a functor $(h, h')^* : MC(X/Y) \to MC(X'/Y')$ that sends integrable sheaves to integrable sheaves.

Finally, all of these constructions are also compatible with the above construction of a connection on $\mathcal{O}_X \otimes_{p^{-1}\mathcal{O}_Y} \Lambda$ for Λ a locally free $p^{-1}\mathcal{O}_Y$-module on Y of finite rank. For instance, if we denote ∇_Λ to be the connection on $\mathcal{E}_\Lambda := O_X \otimes_{p^{-1}\mathcal{O}_Y} \Lambda$, then

$$\nabla_\Lambda^\vee = \nabla_{\Lambda^\vee}.$$

Remark 2.34. There is actually a neat way to see the abelian category structure on $MC_{int}(X/Y)$-directly. Suppose $X \to Y$ is fixed, and consider the tangent sheaf $T_{X/Y}$. Let

$$T^\bullet T_{X/Y} := \bigoplus_{n \geq 0} T_{X/Y}^{\otimes n}$$

be the corresponding sheaf of tensor algebras, where $T_{X/Y}^{\otimes 0} := \mathcal{O}_X$. We then define the sheaf $D_{X/Y}$, the sheaf of *PD differential operators*, by taking the sheafification of the quotient of $T^\bullet T_{X/Y}$ under the equivalence relation generated by

$$\partial \cdot f - f \cdot \partial = \partial(f) \quad \text{and} \quad \partial \otimes \partial' - \partial' \otimes \partial = [\partial, \partial']$$

for local sections ∂, ∂' of $T_{X/Y}$ and f a local section of $\mathcal{O}_X$. It turns out that a quasi-coherent sheaf with integrable connection can be viewed as a quasi-coherent $D_{X/Y}$-modules on X, and this gives an equivalence of categories between $MC_{int}(X/Y)$ and $\mathrm{QCoh}_X(\mathcal{D}_{X/Y})$. This perspective also immediately shows that $MC_{int}(X/Y)$ has enough injectives.

See [EG17, Section 2.3] for more on PD differential operators, and see [Kat70b, (1.2)] for references on $MC_{int}(X/Y)$ being the category of quasi-coherent modules over some sheaf of algebras.

2.7 de Rham Cohomology Sheaves

Suppose $\pi : X \to Y$ is a morphism of schemes. Recall that we defined the de Rham cohomology group $H^p_{dR}(X/Y)$ as the image of $\Omega^\bullet_{X/Y}$ under the composition of the derived functors

$$D_{Coh}(X) \xrightarrow{R\Gamma} D(\mathrm{Ab}) \xrightarrow{H^p} \mathrm{Ab}.$$

Here, for a scheme S, we write $D_{Coh}(S)$ to be the derived category of coherent sheaves on S. We can do a similar thing for $R\pi_*$ instead.

Definition. The *p'th relative de Rham cohomology sheaf* $\mathcal{H}^p_{dR}(X/Y)$ is the image of $\Omega^\bullet_{X/Y}$ under the composition

$$D_{Coh}(X) \xrightarrow{R\pi_*} D_{Coh}(Y) \xrightarrow{H^p} \mathrm{Coh}(Y).$$

Notice that the relative de Rham cohomology sheaf is a coherent sheaf on Y.

The $\mathcal{O}_Y$-modules $\mathcal{H}^{p,q}_{Hodge}(X/Y) := R^q\pi_*\Omega^p_{X/Y}$ on Y are the *Hodge cohomology sheaves.*

This has all the usual properties that one would expect from cohomology sheaves.

Proposition 2.35. *The sheafification of the pre-sheaf on Y*

$$V \mapsto H^p_{dR}(\pi^{-1}(V)/Y)$$

is the sheaf $\mathcal{H}^p_{dR}(X/Y)$ *(where recall that* $\pi^{-1}(V) = X \times_Y V$*).*

Proof. Let $\mathcal{I}^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $\Omega^\bullet_{X/Y}$, i.e. $\mathcal{I}^{\bullet,\bullet}$ is a double complex of injective sheaves on X such that for every $p \geq 0$, we have an injective resolution $\Omega^p_{X/Y} \to \mathcal{I}^{p,\bullet}$. Then $R\pi_*\Omega^\bullet_{X/Y}$ is quasi-isomorphic to the total complex of the double complex $\pi_*\mathcal{I}^{\bullet,\bullet}$. Thus, $\mathcal{H}^p_{dR}(X/Y)$ can be computed by computing $\pi_*\mathcal{I}^{p,q}$. Notice that $\pi_*\mathcal{I}^{p,q}$ is the sheafification of the pre-sheaf $V \mapsto \Gamma(\pi^{-1}(V), \mathcal{I}^{p,q})$ on Y. Since sheafification is exact and so commutes with cohomology, it follows that $\mathcal{H}^p_{dR}(X/Y)$ is the sheafification of the pre-sheaf

$$V \mapsto H^p\left(\mathrm{Tot}\left(\Gamma(V, \mathcal{I}^{\bullet,\bullet})\right)\right).$$

Here, we also need to use the fact that sheafification commutes with totalisation of a first quadrant double complex, which $\mathcal{I}^{\bullet,\bullet}$ is. This last cohomology group is precisely $H^p_{dR}(\pi^{-1}(V), \Omega^\bullet_{X/Y})$ as required. □

We can relativize the previous two spectral sequences in de Rham cohomology.

Proposition 2.36. *Suppose* $\pi : X \to Y$ *is a morphism of schemes, with associated de Rham cohomology sheaves* $\mathcal{H}^*_{dR}(X/Y)$*. We have the following two spectral sequences:*

- *(*Hodge to de Rham SS*)* $E_1^{p,q} = \mathcal{H}^{p,p+q}_{Hodge}(X/Y) \Rightarrow \mathcal{H}^{p+q}_{dR}(X/Y)$.
- *(*Conjugate SS*)* $E_2^{p,q} = R^q\pi_*\big(H^p\Omega^\bullet_{X/Y}\big) \Rightarrow \mathcal{H}^{p+q}_{dR}(X/Y)$.

Proposition 2.37. *If* $\pi : X \to Y$ *is a proper smooth morphism of schemes of characteristic 0, then the Hodge-to-de Rham spectral sequence degenerates at the* E_1 *page.*

As a result of an abstract proper base change theorem (see [Kat70b, Section 8] for more), we have the following result on the de Rham cohomology sheaves.

Proposition 2.38. *Suppose $\pi : X \to Y$ is a proper smooth morphism, and suppose Y is Noetherian.*

1. *There is a non-empty open $V \subset Y$ such that each of the coherent sheaves $\mathcal{H}^{p,q}_{Hodge}(X/Y)$ and $\mathcal{H}^p_{dR}(X/Y)$ (for $p, q \geq 0$) are locally free over U.*
2. *If Y is of characteristic 0, then we may take $U = Y$.*
3. *Suppose further that all the Hodge and de Rham cohomology sheaves are locally free. Then, for any $g : Y' \to Y$, the canonical morphisms*
$$g^*\mathcal{H}^{p,q}_{Hodge}(X/Y) \to \mathcal{H}^{p,q}_{Hodge}(X'/Y') \quad \text{and} \quad g^*\mathcal{H}^n_{dR}(X/Y) \to \mathcal{H}^n_{dR}(X'/Y')$$
are isomorphisms for all $p, q, n \geq 0$, where $X' := X \times_Y Y'$.

2.8 Gauss-Manin Connections

It turns out that we can endow the de Rham cohomology sheaves with a canonical integrable connection, the *Gauss-Manin connection.* Let us restrict to the case $\pi : X \to Y$ a smooth map where X and Y are smooth k-schemes (k some field); we reproduce the construction given in [KO68, Section 2]. For an example of some explicit computations, see Section 3 of loc. cit.

The complex admits $\Omega^\bullet_{X/k}$ admits a canonical filtration

$$\Omega^\bullet_{X/k} = F^0\Omega^\bullet_{X/k} \supset F^1\Omega^\bullet_{X/k} \supset \cdots$$

where $F^i = F^i\Omega^\bullet_{X/k}$ is the complex with terms

$$(F^i)^p = \operatorname{Im}\left(\Omega^{p-i}_{X/k} \otimes_{\mathcal{O}_X} \pi^*\Omega^i_{Y/k} \to \Omega^p_{X/k}\right).$$

Since we have assumed X and Y are smooth over k, the sheaves $\Omega^i_{X/k}$ and $\Omega^i_{Y/k}$ are both locally free of finite type. Also, since π is smooth, we have the exact sequence

$$0 \to \pi^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0.$$

These two facts together imply that the associated graded objects

$$gr^i = gr^i\Omega^\bullet_{X/k} := F^i/F^{i+1}$$

have pth term

$$(gr^i)^p = \pi^*\Omega^i_{Y/k} \otimes_{\mathcal{O}_X} \Omega^{p-i}_{X/Y}.$$

We need to use the following lemma, which is a consequence of abstract nonsense in abelian categories (see [Sta23, 012K] for instance).

Lemma 2.39. *Suppose $K^\bullet$ is a filtered complex with a finite filtration*

$$K^\bullet = F^0K^\bullet \supset F^1K^\bullet \supset F^2K^\bullet \supset \cdots.$$

Set $gr^pK^\bullet = F^pK^\bullet/F^{p+1}K^\bullet$; notice that $gr^pK^\bullet$ is also a complex, and so taking cohomology of $gr^pK^\bullet$ makes sense. There is a spectral sequence

$$E_1^{p,q} = H^{p+q}(gr^pK^\bullet) \Rightarrow H^{p+q}(K^\bullet).$$

We take $K^\bullet = \pi_*\Omega^\bullet_{X/k}$; then we have a spectral sequence as in the lemma with first page

$$E_1^{p,q} = R^{p+q}\pi_*(gr^p),$$

which some computation shows that

$$E_1^{p,q} = \Omega^p_{Y/k} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y).$$

Now, we have a map

$$\nabla = d_1^{0,q} : \mathcal{H}^q_{dR}(X/Y) = E_1^{0,q} \to E_1^{1,q} = \Omega_{Y/k} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y).$$

After some computation with the product structure on the spectral sequence induced by the wedge product $\wedge : F^p \otimes F^q \to F^{p+q}$, one checks that this map ∇ satisfies Liebniz' rule, and so is a connection. However, we see from the explicit form of $E_1^{p,q}$ that the complex $E_1^{\bullet,q}$ is precisely the de Rham complex, which implies that the connection is integrable. This is the Gauss-Manin connection.

There is another (possibly slightly more concrete) description. In fact, the map $d_1^{0,q}$ is the connecting homomorphism of the functor $R^q\pi_*$ in the long exact sequence associated to the short exact sequence

$$0 \to gr^1 \to F^0/F^2 \to gr^0 \to 0.$$

Remark 2.40. In fact, one can upgrade the above construction of the de Rham cohomology sheaf with its Gauss-Manin connection to the following very general setting. Suppose $f : X \to Y$ is a smooth morphism of smooth S-schemes for some base scheme S. We have a functor

$$\mathcal{H}^q_{dR}(X/S, -) : MC_{int}(X/S) \to MC_{int}(Y/S)$$

that sends a flat bundle $\mathcal{E}$ with connection ∇ to $H^q \circ Rf_*(\Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$, endowed with the Gauss-Manin connection. The functor $\mathcal{H}^q_{dR}(X/S, -)$ is in fact the q'th right-derived functor of $\mathcal{H}^0_{dR}(X/S, -)$. The construction of the Gauss-Manin connection is essentially the same as above.

See [Kat70b, (3.0)] for all the detail you may need.

3 Differential Geometry in Positive Characteristic

Recall that a scheme is of *characteristic* p if it is a scheme over $\operatorname{Spec}\mathbb{F}_p$, or equivalently, if $p\mathcal{O}_X = 0$. Throughout, unless otherwise stated, we will assume all schemes to be over $\mathbb{F}_p$ for a fixed prime p.

Our main reference is [Kat70b].

3.1 Frobenius and Cartier Morphisms

For this subsection, see [Ill02, Section 3].

Definition. Suppose X is a $\mathbb{F}_p$-scheme. The *absolute Frobenius* F_X is the endomorphism of the ringed space $(X, \mathcal{O}_X)$ that is the identity on the underlying space X, and acts on $\mathcal{O}_X$ by raising to the pth power.

X is said to be *perfect* if F_X is an isomorphism.

It is easy to see that if $X = \operatorname{Spec} A$, then F_X corresponds to the usual Frobenius morphism $A \to A, a \mapsto a^p$. One can also check easily that if $f : X \to Y$ is a morphism of $\mathbb{F}_p$-schemes, then we have a commuting square

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{F_Y} & Y. \end{array}$$

Definition. Suppose X is a Y-scheme with structure map $f : X \to Y$. The *relative Frobenius twist* is the Y-scheme

$$X^{(p)} = X \times_{Y, F_Y} Y,$$

i.e. it is defined by the commuting square

$$\begin{array}{ccc} X^{(p)} & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{F_Y} & Y. \end{array}$$

For any $\mathbb{F}_p$-scheme X, the *absolute Frobenius twist* is the Frobenius twist of X relative to $Y = \operatorname{Spec}\mathbb{F}_p$.

Due to the fibre product, the morphisms $F_X : X \to X$ and $f : X \to Y$ induce a map $F_{X/Y} : X \to X^{(p)}$ sitting inside the following diagram.

$$\begin{array}{cccc} X & & & \\ & \searrow^{F_{X/Y}} & & \searrow^{F_X} \\ & & X^{(p)} & \xrightarrow{W} & X \\ & {\scriptstyle f}\searrow & \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ & & Y & \xrightarrow[F_Y]{} & Y \end{array}$$

It is a fact that the endomorphism $F_{X/Y} \circ W$ on $X^{(p)}$ is the absolute Frobenius on $X^{(p)}$.

Definition. The morphism $F_{X/Y} : X \to X^{(p)}$ is the *relative Frobenius of X over Y*.

Notice that the relative Frobenius is a homeomorphism on the underlying topological space.

Example 3.1. If $Y = \operatorname{Spec} A$ and $X = \mathbb{A}^n_A = \operatorname{Spec} A[t_1, ..., t_n]$, then $X' = \operatorname{Spec} A[t_1, ..., t_n]$ with the morphism $F_{X/Y} : X \to X^{(p)}$ is induced by the ring morphism

$$A[t_1, ..., t_n] \to A[t_1, ..., t_n], \quad t_i \mapsto t_i^p$$

whereas the canonical projection morphism $X^{(p)} \to X$ is induced by the ring morphism

$$A[t_1, ..., t_n] \to A[t_1, ..., t_n], \quad at_i \mapsto a^p t_i.$$

Remark 3.2. While in the above affine case it is true that the relative Frobenius twist of X is isomorphic as a scheme to X, this is not always the case.

Proposition 3.3. *Suppose $f : X \to Y$ is a smooth morphism of pure relative dimension n. Then, the relative Frobenius $F_{X/Y} : X \to X^{(p)}$ is a finite flat morphism, and the $\mathcal{O}_{X^{(p)}}$-algebra $F_{X/Y,*}\mathcal{O}_X$ is locally free of rank p^n.*

Corollary 3.3.1. *If f is étale, then $F_{X/Y}$ is an isomorphism and the square*

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{F_Y} & Y. \end{array}$$

is Cartesian.

Let us now see the interplay of the relative Frobenius morphism with the relative de Rham complex. The basic fact at play is that $d(s^p) = ps^{p-1}ds = 0$ in characteristic p. In particular, the canonical morphism $F_X^*\Omega_{X/Y} \to \Omega_{X/Y}$ coming from the square

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{F_Y} & Y \end{array}$$

is the zero map. Similarly, the canonical morphism $F_{X/Y}^*\Omega_{X^{(p)}/Y} \to \Omega_{X/Y}$ coming from the commuting square

$$\begin{array}{ccc} X & \xrightarrow{F_{X/Y}} & X^{(p)} \\ \downarrow f & & \downarrow \\ Y & \xrightarrow{id} & Y \end{array}$$

is the zero map as well.

Another consequence is that the differential of the complex $F_{X/Y,*}\Omega^\bullet_{X/Y}$ is $\mathcal{O}_{X^{(p)}}$-linear. In particular, this forces that the cohomology sheaves $H^qF_{X/Y,*}\Omega^\bullet_{X/Y}$ are all $\mathcal{O}_{X^{(p)}}$-modules. The exterior product then induces a graded anti-commutative $\mathcal{O}_{X^{(p)}}$-algebra structure on

$$\bigoplus_q H^qF_{X/Y,*}\Omega^\bullet_{X/Y}.$$

The following theorem of Cartier is an important fundamental result in the differential theory in positive characteristic. In order to state it, labelling the canonical projection map $X^{(p)} \to X$ (coming from the fibre product) by F', we know that

$$\Omega^1_{X^{(p)}/Y} \cong F'^*\Omega^1_{X/Y} = \mathcal{O}_{X^{(p)}} \otimes_{F'^{-1}\mathcal{O}_X} F'^{-1}\Omega^1_{X/Y}.$$

In particular, if ds is a local section of $\Omega^1_{X/Y}$, we can denote the local section of $\Omega^1_{X^{(p)}/Y}$ by $1 \otimes ds$.

Theorem 3.4. *Suppose Y is a $\mathbb{F}_p$-scheme and $f : X \to Y$ a morphism.*

1. *There exists a unique homomorphism of graded $\mathcal{O}_{X^{(p)}}$-algebras*

$$\gamma : \bigoplus_q \Omega^q_{X^{(p)}/Y} \to \bigoplus_q H^qF_{X/Y,*}\Omega^\bullet_{X/Y}$$

   satisfying the two conditions:

   - *for $i = 0$, γ is given by the homomorphism $F_{X/Y}^* : \mathcal{O}_{X^{(p)}} \to F_{X/Y,*}\mathcal{O}_X$; and*
   - *for $i = 1$, γ sends $1 \otimes ds \in \Omega^1_{X^{(p)}/Y}$ to the class of $s^{p-1}ds$ in $H^1F_{X/Y,*}\Omega^\bullet_{X/Y}$.*

2. *If f is smooth, then γ is an isomorphism.*

The isomorphism γ is constructed in [Kat70b, Theorem 7.2].

Definition. For X a smooth Y-scheme, the isomorphism γ in the above theorem is called the *Cartier isomorphism*, and is denoted by C^{-1}. The map $C = \gamma^{-1}$ is often called the *Cartier operation.*

Suppose Y is perfect and $f : X \to Y$ is smooth. We have an isomorphism

$$\bigoplus_q \Omega^q_{X/Y} \to \bigoplus_q F'_*\Omega^q_{X^{(p)}/Y}$$

where $F' : X^{(p)} \to X$ is the canonical projection morphism. Composing with the Cartier isomorphism, we get an isomorphism

$$C^{-1}_{abs} : \bigoplus_q \Omega^q_{X/Y} \to \bigoplus_q H^qF_{X,*}\Omega^\bullet_{X/Y}$$

of graded $\mathcal{O}_X$-algebras

Definition. The above isomorphism C^{-1}_{abs} is called the *absolute Cartier isomorphism.*

As a result of the Cartier isomorphism, one can rewrite the conjugate spectral sequence for smooth $\pi : X \to Y$, where Y is such that F_Y is flat, as

$$E_2^{p,q} = F_Y^*(R^q\pi_*(\Omega^p_{X/Y})) \Rightarrow \mathcal{H}^{p+q}_{dR}(X/Y).$$

For more information, see [Kat70b, Section 7].

3.2 p-Curvature and Nilpotence

As usual, fix $\pi : X \to Y$. Throughout, we will use the identification $T_{X/Y} = \mathrm{Der}_Y(\mathcal{O}_X, \mathcal{O}_X)$.

Suppose D is any derivation of $\mathcal{O}_X$. One can check, by using the fact that $p|\binom{p}{i}$ for $1 \le i \le p-1$, that D^p also satisfies Liebniz rule. Thus, the pth iterate of a derivation is also a derivation. Thus, $T_{X/Y}$ is a sheaf of a restricted p-Lie algebra, where we recall the definition of a restricted p-Lie algebra below.

Definition. Suppose V is a Lie algebra over a field k of characteristic p. A *p-operation* is a map $()^p : V \to V$ satisfying the following axioms:

1. $ad_{X^p} = ad_X^p$ the composition of ad_X with itself p times;
2. $(aX)^p = a^pX^p$ for all $a \in k$;
3. $(X+Y)^p = X^p + Y^p + \sum_{i=1}^{p-1} \frac{1}{i} s_i(X,Y)$, where $s_i(X,Y)$ is the coefficient of t^{i-1} in the formal expression $ad^{p-1}_{tX+Y}(X)$.

Here, ad_X is the linear map $ad_X : V \to V, Y \mapsto [X,Y]$.

Notice that for a quasi-coherent sheaf $\mathcal{E}$ on X, the sheaf $\mathcal{E}nd(\mathcal{E})$ is also a sheaf of restricted p-Lie algebras, with the p-operation simply iterating an endomorphism p-times. Thus if ∇ is a flat connection on $\mathcal{E}$, we have a Lie algebra morphism of sheaves

$$\nabla : T_{X/Y} \to \mathcal{E}nd(\mathcal{E})$$

between sheaves of restricted p-Lie algebras, and so one may ask whether ∇-preserves the p-operation. This yields the following definition.

Definition. The *p-curvature* ψ of a connection ∇ is the mapping of sheaves

$$\psi : T_{X/Y} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}), \quad D \mapsto \nabla_D^p - \nabla_{D^p}.$$

Here, it requires a computation to check that $\psi(D)$ is in fact a $\mathcal{O}_X$-linear map, as *a priori* it is only $\pi^{-1}\mathcal{O}_Y$-linear.

The significance of p-curvature comes from the following theorem of Cartier.

Theorem 3.5. *Suppose $\pi : X \to Y$ is a smooth morphism of schemes. Recall the relative Frobenius twist $X^{(p)}$ of X with respect to Y, with relative Frobenius $F_{X/Y} : X \to X^{(p)}$.*

There is an equivalence of categories between the category of quasicoherent sheaves on $X^{(p)}$ and the full subcategory of $MC_{int}(X/Y)$ consisting of sheaves on X with connections of zero p-curvature. The equivalence is given as follows:

- *Given a quasi-coherent sheaf $\mathcal{E}$ on $X^{(p)}$, there is a unique integrable Y-connection ∇ of p-curvature 0 on $F^*_{X/Y}(\mathcal{E})$ such that $\mathcal{E} \cong \ker \nabla$.*
- *For a flat quasi-coherent integrable sheaf $(\mathcal{E}, \nabla)$ with zero p-curvature, the sheaf* $\ker \nabla$ *is naturally a quasi-coherent sheaf on $X^{(p)}$.*

We have some basic properties of p-curvature.

Proposition 3.6. *Suppose ψ is the p-curvature of a flat connection ∇ on $\mathcal{E} \in \mathrm{QCoh}(X)$.*

1. *(p-linearity) $\psi(gD) = g^p\psi(D)$ for all local sections g of $\mathcal{O}_X$ and D of $T_{X/Y}$.*
2. *For $U \subset X$ open and $D \in T_{X/Y}(U)$, the three elements ∇_D, ∇_{D^p}, and $\psi(D)$ of $\mathcal{E}nd_{\mathcal{O}_U}(\mathcal{E}|_U)$ mutually commute.*
3. *Suppose $X \to Y$ is smooth. If D, D' are any two local sections of $T_{X/Y}$, then $\psi(D)$ and $\psi(D')$ commute.*

4. Suppose $X \to Y$ is smooth. $\psi(D) : \mathcal{E}|_U \to \mathcal{E}|_U$ is a horizontal map, for all $D \in T_{X/Y}(U)$.

Corollary 3.6.1. *Suppose ψ is the p-curvature of a flat connection ∇ on $\mathcal{E} \in \mathrm{QCoh}(X)$, and suppose $X \to Y$ is smooth. Let $n \geq 1$ be a given integer. The following are equivalent.*

- *There exists a filtration of $(\mathcal{E}, \nabla)$ of length $\leq n$ whose associated graded objects all have p-curvature 0.*
- *Whenever $D_1, ..., D_n$ are local sections of $T_{X/Y}$, we have $\psi(D_1) \cdots \psi(D_n) = 0$.*
- *There exists a covering of S by affine opens, and on each such open affine U there exist sections $u_1, ..., u_r \in \mathcal{O}_X(U)$ such that $\Omega^1_{X/Y}|_U$ is free on $du_1, ..., du_r$ (these u_i can be viewed as coordinates) such that for every r-tuple $(w_1, ..., w_r)$ of integers with $\sum_i w_i = n$, we have*

  $$\nabla^{pw_1}_{\partial_{u_1}} \cdots \nabla^{pw_r}_{\partial_{u_r}} = 0$$

  where $\{\partial_{u_i}\}$ is dual to the basis $\{du_i\}$.
- *The map $\psi^n : F_{X/Y,*}\mathcal{E} \to F_{X/Y,*}\mathcal{E} \otimes_{\mathcal{O}_{X^{(p)}}} (\Omega^1_{X^{(p)}/Y})^{\otimes n}$ is the zero map.*

Definition. Say that $(\mathcal{E}, \nabla)$ is *nilpotent of exponent* $\leq n$ when one of the above equivalent conditions holds. Say that $(\mathcal{E}, \nabla)$ is nilpotent if it is nilpotent of exponent $\leq n$ for some $n \geq 1$.

Let $Nil(X/Y)$ be the full subcategory of $MC_{int}(X/Y)$ of nilpotent flat sheaves, and let $Nil^n(X/Y)$ be the full subcategory consisting of those flat sheaves that are nilpotent of exponent $\leq n$.

Proposition 3.7. *Suppose $(\mathcal{E}, \nabla) \in MC_{int}(X/Y)$. If $(\mathcal{E}, \nabla)$ is nilpotent, then for any $D \in T_{X/Y}(U)$ (viewed as $\pi^{-1}\mathcal{O}_Y$-linear endomorphism of $\mathcal{O}_X$) which is nilpotent, the corresponding $\pi^{-1}\mathcal{O}_Y|_U$-linear endomorphism ∇_D of $\mathcal{E}|_U$ is nilpotent.*

If $\pi : X \to Y$ is smooth, then the converse holds.

Proposition 3.8. *The category $Nil(X/Y)$ is an exact abelian subcategory of $MC_{int}(X/Y)$, that is stable under internal Hom and the tensor product.*

Each $Nil^n(X/Y)$ is stable under taking sub-objects and taking quotients. If $\mathcal{E} \in Nil^n(X/Y)$ and $\mathcal{F} \in Nil^m(X/Y)$, then $\mathcal{E} \otimes \mathcal{F}$ and $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ (with their corresponding flat connections) are in $Nil^{n+m-1}(X/Y)$

Proposition 3.9. *Suppose $\pi : X \to Y$ is a smooth morphism, and suppose we have a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{h} & Y \end{array}$$

such that π' is also smooth. We assume (as always) that all schemes are over $\mathbb{F}_p$. Then, under the inverse image functor

$$(g, h)^* : MC_{int}(X/Y) \to MC_{int}(X'/Y'),$$

we have for any $n \geq 1$,

$$(g, h)^*\big(Nil^n(X/Y)\big) \subseteq Nil^n(X'/Y').$$

There is also a very strong statement ([Kat70b, Theorem 5.10]) on the stability of nilpotence under the de Rham cohomology sheaf functor given in Section 2.8. In fact, it not only gives bounded on the nilpotence exponent for the de Rham cohomology sheaves, but also gives information about a certain spectral sequence computing the de Rham cohomology sheaf. We do not reproduce it here, since we did not introduce the de Rham cohomology sheaf functor in that level of generality. However, we do have the following result as a consequence of Katz's theorem. The statement given below is adapted from [EG17, Theorem 2.7].

Proposition 3.10. *Suppose $X \to Y$ is a proper smooth morphism of pure relative dimension n between smooth k-varieties, for k a perfect field of characteristic p. Then, the Gauss-Manin connection on $\mathcal{H}^q_{dR}(X/Y)$ has nilpotent p-curvature.*

3.3 Nilpotence over Global Bases

Suppose $Y = \operatorname{Spec} R$ where R is an integral domain, finitely generated as a ring over $\mathbb{Z}$, whose field of fractions has characteristic 0. Such a Y is often referred to as a global affine variety. Fix a morphism $\pi : X \to Y$.

Suppose p is a prime that is not invertible on X, i.e. p is not invertible as an element of $\mathcal{O}_X$. Write $Y_p := \operatorname{Spec}(R/pR) = Y \times_{\mathbb{Z}} \mathbb{F}_p$ and $X_p := X \times_{\mathbb{Z}} \mathbb{F}_p$ for the reduction modulo p. We obviously have a commuting diagram

$$\begin{array}{ccc} X_p & \longrightarrow & X \\ \big\downarrow{\scriptstyle \pi_p} & & \big\downarrow{\scriptstyle \pi} \\ Y_p & \longrightarrow & Y \end{array}$$

which thus induces a functor

$$MC_{int}(X/Y) \to MC_{int}(X_p/Y_p).$$

We denote this functor by $(\mathcal{E}, \nabla) \mapsto (\mathcal{E}_p, \nabla_p)$. One can now ask about the nilpotence properties of $\mathcal{E}_p$.

Definition. Say $(\mathcal{E}, \nabla)$ is *globally nilpotent on* X/Y if, for every prime p not invertible on S, we have $(\mathcal{E}_p, \nabla_p) \in Nil(X_p/Y_p)$.

Suppose $n \geq 1$ is fixed. Say $(\mathcal{E}, \nabla)$ is *globally nilpotent of exponent* n *on* X/Y if, for every prime p not invertible on S, we have $(\mathcal{E}_p, \nabla_p) \in Nil^n(X_p/Y_p)$.

We have some basic properties. The following is obvious from Proposition 3.9.

Proposition 3.11. *Suppose $\pi : X \to Y$ and $\pi' : X' \to Y'$ are smooth, with Y and Y' global affine varieties. Suppose we have a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\;g\;} & X \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\;h\;} & Y. \end{array}$$

Suppose $(\mathcal{E}, \nabla)$ is an object of $MC_{int}(X/Y)$ with $\mathcal{E}$ locally free of finite rank on X. If $(\mathcal{E}, \nabla)$ is globally nilpotent (resp. globally nilpotent of exponent n) on X/Y then its inverse image $(g, h)^*(\mathcal{E}, \nabla)$ is also globally nilpotent (resp. globally nilpotent of exponent n) on X'/Y'.

Proposition 3.12. *Suppose Y a global affine variety, $\pi : X \to Y$ smooth, and $g : X' \to X$ a proper étale morphism. Suppose $(\mathcal{E}, \nabla)$ is an object of $MC_{int}(X'/Y)$ with $\mathcal{E}$ locally free of finite rank on X. Then, $(\mathcal{E}, \nabla)$ is globally nilpotent (resp. globally nilpotent of exponent n) on X'/Y if and only if $(g_*\mathcal{E}, \nabla)$ is also globally nilpotent (resp. globally nilpotent of exponent n) on X/Y.*

It turns out that the de Rham cohomology sheaves equipped with the Gauss-Manin connection are always globally nilpotent.

Proposition 3.13. *Suppose $\pi : X \to S$ is proper and smooth with S connected, where X and S are smooth over a global affine variety Y. Assume that all the coherent sheaves $\mathcal{H}^{p,q}_{Hodge}(X/S)$ and $\mathcal{H}^n_{dR}(X/S)$ are locally free on S. For each integer $n \geq 0$, let $h(n)$ denote the number of integers i such that $\mathcal{H}^{i,n-i}_{Hodge}(X/S)$ is non-zero. Then, $\mathcal{H}^n_{dR}(X/S)$ with the Gauss-Manin connection is globally nilpotent of exponent $h(n)$ on S/Y.*

4 Logarithmic Singularities and Monodromy 'at ∞'

Suppose $\pi : X \to S$ is a smooth morphism of schemes that is pure of relative dimension n. Thus, $\Omega_{X/S}$ is a locally free $\mathcal{O}_X$-module of rank n. Let $i : Y \hookrightarrow X$ the closed immersion of a divisor Y in X over S. Let $j : X - Y \hookrightarrow X$ be the open immersion of the complement.

Definition. The divisor Y is said to have *normal crossings* if X admits a cover $\mathcal{U}$ by open affines such that for any $U \in \mathcal{U}$,

- U is étale over $\mathbb{A}^n_S$ (via coordinates $x_1, ..., x_n$ where $x_i : U \to S$ are morphisms such that $dx_1, ..., dx_n$ form a basis for $\Omega_{X/S}|_U$); and
- there exists $r \geq 0$ such that $Y|_U$ is defined by the equation $x_1 \cdots x_r = 0$ (i.e. Y is the inverse image under $U \to \mathbb{A}^n_S$ of the first r-coordinate hyperplanes in $\mathbb{A}^n_S$).

Say that an open affine $\mathcal{U}$ satisfying the above two conditions is an open cover of X *adapted* to Y, and that $U \in \mathcal{U}$ is an open affine *adapted* to Y.

In some sense, divisors with normal crossings are those divisors such that their prime divisors (i.e. subschemes) all intersect transversely.

Throughout this section, we will be in the above setup, with the added assumption that Y is a normal crossings divisor. The main reference is [Kat70b].

4.1 de Rham Cohomology and Connections with Singularities

We define a locally free $\mathcal{O}_X$-module $\Omega_{X/S}(\log Y)$. Let $\mathcal{U}$ be an open cover of X adapted to Y. For any $U \in \mathcal{U}$ with coordinates $x_1, ..., x_n$, let r be such that $Y|_U$ is defined by $x_1 \cdots x_r = 0$. Then, we set

$$\Omega_{X/S}(\log Y) := \mathcal{O}_U(\tfrac{dx_1}{x_1}) \oplus \cdots \oplus \mathcal{O}_U(\tfrac{dx_r}{x_r}) \oplus \mathcal{O}_U dx_{r+1} \oplus \cdots \oplus \mathcal{O}_U dx_n.$$

We again have the exterior differential map $d : \mathcal{O}_X \to \Omega_{X/S}(\log Y)$ given in the obvious way. Set $\Omega^p_{X/S}(\log Y) := \bigwedge^p_{\mathcal{O}_X} \Omega_{X/S}(\log Y)$ with $\Omega^0_{X/S}(\log Y) := \mathcal{O}_X$. As before, this map induces a complex structure on $\Omega^\bullet_{X/S}(\log Y)$. One checks that the canonical inclusion $\mathcal{O}_X \hookrightarrow j_*\mathcal{O}_{X-Y}$ extends so that we can view $\Omega^p_{X/S}(\log Y)$ as a subsheaf of $j_*(\Omega^p_{(X-Y)/S})$, and in fact this map

$$\Omega^p_{X/S}(\log Y) \hookrightarrow j_*(\Omega^p_{(X-Y)/S})$$

is a map of complexes. In other words, the usual exterior differentiation in $j_*\Omega^\bullet_{(X-Y)/S}$ preserves the complex $\Omega^\bullet_{X/S}(\log Y)$.

Definition. The complex $\Omega^\bullet_{X/S}(\log Y)$ is the *de Rham complex of X/S with logarithmic singularities along Y*.

We can now define connections as before.

Definition. Suppose $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_X$-module. An *S-connection on $\mathcal{E}$ with logarithmic singularities along Y* is a homomorphism of abelian sheaves

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}(\log Y)$$

satisfying the Liebniz rule.

Let $MC(X/S, \log Y)$ be the category whose objects are quasicoherent sheaves on X with a S-connection having logarithmic singularities along Y, and with morphisms the horizontal morphisms.

As usual, a connection with logarithmic singularities can be extended to maps

$$\nabla : \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^p_{X/S}(\log Y) \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{p+1}_{X/S}(\log Y).$$

Definition. A connection with logarithmic singularities is *integrable* if $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}(\log Y)$ is a complex. In such a case, this complex is called the *de Rham complex of $(\mathcal{E}, \nabla)$ with logarithmic singularities along Y*.

Let $MC_{int}(X/S, \log Y)$ be the full subcategory of $MC(X/S, \log Y)$ consisting of sheaves with integrable connections.

As before, $MC_{int}(X/S, \log Y)$ is an abelian category that has enough injectives and contains a tensor product and (not everywhere defined) internal Hom.

Consider the sheaf of $\pi^{-1}(\mathcal{O}_S)$-Lie algebras on X given by

$$T_{X/S}(\log Y) := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/S}(\log Y), \mathcal{O}_X).$$

It is clear that for U an open affine adapted to Y, we have

$$T_{X/S}(\log Y) = \mathcal{O}_U(x_1\partial_{x_1}) \oplus \cdots \oplus \mathcal{O}_U(x_r\partial_{x_r}) \oplus \mathcal{O}_U\partial_{x_{r+1}} \oplus \cdots \oplus \mathcal{O}_U\partial_{x_n},$$

where $\{\partial_{x_i}\}$ is dual to $\{dx_i\}$. Moreover, as before, one sees that the data of an integrable S-connection on $\mathcal{E}$ with logarithmic singularities along Y is the same as the data of an $\mathcal{O}_X$-linear map

$$\nabla : T_{X/Y}(\log Y) \to \mathcal{E}nd_{\pi^{-1}\mathcal{O}_S}(\mathcal{E})$$

that is also a morphism of sheaves of $\pi^{-1}\mathcal{O}_S$-Lie algebras, satisfying Liebniz rule on local sections.

Again, we can define de Rham and Hodge cohomology with logarithmic singularities as before:

$$H^p_{dR,\log Y}(X/S) := H^i\big(R\Gamma(X, \Omega^\bullet_{X/S}(\log Y))\big)$$
$$\mathcal{H}^p_{dR,\log Y}(X/S) := R^i\pi_*\big(\Omega^\bullet_{X/S}(\log Y)\big)$$
$$\mathcal{H}^{p,q}_{Hodge,\log Y}(X/S) := R^q\pi_*(\Omega^p_{X/S}(\log Y))$$

Here, $\mathcal{H}^p_{dR,\log Y}(X/S)$ is sheaf on S, and as before can be endowed with a Gauss-Manin connection. These are of course right-derived functors of the corresponding functor for $p = 0$.

More generally, as in Remark 2.40, suppose $X \to S$ is a smooth morphism over a base-scheme T such that $S \to T$ is also smooth, and $i : Y \hookrightarrow X$ is a divisor with normal crossings relative to T. We then have the functor

$$\mathcal{H}^p_{dR,\log Y}(X/S, -) : MC_{int}(X/T, \log Y) \to MC_{int}(S/T, \log Y)$$

that sends $(\mathcal{E}, \nabla)$ to the shead $R^q\pi_*\big(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}(\log Y)\big)$ endowed with the Gauss-Manin connection.

We also have the analog of Proposition 2.38.

Proposition 4.1. *In addition to the assumptions at the beginning of the section, suppose Y is Noetherian.*

1. *There is a non-empty open $V \subset S$ such that each of the coherent sheaves $\mathcal{H}^{p,q}_{Hodge,\log Y}(X/S)$ and $\mathcal{H}^p_{dR}(X/S)$ (for $p, q \geq 0$) are locally free over V.*
2. *If S is of characteristic 0, then we may take $V = S$.*
3. *Suppose further that all the Hodge and de Rham cohomology sheaves are locally free. Suppose $g : S' \to S$ is any morphism, and let $X' := X \times_S S'$. Let Y' be the fibre product of $i : Y \hookrightarrow X$ and the canonical projection $X' \to X$. Then, Y' is a divisor in X' with normal crossings relative to S', and the canonical morphisms*

$$g^*\mathcal{H}^{p,q}_{Hodge,\log Y}(X/S) \to \mathcal{H}^{p,q}_{Hodge,\log Y'}(X'/S') \quad \textit{and} \quad g^*\mathcal{H}^m_{dR,\log Y}(X/S) \to \mathcal{H}^m_{dR,\log Y'}(X'/S')$$

are isomorphisms for all $p, q, m \geq 0$.

For S characteristic 0 and $\pi : X \to S$ proper and smooth, the canonical morphism of complexes of sheaves on X

$$\Omega^\bullet_{X/S}(\log Y) \to j_*\Omega^\bullet_{(X-Y)/S}$$

is in fact a quasi-isomorphism in the derived category. In particular, we have isomorphisms

$$H^i_{dR,\log Y}(X/S) \cong H^*_{dR}((X - Y)/S)$$

of de Rham cohomology sheaves on S. Thus, de Rham cohomology with logarithmic singularities along Y is only really interesting in positive characteristic.

4.2 Logarithmic Singularities for Positive Characieristic

Let us now suppose that all of our schemes are of characteristic p.

Definition. Suppose $n \geq 1$. An object $(\mathcal{E}, \nabla) \in MC_{int}(X/S, \log Y)$ is *nilpotent with exponent* $\leq n$ if there exists a filtration of $\mathcal{E}$ of length $\leq n$ whose associated graded objects all have p-curvature 0.

A sheaf with connection is *nilpotent* if it is nilpotent with exponent $\leq n$ for some $n \geq 1$.

Let $Nil^n(X/S, \log Y)$ be the full subcategory of $MC_{int}(X/S, \log Y)$ consisting of nilpotent sheaves with exponent $\leq n$. Let $Nil(X/S, \log Y)$ be the full subcategory of nilpotent sheaves.

As before, one can check that Gauss-Manin connections are nilpotent, and that one can explicitly compute the exponent of nilpotence (see [Kat70b, Theorem 6.1]).

The Cartier isomorphism also extends as expected.

Proposition 4.2. *The (unique) Cartier isomorphism*

$$C^{-1} : \Omega^p_{(X^{(p)} - Y^{(p)})/S} \xrightarrow{\sim} H^q(F_{(X-Y)/S,*}\Omega^\bullet_{(X-Y)/S})$$

induces an isomorphism of $\mathcal{O}_{X^{(p)}}$-modules

$$C^{-1} : \Omega^p_{X^{(p)}/S}(\log Y^{(p)}) \xrightarrow{\sim} H^q\big(F_{X/S,*}\Omega^\bullet_{X/S}(\log Y)\big).$$

The Cartier isomorphism yields the following result, as before.

Proposition 4.3. *Suppose S is such that $F_S : S \to S$ is flat. Then, the conjugate spectral sequence can be rewritten as*

$$E_2^{p,q} = F_S^*\big(R^q\pi_*\Omega^q_{X/S}(\log Y)\big) \Rightarrow \mathcal{H}^{p+q}_{dR,\log Y}(X/S).$$

If $h_Y(m)$ denotes the number of i such that $F_S^*\big(R^i\pi_*\Omega^{m-i}_{X/S}(\log Y)\big) \neq 0$, then for any smooth base $S \to T$ we have

$$\mathcal{H}^m_{dR,\log y}(X/S) \in Nil^{h_Y(n)}(S/T),$$

where as usual we equip the de Rham cohomology sheaves with the Gauss-Manin connection.

We also have the following analog of Proposition 3.13, which is in some sense a corollary of the preceding result.

Proposition 4.4. *Suppose, in addition to the assumptions listed at the beginning of the section, that $\pi : X \to S$ is proper and S is connected. Suppose also that X and S are smooth over a global affine variety T. Assume that all the coherent sheaves $\mathcal{H}^{p,q}_{Hodge,\log Y}(X/S)$ and $\mathcal{H}^m_{dR,\log Y}(X/S)$ are locally free on S. For each integer $m \geq 0$, let $h_Y(m)$ denote the number of integers i such that $\mathcal{H}^{i,m-i}_{Hodge,\log Y}(X/S)$ is non-zero. Then, $\mathcal{H}^m_{dR,\log Y}(X/S)$ with the Gauss-Manin connection is globally nilpotent of exponent $h_Y(m)$ on S/T.*

4.3 Monodromy

The idea of monodromy is simple: by probing with loops and seeing what changes as we move around the loop, we learn more about the underlying space. More precisely, if X is a path-connected and locally simply path-connected topological space, then we have the following proposition.

Proposition 4.5. *The functor $\mathcal{F} \to \mathcal{F}_{x_0}$ is an equivalence of categories between the category of complex local systems on X and the category of complex finite-dimensional representations of $\pi_1(X, x_0)$.*

What is this representation? Suppose $\alpha : [0,1] \to X$ is a path. Then, $\alpha^*\mathcal{F}$ is a constant sheaf on $[0,1]$, and so α induces an isomorphism $[\alpha] : \mathcal{F}_{\alpha(0)} \to \mathcal{F}_{\alpha(1)}$. This isomorphism only depends on the homotopy class of α. In particular, if α is a loop, we get an automorphism of the vector space $\mathcal{F}_{x_0}$, which then yields the representation $\pi_1(X, x_0) \to \mathrm{Aut}(\mathcal{F}_{x_0})$.

By Riemann-Hilbert, it thus follows that sheaves with connections carry representation theoretic information about the underlying space X. This version of monodromy is closely linked to differential equations. See for instance Example 4.15 and the *local monodromy theorem.* For a good discussion of this in the analytic setting, see [Del70].

4.4 Regular Singular Points

We now need to briefly discuss the theory of regular singular points, in order to discuss local monodromy. For this, we suppose that k is a field of characteristic 0, and that $K = k(C)$ is the function field of a projective smooth absolutely irreducible curve C over k. By [Har77, Theorem II.8.6A], $\Omega_{K/k}$ is a 1 dimensional vector space over K.

Definition. Suppose W is a finite dimensional vector space over K. A *k-connection ∇ on W* is an additive mapping $\nabla : W \to \Omega^1_{K/k} \otimes_K W$ satisfies $\nabla(fw) = df \otimes w + f\nabla w$ for $f \in K, w \in W$.

As usual, a k-connection on W is the same as the data of a K-linear mapping $\nabla : \mathrm{Der}(K/k) \to \mathrm{End}_k(W)$ satisfying the Liebniz rule. Such a mapping is necessarily a Lie algebra homomorphism since $\Omega^2_{K/k} = 0$.

As before, we can define the abelian category $MC(K/k)$ of finite dimensional vector spaces W over K equipped with a k-connection, where the morphisms are the horizontal K-linear maps. This category also has an internal Hom and a tensor product, the construction of which is analagous to that given in Section 2.6.

How does this relate to the previous notion of connections on sheaves over schemes? Suppose S is a sufficiently nice scheme over k (with k of characteristic 0), and $f : X \to S$ a smooth morphism of relative dimension 1 whose generic fibres are geometrically connected. Let K be the function field of X. Then, we have the following diagram

$$\begin{array}{ccc} \mathrm{Spec}\, K & \xhookrightarrow{\eta'} & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}\, k & \xhookrightarrow{\eta} & S \end{array}$$

where η and η' are inclusions of the generic point. This diagram induces the inverse image functor

$$MC_{int}(X/S) \to MC_{int}(\mathrm{Spec}\, K/\mathrm{Spec}\, k) = MC(\mathrm{Spec}\, K/\mathrm{Spec}\, k).$$

Suppose $(\mathcal{V}, \nabla) \in MC_{int}(X/S)$, with $\mathcal{E}$ only quasi-coherent for now. Then, under the pull-back, we get a K-vector space V (recall that a quasi-coherent sheaf on $\mathrm{Spec}\, K$ is the same as a K-vector space). In order to ensure that V is finite-dimensional, we need to assume that $\mathcal{E}$ is locally free of finite rank. If that is the case, the one sees easily that V with the connection induced by the pull-back satisfies the above definitions. In fact, $MC(K/k) \subset MC(\mathrm{Spec}\, K/\mathrm{Spec}\, k)$ is the full subcategory of finite rank (equivalently, coherent) sheaves.

Now, let $\mathfrak{p}$ be a place of K/k, i.e. a closed point of C. Let $\mathcal{O}_\mathfrak{p}$ be the local ring, $\mathfrak{m}_\mathfrak{p}$ the maximal ideal, and let $\mathrm{ord}_\mathfrak{p} K \to \mathbb{Z} \cup \{\infty\}$ be the associated valuation of the order of zero at $\mathfrak{p}$. Consider the $\mathcal{O}_\mathfrak{p}$-submodule of $\mathrm{Der}(K/k)$ given by

$$\mathrm{Der}_\mathfrak{p}(K/k) = \{D \in \mathrm{Der}(K/k) : D(\mathfrak{m}_\mathfrak{p}) \subset \mathfrak{m}_\mathfrak{p}\}.$$

Lemma 4.6. *If $f \in K^*$ is not a unit at $\mathfrak{p}$ (i.e. $\mathrm{ord}_\mathfrak{p} f \neq 0$), then $f\frac{d}{df}$ is an $\mathcal{O}_\mathfrak{p}$-basis for $\mathrm{Der}_\mathfrak{p}(K/k)$.*

Definition. Let $(W, \nabla) \in MC(K/k)$. Say that (W, ∇) has a *regular singular point* at $\mathfrak{p}$ if there exists an $\mathcal{O}_\mathfrak{p}$-lattice $W_\mathfrak{p}$ of W such that

$$\mathrm{Der}_\mathfrak{p}(K/k) \cdot W_\mathfrak{p} \subset W_\mathfrak{p}.$$

In elementary terms, $\mathfrak{p}$ is a regular singular point of (W, ∇) if there exists a K-basis $\mathbf{e} = (e_1, ..., e_m)$ such that, for some (equivalently any) uniformising parameter h at $\mathfrak{p}$, one has

$$\nabla(h\tfrac{d}{dh}) \cdot \mathbf{e} = \mathbf{e}B$$

for some $B \in M_n(\mathcal{O}_\mathfrak{p})$.

Remark 4.7. Suppose $\mathcal{V}$ is a coherent sheaf of $\mathcal{O}_C$-modules, where C is the nice curve over k with function field K. Suppose we have a flat connection on $\mathcal{V}$. Taking the fibre at the generic point, we get a K-vector space V with k-connection. As far as I can tell, having a regular singular point at a closed point $\mathfrak{p} \in C$ means that the connection on $\mathcal{V}$ has a regular singular point at $\mathfrak{p}$. This notion essentially means that the ODE defined by the connection has a regular singular point at $\mathfrak{p}$, i.e. the solution has a pole of certain bounded order, so that there are still 'enough' independent meromorphic solutions to the ODE in a neighbourhood of $\mathfrak{p}$.

In some sense, the requirement that $\mathrm{Der}_\mathfrak{p}(K/k) = K \cdot y\frac{d}{dy}$ (for $y \in K^* \setminus \mathcal{O}^*_\mathfrak{p}$) preserves a lattice in $W_\mathfrak{p}$ is supposed to encode that the order of poles are nice enough.

The property of having a regular singular point at $\mathfrak{p}$ is nice.

Proposition 4.8. *Suppose $\mathfrak{p}$ is a place of K/k.*

1. *If $0 \to (V, \nabla') \to (W, \nabla) \to (U, \nabla'') \to 0$ is an exact sequence in $MC(K/k)$. Then (W, ∇) has a regular singular point at $\mathfrak{p}$ if and only if both (V, ∇') and (U, ∇'') have a regular singular point at $\mathfrak{p}$.*

2. *The full subcategory of $MC(K/k)$ consisting of those (W, ∇) having a regular singular point at $\mathfrak{p}$ is stable under taking internal Homs and tensor products.*

Definition. Say that (W, ∇) is *cyclic* if there is a vector $w \in W$ such that for some (hence any) non-zero derivation $D \in \mathrm{Der}(K/k)$, the set $\{\nabla_D^i w : i \geq 0\}$ spans W over K.

Since ∇ satisfies the Liebniz rule, one can check that the K-span of $\{\nabla_D^i w : i \geq 0\}$ is independent of $D \in \mathrm{Der}(K/k)$, and so defines a $\mathrm{Der}(K/k)$-invariant subspace of W.

Lemma 4.9. *Suppose $(W, \nabla) \in MC(K/k)$. Then (W, ∇) has a regular singular point at $\mathfrak{p}$ if and only if every cyclic subobject of (W, ∇) has a regular singular point at $\mathfrak{p}$.*

There are various other necessary and sufficient conditions on (W, ∇) for it to have a regular singular point at $\mathfrak{p}$. Some of these conditions are explicit conditions on a matrix for the connection with respect to a given base, and are thus very handy for explicit computation. See [Kat70b, Section 11] for more.

Now, suppose L is a finite extension of K. Then, there is a natural 'inverse image' functor

$$MC(K/k) \to MC(L/k), \quad (W, \nabla : W \to W \otimes_K \Omega_{K/k}) \mapsto (W \otimes_K L, \nabla \otimes L : W \otimes_K L \to (W \otimes_K L) \otimes_L \Omega_{L/k}).$$

Here, we need to use the fact that $\Omega_{L/k} \cong \Omega_{K/k} \otimes_K L$ (see [Har77, Proposition II.8.4A]). We also have another functor, the 'direct image' functor

$$MC(L/k) \to MC(K/k)$$

that takes $(W, \nabla) \in MC(L/k)$ to W regarded as a K-vector space, and that takes ∇ to $\nabla|_{\mathrm{Der}(K/k)}$.

Proposition 4.10. *Suppose $F \to K \to L$ is a tower of function fields over k, with L/F a finite extension. Let $\mathfrak{p}$ be a place of L/k, and let the corresponding place of K/k (resp. F/k) be $\mathfrak{p}'$ (resp. $\mathfrak{p}''$). Let also $\mathfrak{p}'_1, ..., \mathfrak{p}'_r$ be all the places of K/k lying over the place $\mathfrak{p}''$ of F/k. Suppose $(W, \nabla) \in MC(K/k)$.*

1. *(W, ∇) has a regular singular point at $\mathfrak{p}'$ if and only if the inverse image of $(W \otimes_K L, \nabla \otimes L)$ of (W, ∇) in $MC(L/k)$ has a regular singular point at $\mathfrak{p}$.*

2. *The direct image of (W, ∇) in $MC(F/k)$ has a regular singular point at $\mathfrak{p}''$ if and only if (W, ∇) has a regular singular point at each of the places $\mathfrak{p}'_i$ of K/k lying over $\mathfrak{p}''$.*

Corollary 4.10.1. *Suppose $\overline{k}$ is an algebraic closure of k, and let $\overline{\mathfrak{p}}$ be the place of $K\overline{k}/\overline{k}$ induced by the place $\mathfrak{p}$ of K/k. Suppose $(W, \nabla) \in MC(K/k)$, and let $(W_{\overline{k}}, \nabla_{\overline{k}})$ be its inverse image in $MC(K\overline{k}/\overline{k})$. Then (W, ∇) has a regular singular point at $\mathfrak{p}$ if and only if $(W_{\overline{k}}, \nabla_{\overline{k}})$ has a regular singular point at $\overline{\mathfrak{p}}$.*

4.5 Monodromy Around a Regular Singular Point

We start with the following theorem of Manin. The setup is the same as before, i.e. K/k is a function field of a curve, with k characteristic 0. We suppose $\mathfrak{p}$ is a place of K/k.

Theorem 4.11 ([Kat70b, Theorem (12.0)])**.** *Suppose $\mathfrak{p}$ is a rational place of K/k, i.e. the residue field at $\mathfrak{p}$ is k. Suppose $(W, \nabla) \in MC(K/k)$ has a regular singular point at $\mathfrak{p}$. Let t denotes a uniformising parameter at $\mathfrak{p}$, and let $\mathbf{e}$ is a K-basis of W such that the corresponding $\mathcal{O}_\mathfrak{p}$-lattice spanned by $\mathbf{e}$ is preserved by $\nabla_{t\frac{d}{dt}}$. Write*

$$\nabla_{t\frac{d}{dt}} \cdot \mathbf{e} = \mathbf{e} \cdot B$$

for $B \in M_n(\mathcal{O}_\mathfrak{p})$. Let the reduction modulo $\mathfrak{m}_\mathfrak{p}$ be $B(\mathfrak{p}) \in M_n(k)$.

Suppose all eigenvalues of $B(\mathfrak{p})$ are in k. Then:

- *The set of images in the additive group $k/\mathbb{Z}$ of the eigenvalues of $B(\mathfrak{p})$ is independent of the above choice of basis $\mathbf{e}$.*

- *The non-equal eigenvalues (which only depend on $\mathfrak{p}$ and (W, ∇)) do not differ by integers. More precisely, fix a set-theoretic section $\varphi : k/\mathbb{Z} \to k$ of the projection mapping $k \to k/\mathbb{Z}$. Then, there exists a unique $\mathcal{O}_\mathfrak{p}$-lattice $W_\mathfrak{p}$ of W, stable under $\nabla_{t\frac{d}{dt}}$, and there is a basis $\mathbf{e}'$ of $W_\mathfrak{p}$ such that, writing*

$$\nabla_{t\frac{d}{dt}} \cdot \mathbf{e} = \mathbf{e} \cdot C$$

for $C \in M_n(\mathcal{O}_\mathfrak{p})$, all eigenvalues of $C(\mathfrak{p}) \in M_n(k)$ are fixed by the composition $k \to k/\mathbb{Z} \xrightarrow{\varphi} k$.

- *The k-space* $\hat{W}_{\mathfrak{p}} = W_{\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}} \hat{\mathcal{O}}_{\mathfrak{p}}$ *(with* $\hat{\mathcal{O}}_{\mathfrak{p}}$ *the* $\mathfrak{m}_{\mathfrak{p}}$*-adic completion of* $\mathcal{O}_{\mathfrak{p}}$*) admits a base* $\hat{\mathbf{e}}$ *in terms of which the conneciton is simply*

$$\nabla_{t\frac{d}{dt}} \hat{\mathbf{e}} = C(\mathfrak{p}) \cdot \hat{\mathbf{e}}.$$

Here, recall that for k characteristic 0, the projection $\mathcal{O}_{\mathfrak{p}} \to k$ to the residue field has a canonical section $k \hookrightarrow \mathcal{O}_{\mathfrak{p}}$, and so we can think of $C(\mathfrak{p}) \in M_n(\hat{\mathcal{O}}_{\mathfrak{p}})$.

Definition. In the above theorem, the image in the additive group $k/\mathbb{Z}$ of the eigenvalues of $B(\mathfrak{p})$ are called the *exponents of* (W, ∇) *at* $\mathfrak{p}$.

Remark 4.12. If we require the set-theoretic section $\varphi : k/\mathbb{Z} \to k$ to send the coset $\mathbb{Z}$ to $0 \in k$, and if $B(\mathfrak{p})$ has all of its eigenvalues in $\mathbb{Z}$ (i.e. the exponents of (W, ∇) at $\mathfrak{p}$ are zero modulo $\mathbb{Z}$), then the matrix $C(\mathfrak{p})$ is nilpotent.

Remark 4.13. More generally, we can write $C(\mathfrak{p}) = D + N$ where D is semi-simple, N is nilpotent, and D and N commute (this is the Jordan decomposition). Then, the conjugacy class of N is independent of the choice of $\mathfrak{p}$. By definition, note that the eigenvalues of D modulo $\mathbb{Z}$ are the exponents of (W, ∇) at $\mathfrak{p}$.

Remark 4.14. To see how this is related to monodromy around the point $\mathfrak{p}$, see [Kat70b, Remark 12.3].

Definition. Suppose $\mathfrak{p}$ is rational and $(W, \nabla) \in MC(K/k)$ as a regular singular point at $\mathfrak{p}$. Say that *the local monodromy at* $\mathfrak{p}$ *is quasi-unipotent* if the exponents at $\mathfrak{p}$ are in $\mathbb{Q}/\mathbb{Z} \subset k/\mathbb{Z}$. We say that *the local monodromy at* $\mathfrak{p}$ *is unipotent* if all the exponents at $\mathfrak{p}$ are integers.

If the local monodromy at $\mathfrak{p}$ is quasi-unipotent, then say that its *exponent of nilpotence* is $\leq n$ if, writing $C(\mathfrak{p}) = D + N$, we have $N^n = 0$.

For $\mathfrak{p}$ any place of K/k (not necessarily rational) at which (W, ∇) has a regular singular point, then say that the *local monodromy at* $\mathfrak{p}$ *is quasi-unipotent* (resp. *quasi-unipotent with exponent of nilpotence* $\leq n$) if the corresponding property holds true for the induced place $\overline{\mathfrak{p}}$ after change of base to $K\overline{k}/\overline{k}$.

Example 4.15. Let $k = \mathbb{C}$ and $K = \mathbb{C}(z)$ (so that the curve is the Riemann sphere). Then $\Omega_{K/k} = \mathbb{C}(z) \cdot dz$ and so $\mathrm{Der}(K/k) = \mathbb{C}(z) \cdot \partial_z$, where ∂_z is the usual derivative of a rational function in z.

Now suppose $W = \mathbb{C}(z)e_1 + \mathbb{C}(z)e_2$, and the connection is defined to act as

$$\nabla_{\partial_z} e_1 = -e_2 \quad \text{and} \quad \nabla_{\partial_z} e_2 = \tfrac{-1}{z} e_2.$$

Consider the place $\mathfrak{p} : z = 0$, so that $\mathrm{Der}_{\mathfrak{p}}(K/k) = \mathbb{C}(z) \cdot z\partial_z$. We have

$$\nabla_{z\partial_z} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & -z \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

One checks that $\mathfrak{p}$ is a regular singular value of (W, ∇) (in fact, every place of $\mathbb{C}(z)/\mathbb{C}$ is a regular singular value, and apart from $z = 0$ every place is not even singular). Notice that the exponents of $\mathfrak{p}$ are integers (i.e. are 0 in $\mathbb{C}/\mathbb{Z}$), and that $B(\mathfrak{p}) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Let us check monodromy. A basis of horizontal sections (i.e. a solution $\nabla e = 0$) is given by $v_1 = ze_2$ and $v_2 = \frac{1}{2\pi i}(e_1 + z \log z e_2)$ (this latter function is multi-valued). Notice that going around $z = 0$ (in the analytic topology) sends v_1 to v_1 and sends v_2 to $v_2 + v_1$.

If we pick the section $\varphi : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ mapping $\mathbb{Z}$ to 0, then the unique $\mathcal{O}_{\mathfrak{p}}$-lattice guaranteed by the theorem is the $\mathcal{O}_{\mathfrak{p}}$-span of $e_1' = e_1$ and $e_2' = -ze_2$. In terms of this basis, we have

$$\nabla_{z\partial_z} \begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_1' \\ e_2' \end{pmatrix}.$$

It is now clear that the exponent of nilpotence is 2.

Proposition 4.16. *Let* $F \to K \to L$ *be a tower of functions fields over* k*. Let* $\mathfrak{p}$ *be a place of* L/k*,* $\mathfrak{p}'$ *the induced place of* K/k*, and* $\mathfrak{p}''$ *the induced place of* F/k*. Let* (W, ∇) *be an object of* $MC(K/k)$ *that has regular singular points at every place of* K/k *above* $\mathfrak{p}''$*. Suppose* $n \geq 1$*.*

1. *The inverse image* $(W \otimes_K L, \nabla \otimes L)$ *of* (W, ∇) *in* $MC(L/k)$ *(which has a regular singular point at* $\mathfrak{p}$*), has quasi-unipotent local monodromy at* $\mathfrak{p}$ *of exponent of nilpotence* $\leq n$ *if and only if* (W, ∇) *has quasi-unipotent local monodromy at* $\mathfrak{p}'$ *of exponent of nilpotence* $\leq n$*.*

2. *The direct image of* (W, ∇) *in* $MC(F/k)$ *has quasi-unipotent local monodromy at* $\mathfrak{p}''$ *of exponent of nilpotence* $\leq n$ *if and only if* (W, ∇) *has quasi-unipotent local monodromy of exponent of nilpotence* $\leq n$ *at every place of* K/k *lying over* $\mathfrak{p}''$*.*

The significance of regular singular points comes from the following theorem of Katz. To set up this theorem, let S be a global affine variety, and $f : X \to S$ a smooth morphism of relative dimension 1 whose generic fibres are geometrically connected. Let k denote the function field of S and K the function field of X. Recall from the previous section that an object $(\mathcal{V}, \nabla) \in MC_{int}(X/S)$ that is locally free of finite rank yields, via pullback, an object $(V, \nabla) \in MC(K/k)$.

Theorem 4.17 ([Kat70b, Theorem (13.0)]). *Suppose we are in the above setup, with $(\mathcal{V}, \nabla) \in MC_{int}(X/S)$ locally free of finite rank. Let (V, ∇) be the corresponding object in $MC(K/k)$ obtained via pull-back.*

1. *Suppose that $(\mathcal{V}, \nabla)$ is globally nilpotent on X/S. Then (V, ∇) has a regular singular point at every place of K/k, and moreover has quasi-unipotent local monodromy at every place of K/k.*

2. *Suppose that $(\mathcal{V}, \nabla)$ is globally nilpotent of exponent $n \geq 1$ on X/S. Then (V, ∇) has quasi-unipotent local monodromy of exponent $\leq n$ at every place of K/k.*

4.6 Regular Sheaves and the Local Monodromy Theorem

Suppose X is a smooth scheme of finite type over a field k of characteristic 0. Let X^* be a smooth proper scheme of finite type over k such that $j : X \hookrightarrow X^*$ as an open dense subscheme. Let $Y = X^* \setminus X$ be a divisor with normal crossings in X^*. We think of Y as being 'at ∞'.

First suppose that X is a curve, so that Y is a finite collection of points. Let $\overline{X}$ be the completion of the curve X (cf. [Har77, Section I.6]). It is a smooth projective curve over k, and moreover it has the property that the normalization of any smooth proper X^* containing X as an open dense subscheme is an open subscheme of $\overline{X}$. Let $(\mathcal{V}, \nabla)$ be a coherent sheaf with connection on X, relative to k.

Definition. Suppose $y \in \overline{X} - X$ is a closed point, and let $K = \mathrm{Frac}(\mathcal{O}_{X,y})$; then K is a function field of transcendence degree 1 over k, and y induces a place $\mathfrak{p}_y$ of K/k. Under the inverse image functor, $(\mathcal{V}, \nabla)$ induces a K-vector space V with connection ∇_K.

We say that $(\mathcal{V}, \nabla)$ is *regular at y* if (V_K, ∇_K) has a regular singular point at $\mathfrak{p}_y$.

We say that $(\mathcal{V}, \nabla)$ is *regular* if it is regular at every closed point $y \in \overline{X} - X$.

Let $S \subset Y = X^* - X$. The normalization of X^* is an open subscheme U of $\overline{X}$; let $\phi : U \to X^*$ be the canonical normalization map. We say that $(\mathcal{V}, \nabla)$ is *regular at (all points in) S* if the pull-back of $(\mathcal{V}, \nabla)$ under ϕ is regular at every point in $\phi^{-1}(S)$.

Let us now go back to the general case of X a smooth scheme of finite type over k.

Definition. Let $(\mathcal{E}, \nabla)$ be a coherent sheaf with connection on X, relative to k. The sheaf with connection $(\mathcal{E}, \nabla)$ is said to be *regular* if for every smooth locally closed curve $f : C \hookrightarrow X$, the pull-back of $(\mathcal{E}, \nabla)$ under f is regular.

We have some basic properties:

Proposition 4.18. *Let X be as above.*

1. *If $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$ is a horizontal exact sequence of coherent sheaves with integrable connections on X (the morphisms must all be horizontal), and if $\mathcal{V}'$ and $\mathcal{V}''$ are regular, then $\mathcal{V}$ is regular.*

2. *If $\mathcal{V}_1$ and $\mathcal{V}_2$ are coherent sheaves with regular flat connections, then $\mathcal{V}_1 \otimes \mathcal{V}_2$, $\mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2)$, $\mathcal{V}_1^\vee$, etc. are all regular.*

3. *Suppose $f : X \to X_1$ is a morphism of smooth k-schemes of finite type, and let $\mathcal{V} \in MC_{int}(Y/k)$. If $\mathcal{V}$ is regular then so is $f^*\mathcal{V}$. Conversely, if $f^*\mathcal{V}$ is regular and f is a dominant map of schemes, then $\mathcal{V}$ is regular.*

The importance of regularity comes from the following result. (cf. [Del70, Théorème 5.9]).

Theorem 4.19. *Suppose X is smooth and finite type over $k = \mathbb{C}$. Then, the functor $\mathcal{V} \to \mathcal{V}^{an}$ gives an equivalence between the category of coherent sheaves on X with regular integrable connection, and the category of holomorphic vector bundles on X^{an} endowed with an integrable connection.*

In particular, we can transport Riemann-Hilbert to the algebraic setting. Moreover, there are various comparison theorems between the cohomology of the local system on the corresponding analytic space, and the de Rham cohomology of the original regular flat sheaf on X. See [Del70, Section 6] for more detail. For instance, we have the following result.

Proposition 4.20. *Suppose X^* is a smooth scheme of finite type over $\mathbb{C}$, Y a normal crossing divisor on X, $\mathcal{V}$ a vector bundle on X^*, and ∇ a regular integrable connection on the restriction of $\mathcal{V}$ to $X = X^* - Y$, with logarithmic singularities along Y. Suppose that the connection ∇ is not unipotent at any point in Y. Then, if V is the local system defined by $\mathcal{V}^{an}$ on X^{an}, we have*

$$H^p\big(R\Gamma(X^*, \Omega^\bullet_{X/\mathbb{C}}(\log Y) \otimes \mathcal{V})\big) \cong H^p(X^{an}, V).$$

It also turns out that regular sheaves on X have canonical extensions to X^* (cf. [Kat70a, Section II-III]).

Theorem 4.21. *Suppose $(\mathcal{V}, \nabla)$ is a regular flat coherent sheaf on a smooth $\mathbb{C}$-scheme X. Let X^* be* any *proper smooth $\mathbb{C}$-scheme with $Y = X^* - X$ a normal crossings divisor. Let $j : X \hookrightarrow X^*$ be the open immersion. Then, there exists a pair $(\overline{\mathcal{V}}, \overline{\nabla})$ consisting of a locally free sheaf $\overline{\mathcal{V}}$ on X^* such that $j^*\overline{\mathcal{V}} \cong \mathcal{V}$, and a homomorphism $\overline{\nabla}$ of abelian sheaves*

$$\overline{\nabla} : \overline{\mathcal{V}} \to \overline{\mathcal{V}} \otimes_{\mathcal{O}_{X^*}} \Omega_{X^*/\mathbb{C}}(\log Y)$$

extending $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_X} \Omega_{X/\mathbb{C}}$.

The Gauss-Manin connections are regular.

Theorem 4.22 ([Del70, Théorème 7.9])**.** *Let $f : X \to S$ be a smooth morphism between smooth schemes of finite type over $\mathbb{C}$ such that there exists a proper smooth S-scheme X^* containing X as an open S-subscheme with $X^* - X$ a normal crossings divisor.*

If $\mathcal{V}$ is a coherent sheaf on X with a regular integrable connection, then the Gauss-Manin connection on $R^i f_I(\Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{V})$ is also regular.

Since the Gauss-Manin connection is regular, we can ask whether the local monodromy is quasi-unipotent. This is the content of the *Local Monodromy Theorem.*

Theorem 4.23 (Local Monodromy Theorem)**.** *Let $S/\mathbb{C}$ a smooth connected curve, $K/\mathbb{C}$ its function field, and $\pi : X \to S$ a proper smooth morphism. Let X_K/K be the generic fibre of π.*

For each $m \geq 0$, let $h(m)$ denote the number of pairs (p, q) such that $p + q = m$ and

$$h^{p,q}(X_K/K) := \dim_K H^q(X_K, \Omega^p_{X_K/K}) = \operatorname{rank}_{\mathcal{O}_S} R^q\pi_*\Omega^p_{X/S}$$

is non-zero. Then, the K-space $H^i_{dR}(X_K/K)$ with the Gauss-Manin connection (equivalently, the inverse image of $H^i_{dR}(X/S)$ with the Gauss-Manin connection) in $MC(K/\mathbb{C})$ has regular singular points at every place of $K/\mathbb{C}$ and has quasi-unipotent local monodromy with exponent of nilpotence $\leq h(i)$.

This theorem can be generalized to Deligne's open local monodromy theorem. To set this up, let $K/\mathbb{C}$ be a function field of a smooth connected curve $S/\mathbb{C}$, and let $\pi : U \to \operatorname{Spec} K$ a smooth morphism. Then, it is a fact that there exists a finite extension L/K, a proper smooth morphism $\rho : X \to \operatorname{Spec} L$, and a divisor $i : Y \hookrightarrow X$ with normal crossings relative to $\operatorname{Spec} L$, such that the morphism

$$\pi_L : U \times_K L \to \operatorname{Spec} L$$

is the morphism

$$\rho_{X-Y} : X - Y \to \operatorname{Spec} L.$$

Notice that

$$H^i_{dR,\log Y}(X/L) = H^i_{dR}((X-Y)/L) = H^i_{dR}(U \times_K L/L) = H^i_{dR}(U/K) \otimes_K L.$$

We get the following result.

Theorem 4.24 (Deligne's Open Local Monodromy Theorem)**.** *In the setup given above, let $h_Y(m)$ denote the number of pairs (p, q) such that $p + q = m$ and*

$$h^{p,q}_Y(X_K/K) := \dim_L H^q(X, \Omega^p_{X/L}(\log Y))$$

is non-zero. Then, the object of $MC(K/\mathbb{C})$ given by $H^i_{dR}(U/K)$ (with the Gauss-Manin connection) has regular singular points at every place of $K/\mathbb{C}$, and at each place the local monodromy is quasi-unipotent with exponent of nilpotence $\leq h_Y(i)$.

We have yet another monodromy theorem by Brieskorn (see [Del70, p. III.2]).

Theorem 4.25. *Suppose $\overline{S}$ is a smooth projective curve of finite type over $\mathbb{C}$. Let $S = \overline{S} - T$ where T is some finite set of points. Suppose $f : X \to S$ is any smooth morphism of $\mathbb{C}$-schemes. Suppose that $R^i f_*\underline{\mathbb{C}}$ is a local system on S. Then, the integrable coherent sheaf on S corresponding to $R^i f_*\underline{\mathbb{C}}$ is regular, and has quasi-unipotent local monodromy at every point on $\overline{S} - S$.*

5 Intersection Theory and Characteristic Classes

We will mostly be following [Sta23, Tag 02P3] (for all proofs consult loc. cit.). Chapter 7 of [Bau15] and Appendix A of [Har77] are good short references as well. [Vai12] has a lot of exercises and examples. Of course, the canonical reference is Grothendieck's [Gro58].

Throughout, we will be working with schemes X locally of finite type over a locally Noetherian 'universally catenary' base scheme S equipped with a dimension function $\delta : S \to \mathbb{Z}$. For concreteness, it suffices to take S to be (the Spec of) a field or a Dedekind domain, or to be a scheme locally of finite type over a field or Dedekind domain. In any of these cases, we have a well-defined notion of dimension of a closed integral subscheme of X relative to S, written $\dim_S(-)$.

We will constantly be using the following two facts about integrality of schemes:

- A scheme is integral if and only if it is reduced and irreducible.
- Every integral scheme has a unique generic point.

Let us also fix some notation. For any closed integral subscheme Y of X with corresponding unique generic point η_Y, we can consider the stalk $\mathcal{O}_{X,\eta_Y}$ of $\mathcal{O}_X$ at η_Y. This is the *local ring of* Y *in* X, denoted simply by $\mathcal{O}_{X,Y}$. The *function field* of Y, denoted $\kappa(Y)$, is the stalk $\mathcal{O}_{Y,\eta_Y}$ of $\mathcal{O}_Y$ at η_Y. Notice that $\mathcal{O}_{X,Y} \hookrightarrow \kappa(Y)$ via pull-back induced by the closed immersion $Y \hookrightarrow X$.

5.1 Algebraic Cycles

Suppose X is a scheme locally of finite type over S.

Definition. A *k-cycle on* X is a formal sum $\sum_Z n_Z[Z]$ where $n_Z \in \mathbb{Z}$, the sum ranges over all integral closed subschemes Z of dimension k over S, and the collection $\{Z : n_Z \neq 0\}$ is locally finite in X.

The *group* $Z_k(X)$ *of cycles of dimension* k is the abelian group of k cycles on X. The *group of cycles* is the graded group $Z_*(X) = \bigoplus_k Z_k(X)$.

Remark 5.1. Note that cycles are formal infinite sums. If X is quasi-compact, then all cycles are formal finite sums, and $Z_k(X)$ is simply the free abelian group generated by all closed integral dimension k subschemes of X.

Remark 5.2. We also write $Z^k(X) = Z_{\dim_S X - k}(X)$ to be the cycles on X of codimension k; this tacitly assumes that every irreducible component of X has the same dimension. The advantage of this notation is that we don't have to write down '$\dim_S X$' everywhere. We will switch freely between these two notations.

Example 5.3. If X is irreducible, then $Z^0 X \cong \mathbb{Z} \cdot [X]$.

Example 5.4. If $X = \mathbb{A}^n_k$, then $Z^1 \mathbb{A}^n_k \cong k(x_1, ..., x_n)^\times / k^\times$.

Suppose Z is a closed subscheme of X and Z' an irreducible component of Z. Then $\mathcal{O}_{Z,Z'}$ is a $\mathcal{O}_{X,Z'}$-module, which turns out to be of finite length if X is locally of finite type.

Definition. Suppose Z is a closed subscheme of X of dimension k (i.e. all of its irreducible components have dimension at most k over S). The *k-cycle associated to* Z is

$$[Z]_k = \sum_{Z'} \operatorname{length}_{\mathcal{O}_{X,Z'}} \mathcal{O}_{Z,Z'} \cdot [Z'],$$

the sum running over all irreducible components Z' of Z such that $\dim_S Z' = k$.

We can also associate a cycle to a coherent sheaf. In this definition, we write η_Z for the generic point of a closed integral scheme Z.

Lemma-Definition. Suppose $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module on X, such that the support $\operatorname{supp}(\mathcal{F})$ of $\mathcal{F}$ has dimension at most k. Then, the formal sum

$$[\mathcal{F}]_k := \sum_Z \operatorname{length}_{\mathcal{O}_{X,Z}} \mathcal{F}_{\eta_Z} \cdot [Z],$$

the sum running over all irreducible components Z of $\operatorname{supp}(\mathcal{F})$ with $\dim_S Z = k$, is a k-cycle called the *k-cycle associated to* $\mathcal{F}$.

Example 5.5. Suppose Z is a closed subscheme of X of dimension at most k. Then $[Z]_k = [\mathcal{O}_Z]_k$.

The k-cycle associated to a coherent sheaf is additive over short exact sequences, i.e. if we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of coherent sheaves such that the dimension of their support has dimension at most k, then $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$.

The group of cycles is functorial in X.

Lemma-Definition. Suppose $f : X \to X'$ is a *proper* k-morphism. Then, we have a homomorphism of abelian groups $f_* Z_k(X) \to Z_k(X')$ by defining

$$f_*[Z] = \begin{cases} 0 & \dim_S f(Z) < k \\ [k(Y) : k(f(Y))] \cdot [f(Y)] & \dim_S f(Z) = k \end{cases}$$

for integral closed subschemes Z with $\dim_S(Z) = k$, and extending by linearity.

For Y a closed subscheme of X, this induces an injection of graded abelian groups $Z_*(Y) \hookrightarrow Z_*(X)$.

Remark 5.6. Here, if $\dim f(Y) = \dim Y$, then the function field $k(Y)$ of Y is a finite extension of the function field $k(f(Y))$ of $f(Y)$, and so the definition makes sense.

It turns out that proper pushforward of cycles coincides with the proper pushforward of coherent sheaves, i.e. $f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k$.

We also have a pull-back functor going the other way. If $f : X \to X'$ is a morphism of S-schemes and $Z \hookrightarrow X'$ a closed subscheme, write $f^{-1}(Z)$ for the closed subscheme $Z \times_{X'} X$ of X.

Lemma-Definition. Suppose $f : X \to X'$ is a flat morphism of relative dimension r of schemes that are locally of finite type over S.

If Z is a closed integral subscheme of X' of dimension k, then let $f^*[Z] = [f^{-1}(Z)]_{k+r}$. Extending via linearity, we then have the *flat pull-back morphism* of abelian groups

$$f^* : Z_k(X') \to Z_{k+r}(X).$$

For $U \subset Z$ an open immersion, the corresponding flat pullback (which is of relative dimension 0) yields a surjection $Z_k(X) \twoheadrightarrow Z_k(U)$., the *restriction to U map.*

It turns out that flat pullback of cycles coincides with flat pullback of coherent sheaves, i.e. $f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$.

Proposition 5.7. *Suppose X is locally of finite type over S, and let Y be a closed subscheme of X with complement the open subscheme $U = X - Y$. Then, we have an exact sequence of graded abelian groups*

$$0 \to Z_*(Y) \to Z_k(X) \to Z_k(U) \to 0.$$

Proposition 5.8. *Suppose we have a Cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & \lrcorner & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of schemes locally of finite type over S. Suppose that f is proper and g flat of relative dimension r. Then f' is proper, g' is flat of relative dimension r, and we have

$$g^* \circ f_* = f'_* \circ (g')^*.$$

5.2 The Chow Group

We now associate a cycle to a rational function. In short, we are taking the cycle associated to the divisor div$f$. Let's recall the procedure. Suppose Y is a closed codimension 1 integral subscheme of X. Then $\mathcal{O}_{X,Y}$ is a DVR, and so it makes sense to talk about orders of elements in its fraction field. If X is also integral, then this fraction field coincides with the function field $k(X)$ of X (this is the local ring at the unique generic point of X). Hence, for X integral, we can define the valuation

$$\mathrm{ord}_Y : k(X) \to \mathbb{Z} \cup \{\infty\}.$$

Lemma-Definition. Fix $f \in \kappa(X)^*$. Then, the formal sum given by

$$\mathrm{div} f = \mathrm{div}_X f := \sum_Y \mathrm{ord}_Y(f) \cdot [Y]$$

where the sum is over all closed integral codimension 1 subschemes Y of X is a codimension 1 cycle on X, the *principal divisor associated to f.*

Here are some properties of principal divisor function.

Proposition 5.9. *Suppose X is locally of finite type over S.*

1. *$div_X : \kappa(X)^* \to Z^1(X)$ is a homomorphism of abelian groups.*
2. *If $p : X \to X'$ is a dominant proper morphism, then*

$$p_* \circ \mathrm{div}_X = \mathrm{div}_{X'} \circ \mathrm{Norm}_{\kappa(X)/\kappa(X')}.$$

3. *If $f : X \to X'$ is a flat morphism of relative dimension r between integral schemes X and X', then*

$$f^* \circ \mathrm{div}_Y = \mathrm{div}_X \circ f^*,$$

where we have the morphism $f^* : \kappa(X') \to \kappa(X)$.

If V is a closed integral subscheme of X of dimension r, then the closed immersion $V \hookrightarrow X$ induces a map $Z_*(V) \hookrightarrow Z_*(X)$, and so we can define a $r-1$-cycle $\mathrm{div}_V f \in Z_{r-1}(X)$ for any $f \in k(V)$.

Definition. Suppose $0 \le r \le \dim X - 1$. The *group of r-cycles rationally equivalent to zero* is the subgroup $\mathrm{Rat}_r(X)$ of $Z_r(X)$ generated by the $\mathrm{div}_V f$ for all $f \in k(V)$ where V ranges over all closed integral subschemes of X of dimension $r+1$.

We also set $\mathrm{Rat}_{\dim X}(X) = 0$.

Definition. Suppose $0 \le r \le \dim X$. The *r'th Chow group of X* is the quotient

$$CH_r(X) := Z_r(X)/\mathrm{Rat}_r(X).$$

The Chow group $CH_*(X)$ is the graded group

$$CH_*(X) := \bigoplus_r CH_r(X).$$

Remark 5.10. The Chow group is meant to be an algebraic analog of the singular homology groups in topology.

Remark 5.11. We also have upper numbering notation for Chow groups, i.e. $CH^r(X) = CH_{\dim X - r}(X)$ is the group of equivalence classes of codimension r cycles modulo rational equivalence.

Of course, $CH^0(X) = Z^0(X)$.

Example 5.12. $CH_{n-1}(\mathbb{A}^n_k) = 0$.

Example 5.13. $CH_*(\mathbb{P}^n_k) \cong \mathbb{Z}[\ell]/\ell^{n+1}$. Here, ℓ stands for the class of the cycle corresponding to any line in $\mathbb{P}^n_k$.

Example 5.14. If X is integral, then $CH^1(X)$ coincides with the Weil divisor class group of X.

The following are properties of the Chow group.

Proposition 5.15. *Suppose X is locally of finite type over S.*

1. *Suppose $f : X \to X'$ is a proper morphism. Then f_* sends $\mathrm{Rat}_k(X)$ to $\mathrm{Rat}_k(X')$, and thus descends to a morphism of graded abelian groups*

$$f_* : CH_*(X) \to CH_*(X').$$

2. *Suppose $f : X \to X'$ is a flat morphism of relative dimension r. Then f^* sends $\mathrm{Rat}_k(X')$ to $\mathrm{Rat}_{k+r}(X)$, and thus descends to a morphism of abelian groups*

$$f^* : CH_k(X') \to CH_{k+r}(X)$$

for all $k \ge 0$.

3. *Suppose Y is a closed subscheme of X and $U = X - Y$ its complement. Then, we have an exact sequence*

$$0 \to CH_*(Y) \to CH_*(X) \to CH_*(U) \to 0$$

of graded abelian groups.

4. *Suppose $f : X' \to X$ is proper. Suppose also that for every closed point $x \in X$, there exists a point $x' \in f^{-1}(x)$ such that the field extension $\kappa(x')/\kappa(x)$ is trivial. Then, we have an exact sequence of graded abelian groups*

$$CH_*(X' \times_X X') \xrightarrow{p_{1,*} - p_{2,*}} CH_*(X') \xrightarrow{f_*} CH_*(X) \to 0,$$

where $p_1, p_2 : X' \times_X X' \to X'$ are the projections.

The Chow group behaves well with vector bundles. Recall that a vector bundle of rank n over X is a scheme V equipped with a morphism $p : V \to X$ and an open covering $\{U_i\}$ of Y with isomorphisms $\psi_i : p^{-1}(U_i) \xrightarrow{\sim} \mathbb{A}^n_{U_i}$ such that for any i, j and for any open affine $\operatorname{Spec} A \subseteq U_i \cap U_j$, the automorphism $\psi_j \circ \psi_i^{-1}$ of $\mathbb{A}^n_V = \operatorname{Spec} A[x_1, ..., x_n]$ corresponds to a linear automorphism of $A[x_1, ..., x_n]$. Here, an automorphism θ of the polynomial ring $A[x_1, ..., x_n]$ is linear if $\theta(a) = a$ for all $a \in A$, and for all i the $\theta(x_i)$ is a linear homogeneous polynomial in the $x_1, ..., x_n$. By [Har77, Exercise II.5.18], the category of rank n vector bundles over X is equivalent to the category of rank n locally free sheaves on X. In fact, if $\mathcal{V}$ is a rank n locally free sheaf on X, then

$$V = \underline{\operatorname{Spec}}_X \left(\operatorname{Sym}^*(\mathcal{V})\right)$$

is the associated vector bundle, where $\operatorname{Sym}^*(\mathcal{V}) := \bigoplus_{k \geq 0} \mathcal{V}^{\otimes k}$. The projection $p : V \to X$ is given by the associated map $\mathcal{O}_X \cong \mathcal{V}^{\otimes 0} \hookrightarrow \operatorname{Sym}^*(\mathcal{V})$.

It is easy to see that the projection map $p : V \to X$ of a rank n vector bundle is a flat morphism of relative dimension n.

Proposition 5.16. *If V is a vector bundle of rank r on X with projection map $p : V \to X$, then the map*

$$p^* : CH_k(X) \to CH_{k+r}(V)$$

is an isomorphism for all $k \in \mathbb{Z}$.

5.3 The First Chern Class

Suppose X is an intgeral scheme locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module, and let s be any non-zero meromorphic section of $\mathcal{L}$. Then, at any point $x \in X$, the germ of s at x is of the form $f s_x$ for s_x a generator of $\mathcal{L}_x \cong \mathcal{O}_{X,x}$ and $f \in \kappa(X)^*$. Up to multiplication by a unit in $\mathcal{O}_{X,x}$, this f is unique. Hence, if Z is a codimension 1 closed integral subscheme with generic point η, then upon taking $x = \eta$ above it makes sense to write

$$\operatorname{ord}_{Z,\mathcal{L}}(s) := \operatorname{ord}_{\mathcal{O}_{X,\eta}}(f).$$

Definition. With the set up as above, the *Weil divisor associated to s* is

$$\operatorname{div}_{\mathcal{L}}(s) = \sum_Z \operatorname{ord}_{Z,\mathcal{L}}(s) \cdot [Z] \in Z^1(X),$$

where the sum runs over all codimension 1 closed integral subschemes Z of X

It turns out that $\operatorname{div}_{\mathcal{L}}(s)$ is rationally equivalent to $\operatorname{div}_{\mathcal{L}}(s')$ for any non-zero meromorphic sections s, s' of $\mathcal{L}$. Hence, the following definition makes sense.

Definition. With set up as above, the *Weil divisor class of $\mathcal{L}$ on X* is

$$c_1(\mathcal{L}) \cap [X] := [\operatorname{div}_{\mathcal{L}}(s)] \in CH^1(X)$$

for any choice of non-zero meromorphic section s of $\mathcal{L}$.

As usual, the Weil divisor class is compatible with pull-backs.

Lemma 5.17. *Suppose X and X' are integral and $f : X \to X'$ is flat of relative dimension r. Then $f^*(c_1(\mathcal{L}) \cap [X']) = c_1(f^*\mathcal{L}) \cap [X]$ as elements of $CH^1(X)$.*

Using the Weil divisor class, we can define a homomorphism of Chow groups.

Lemma-Definition. Suppose X is integral locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For every integer k, there is a homomorphism of Chow groups

$$c_1(\mathcal{L}) \cap - : CH_{k+1}(X) \to CH_k(X)$$

which takes the class $[W]$ of a closed integral subscheme $i : W \hookrightarrow X$ with $\dim_S W = k+1$ to

$$c_1(\mathcal{L}) \cap [W] = i_*\big(c_1(i^*\mathcal{L}) \cap [W]\big)$$

where $c_1(i^*\mathcal{L}) \cap [W]$ is the Weil divisor class of $i^*\updownarrow$ on W. This homomorphism is called the *intersection with the first Chern class of* $\mathcal{L}$.

Remark 5.18. It is actually not at all trivial to show that the above map $Z_{k+1}(X) \to CH_k(X)$ given by $[W] \mapsto i_*\big(c_1(i^*\mathcal{L}) \cap [W]\big)$ and extended to all of $Z_{k+1}(X)$ by linearity actually factors through $CH_{k+1}(X)$.

The rough intuitive picture is that the first Chern class homomorphism gives the locus of where a general section of a line bundle is zero. The intersection with first Chern classes satisfies a lot of nice properties.

Proposition 5.19. *Suppose X is integral locally of finite type over S. Let $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, ...$ be invertible $\mathcal{O}_X$-modules. Fix $k \in \mathbb{Z}$, and let $\alpha \in CH_{k+1}(X)$.*

1. $c_1(\mathcal{O}_X) \cap -$ *is the zero morphism.*
2. $(c_1(\mathcal{L}_1) \cap \alpha) + (c_1(\mathcal{L}_2) \cap \alpha) = c_1(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2) \cap \alpha.$
3. *For $f : X' \to X$ a flat morphism of relative dimension r, we have $f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^*\mathcal{L}) \cap f^*\alpha$.*
4. *For $f : X \to X'$ a proper morphism, we have $f_*(c_1(f^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap f_*\alpha$.*
5. *As morphisms $CH_k(X) \to CH_{k-2}(X)$, we have*

$$c_1(\mathcal{L}_1) \cap (c_1(\mathcal{L}_2) \cap -) = c_1(\mathcal{L}_2) \cap (c_1(\mathcal{L}_1) \cap -).$$

In particular, for s_1 and s_2 non-zero meromorphic sections of $\mathcal{L}_1$ and $\mathcal{L}_2$, we have

$$c_1(\mathcal{L}_2) \cap \operatorname{div}_{\mathcal{L}_1}(s_1) = c_1(\mathcal{L}_1) \cap \operatorname{div}_{\mathcal{L}_2}(s_2)$$

as elements of $CH^2(X)$.

5.4 Projective Bundles and Higher Chern Classes

To a locally free $\mathcal{O}_X$-module $\mathcal{V}$ of rank r, we can also associate the so-called *projective bundle*, given by

$$\mathbb{P}(\mathcal{V}) := \underline{\operatorname{Proj}}_X(\operatorname{Sym}^*(\mathcal{V}) \xrightarrow{\pi} X.$$

Here, notice that $\operatorname{Sym}^*(\mathcal{V})$ is a graded $\mathcal{O}_X$-algebra so the relative Proj makes sense. The map π corresponds to the usual inclusion $\mathcal{O}_X \hookrightarrow \operatorname{Sym}^*(\mathcal{V})$, and is a projective (thus proper) map. It is also flat of relative dimension $r-1$. For any open affine $U = \operatorname{Spec} A$ of X, the subscheme $\pi^{-1}(U)$ of $\mathbb{P}(\mathcal{V})$ is (essentially by definition) the projective scheme $\operatorname{ProjSym}^*(\mathcal{V})(U)$ where $\operatorname{Sym}^*(\mathcal{V})(U)$ is a graded A-algebra. In particular, it is canonically equipped with an invertible sheaf $\mathcal{O}_{\pi^{-1}(U)}(1)$. These invertible sheaves glue to give an invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$. This sheaf satisfies

$$\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)) = \mathcal{V},$$

and so is there is a surjection $\pi^*\mathcal{V} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$. If Y is a X-scheme with structure map $g : Y \to X$, a X-morphism $Y \to \mathbb{P}(\mathcal{V})$ is the same as an invertible sheaf $\mathcal{L}$ on Y and a surjective map $g^*\mathcal{V} \to \mathcal{L}$ of sheaves on Y. See [Har77, Section II.7] for more details on projective bundles.

We have the following formula associated to a projective space bundles.

Proposition 5.20 (Projective Space Bundle Formula)**.** *Suppose X is locally of finite type over S and $\mathcal{V}$ a locally free $\mathcal{O}_X$-module $\mathcal{V}$ of rank r. Let $\pi : P = \mathbb{P}(\mathcal{V}) \to X$ be the associated projective bundle with associated twisting sheaf $\mathcal{O}_P(1)$. Then, the map*

$$(\alpha_0, ..., \alpha_{r-1} \mapsto \pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \cdots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

gives an isomorphism

$$\bigoplus_{i=0}^{r-1} CH_{k+i}(X) \xrightarrow{\sim} CH_{k+r-1}(P)$$

for all $k \in \mathbb{Z}$. Here, by $c_1(\mathcal{L})^i \cap (-)$ we mean the i'th iterate of $c_1(\mathcal{L}) \cap (-)$.

This formula in fact allows us to formulate the following definition.

Definition. Suppose X is integral and locally of finite type over S. Let $\mathcal{V}$ be a rank r locally free sheaf on X with associated projective space bundle $\pi : P \to X$ with twisting sheaf $\mathcal{O}_P(1)$.

By the projective space bundle formula, there exist elements $c_i \in CH^i(X)$ for $0 \leq i \leq r$ such that $c_0 = [X]$, and

$$\sum_{i=0}^{r}(-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^* c_{r-i} = 0.$$

We set $c_i(\mathcal{V}) \cap [X] := c_i \in CH^i(X)$, and call this the *i'th Chern class* of $\mathcal{V}$ on X.

The *total Chern class* of $\mathcal{V}$ is

$$c(\mathcal{V}) \cap [X] := \sum_{i=0}^{r} c_i(\mathcal{V}) \cap [X] \in CH_*(X).$$

As one would expect, for $\mathcal{L}$ an invertible sheaf, the above defined Chern class coincides with the previously defined Chern class for $\mathcal{L}$.

As with Chern classes of line bundles, we can define homomorphisms between Chow groups.

Lemma-Definition. Suppose $\mathcal{V}$ is a locally free sheaf of rank r on X. For every $k \in \mathbb{Z}$ and every $0 \leq j \leq r$, we have a homomorphism of abelian groups

$$c_j(\mathcal{V}) \cap - : CH_k(X) \to CH_{k-j}(X)$$

satisfying

$$c_j(\mathcal{E}) \cap [Z] := i_*(c_j(i^*\mathcal{E}) \cap [Z]) \in CH_{k-j}(X)$$

for any integral closed subscheme $i : Z \hookrightarrow X$, and extended to the entire Chow group by linearity. The above homomorphism is called the *intersection with the j'th Chern class of $\mathcal{E}$*.

Intersecting with the j'th Chern class satisfies all the usual properties one would expect.

Proposition 5.21. *Suppose X and $\mathcal{V}$ are as before.*

1. *If $p : X \to X'$ is a proper morphism and $\alpha \in CH_*(X)$, then*
$$p_*(c_j(p^*\mathcal{E}) \cap \alpha) = c_j(\mathcal{E}) \cap p_*(\alpha).$$
2. *If $f : X' \to X$ is a flat morphism of fixed relative dimension and $\alpha \in CH_*(X)$, then*
$$f^*(c_j(\mathcal{E}) \cap \alpha) = c_j(f^*\mathcal{E}) \cap f^*(\alpha).$$
3. *If $\mathcal{V}'$ is another locally free $\mathcal{O}_X$-module on C of finite rank, then for all $\alpha \in CH_k(X)$ we have*
$$c_i(\mathcal{V}) \cap (c_j(\mathcal{V}') \cap \alpha) = c_j(\mathcal{V}') \cap (c_i(\mathcal{V}) \cap \alpha) \in CH_{k-i-j}(X).$$
4. *We have $c_i(\mathcal{V}^\vee) = (-1)^i c_i(\mathcal{V})$ for all i.*

5.5 The Chow Cohomology Ring

In order to better talk about Chern classes, and in particular to talk about polynomial relations and so on, we need to introduce the Chow cohomology ring. This ring is essentially a graded ring of morphisms of Chow groups satisfying certain properties. In order to formulate one of the properties, we need to first talk about Gysin maps. Throughout, as usual, we suppose X is locally of finite type over S.

Lemma-Definition. Let $\mathcal{L}$ be an invertible sheaf on X and $s \in \Gamma(X, \mathcal{L})$ a global section. Let Z be the zero scheme of s, and denote $i : Z \hookrightarrow X$ the corresponding closed immersion. Then, for every integer k, there is a unique homomorphism

$$i^* : CH_{k+1}(X) \to CH_k(Z)$$

such that, for any integral closed subscheme W with $\dim_S W = k+1$, we define

$$i^*[W] := [Z \cap W]_k$$

as a k-cycle on Z if $W \not\subset Z$, and otherwise set

$$i^*[W] = i'_*(c_1(\mathcal{L}|_W) \cap [W]$$

where $i' : W \hookrightarrow Z$ is the corresponding closed immersion. This homomorphism corresponding to $(\mathcal{L}, s)$ is called the *Gysin homomorphism*.

This essentially extends the notion of pull-back to non-flat morphisms. We have the following properties.

Proposition 5.22. *Suppose $\mathcal{L}$ is an invertible sheaf on X with global section s with vanishing locus Z, so that we have the Gysin homomorphism i^*.*

1. *The composite $i^* \circ i_* : CH_k(Z) \to CH_{k-1}(Z)$ is given by*

$$\alpha \mapsto c_1(i^*\mathcal{L}) \cap \alpha.$$

2. *If $\mathcal{N}$ is another invertible $|mcO_X$-module, then for any $\alpha \in CH_k(X)$,*

$$i^*(c_1(\mathcal{N}) \cap \alpha) = c_1(i^*\mathcal{N}) \cap i^*\alpha$$

Proposition 5.23. *Suppose $\mathcal{L}$ is an invertible sheaf on X with global section s with vanishing locus Z. Suppose $f : X' \to X$ is a morphism, and set $\mathcal{L}' := f^*\mathcal{L}$ and $s' = f^*s$. Then the vanishing locus Z' of s' sits inside the Cartesian square*

$$\begin{array}{ccc} Z' & \overset{i'}{\hookrightarrow} & X' \\ \downarrow g & \lrcorner & \downarrow f \\ Z & \overset{i}{\hookrightarrow} & X. \end{array}$$

We thus form the Gysin homomorphisms i^* and $(i')^*$.

1. *If f is proper, then*

$$i^* \circ f_* = g_* \circ (i')^*.$$

2. *If f is flat of fixed relative dimension, then*

$$(i')^* \circ f^* = g^* \circ i^*$$

We can now define the Chow cohomology groups.

Definition. A *bivariant class c of degree p* on X is a rule which assigns a map

$$c \cap - : CH_k(X') \to CH_{k-p}(X')$$

to every locally of finite type morphism $f : X' \to X$ and every $k \in \mathbb{Z}$, satisfying the following conditions:

1. If $g : X'' \to X'$ is proper, then for all $\alpha'' \in CH_*(X'')$,

$$c \cap g_*\alpha'' = g_*(c \cap \alpha'').$$

2. If $g : X'' \to X'$ is flat of fixed relative dimension and locally of finite type, then for all $\alpha' \in CH_*(X')$,

$$c \cap g^*\alpha' = g^*(c \cap \alpha').$$

3. If $\mathcal{L}$; is an invertible sheaf, $s' \in \Gamma(X', \mathcal{L}')$ with zero locus $i : Z' \hookrightarrow X'$, then for all $\alpha' \in CH_*(X')$,

$$c \cap (i')^*\alpha' = (i')^*(c \cap \alpha').$$

The abelian group of all bivariant classes of degree p on X is denoted $A^p(X)$.

We have an obvious bilinear associative composition map

$$\circ : A^p(X) \times A^q(X) \to A^{p+q}(X).$$

Hence, $A^*(X)$ is a graded ring.

We also have functoriality. Indeed, if $X' \to X$ is a morphism of schemes locally of finite type over S, then by viewing a scheme X'' locally of finite type over X' as a scheme locally of finite type over X, we get an obvious restriction map

$$A^p(X) \to A^p(X'), c \mapsto res(c).$$

This map clearly induces a graded ring homomorphism $A^*(X) \to A^*(X')$.

Definition. The graded (possibly non-commutative ring) $A^*(X)$ associated to X is the *Chow cohomology of* X.

Notice that the intersection with i'th Chern class homomorphism associated to a locally free sheaf $\mathcal{V}$ is a bivariant class on X of degree i; indeed, to any $f : X' \to X$ we have the associated map $c_i(f^*\mathcal{V}) : CH_k(X') \to CH_{k-i}(X')$. This gives the following definition.

Definition. For a locally free sheaf $\mathcal{V}$ of fixed rank on X, the *i'th Chern class* $c_i(\mathcal{V})$ is the bivariant class associated to the homomorphism $c_i(\mathcal{V}) \cap - : CH_k(X) \to CH_{k-i}(X)$ described above.

The *total Chern class* is the non-homogeneous element

$$c(\mathcal{V}) := \sum_{i=0}^{r} c_i(\mathcal{V}) \in A^*(X).$$

Example 5.24. $c_0(\mathcal{O}_X) = 1$ and $c_1(\mathcal{O}_X) = 0$ in the ring $A^*(X)$. More generally, we have $c_0(\mathcal{V}) = 1$ in $A^*(X)$ for all locally free sheaves $\mathcal{V}$.

Now that the Chern classes live inside a graded (possibly non-commutative) ring, we can talk about polynomial relations between them. We have the following properties of Chern classes. Here, we will write $P_p \in \mathbb{Z}[y_1, y_2, ...]$ to be the unique degree p homogeneous polynomial (where we set $\deg y_i := i$) such that for all $n \geq p$, we have

$$P_p(s_1(n), ..., s_p(n)) = \sum_{i=1}^{n} x_i^p$$

where the $s_i(n)$ are the elementary symmetric polynomials in $x_1, ..., x_n$. For instance, we have

$$P_1 = y_1, \quad P_2 = y_1^2 - 2y_2, \quad P_3 = y_1^3 - 3y_1y_2 + 3y_3,$$

and so on. We set

$$P_p(\mathcal{V}) := P_p(c_1, ..., c_r) \in A^p(X)$$

for $p \leq r$, for any locally free sheaf $\mathcal{V}$ of rank r. For convenience, we also set $P_0(\mathcal{V}) := \mathrm{rank}\mathcal{V} \in \mathbb{Z} \hookrightarrow A^0(X)$.

Proposition 5.25. *Let X be locally of finite type over S, and let $\mathcal{V}, \mathcal{V}', \mathcal{V}_1, ...$ be locally free sheaves on X of finite rank.*

1. *For all i, the i'th Chern class $c_i(\mathcal{V})$ lies in the center of $A^*(X)$.*
2. *For any exact sequence*
$$0 \to \mathcal{V}_1 \to \mathcal{V} \to \mathcal{V}_2 \to 0,$$
we have
$$P_p(\mathcal{V}) = P_p(\mathcal{V}_1) + P_p(\mathcal{V}_2)$$
in $A^*(X)$.
3. *For any filtration*
$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_r = \mathcal{V}$$
where $\mathcal{V}_i/\mathcal{V}_{i-1} =: \mathcal{L}_i$ is a line bundle, then we have
$$c(\mathcal{V}) = \prod_{i=1}^{r} \big(1 + c_1(\mathcal{L}_i)\big)$$
in $A^*(X)$.
4. *If $\mathcal{V}$ is of rank r and $\mathcal{L}$ is an invertible sheaf, then*
$$c_i(\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{L}) = \sum_{j=0}^{i} \binom{r - i - +j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j$$
in $A^*(X)$.
5. *More generally, if $\mathcal{V}_i$ is of rank r_i, then we have*
$$P_p(\mathcal{V}_1 \otimes_{\mathcal{O}_X} \mathcal{V}_2) = \sum_{i=0}^{p} \binom{p}{i} P_i(\mathcal{V}_1) P_{p-i}(\mathcal{V}_2)$$
as elements of $A^p(X)$.
6. *Recall that $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{V})$ is a locally free sheaf of rank r^2. We have*
$$c_2\big(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{V})\big) = 2rc_2(\mathcal{V}) - (r-1)c_1(\mathcal{V})^2.$$

5.6 Chern Roots, Chern Characters, Todd Classes, and Grothendieck-Riemann-Roch

Lemma 5.26 (Splitting Principle)**.** *Suppose X is locally of finite type over S, and $\mathcal{V}_i$ $(1 \le i \le k)$ is a locally free $\mathcal{O}_X$-module of rank r_i. Then, there exists a projective flat morphism $\pi : P \to X$ of fixed relative dimension $d = r_1 + \cdots + r_k - k$ such that*

- *for any $f : X' \to X$ the map $\pi^*_{X'} : CH_*(X') \to CH_{*+d}(X' \times_X P)$ is injective;*
- *the restriction map $res_\pi : A^*(X) \to A^*(P)$ associated to π is injective; and*
- *for each $1 \le i \le k$, the bundle $\pi^*\mathcal{V}_i$ has a filtration*

$$0 = \mathcal{V}_0^{(i)} \subset \mathcal{V}_1^{(i)} \subset \mathcal{V}_2^{(i)} \subset \cdots \subset \mathcal{V}_r^{(i)} = \mathcal{V}_i$$

where each graded piece $\mathcal{L}_j^{(i)} := \mathcal{V}_j^{(i)}/\mathcal{V}_{j-1}^{(i)}$ is a line bundle.

For $k = 1$ and $r \ge 2$, the projective space bundle $P = \mathbb{P}(\mathcal{V})$ works. Of course, for $k = 1$ and $r = 1$, $P = X$ trivially works. The general P is constructed by induction on k.

In particular, in the notation of the lemma for $k = 1$, we have

$$res_\pi(c(\mathcal{V})) = c(\pi^*\mathcal{V}) = \prod_{i=1}^{r}(1 + c_1(\mathcal{L}_i)).$$

Since res_π is injective by construction, for any loaclly free sheaf $\mathcal{V}$ of rank r on X, it makes sense to write

$$c(\mathcal{V}) =: \prod_{i=1}^{r}(1 + x_i)$$

for formal symbols $x_1, ..., x_n$.

Definition. The formal symbols $x_1, ..., x_n$ such that $c(\mathcal{V}) =: \prod_{i=1}^{r}(1 + x_i)$ are called *Chern roots*.

The splitting principle for general k then says that we can compute (homogeneous) polynomials in various Chern classes by decomposing them into Chern roots and then formally manipulating the Chern roots of the various locally free sheaves in question. In fact, pretty much every statement in Proposition 5.25 can be proven in this way.

Remark 5.27. From now on, rather than fixing such projective maps $\pi : P \to X$ and considering the Chern roots as elements of $A^1(P)$, we can simply consider them as formal symbols of degree 1.

Remark 5.28. The i'th Chern class $c_i(\mathcal{V})$ is the i'th symmetric polynomial in the Chern roots $x_1(\mathcal{V}), ..., x_r(\mathcal{V})$.

Definition. Let $\mathcal{V}$ be a locally free sheaf on X of rank r with Chern roots $x_1, ..., x_r$. The *Chern character* $ch(\mathcal{V})$ is the formal expression

$$ch(\mathcal{V}) = \sum_{i=1}^{r} e^{x_i},$$

viewed as a formal power series in the x_i.

Notice first that

$$ch(\mathcal{V}) = \sum_{p \ge 0} \frac{1}{p!} P_p(\mathcal{V})$$

with P_p the polynomials defined previously. Thus, for instance, Proposition 5.25(2) can be written simply as

$$ch(\mathcal{V}) = ch(\mathcal{V}_1) + ch(\mathcal{V}_2)$$

for any short exact sequence $0 \to \mathcal{V}_1 \to \mathcal{V} \to \mathcal{V}_2 \to 0$, while Proposition 5.25(5) can be written simply as

$$ch(\mathcal{V}_1 \otimes_{\mathcal{O}_X} \mathcal{V}_2) = ch(\mathcal{V}_1)ch(\mathcal{V}_2).$$

Remark 5.29. If $\pi : P \to X$ is as above, then we view $ch(\mathcal{V})$ as living in

$$\prod_{p \ge 0} A^p(X) \otimes \mathbb{Q},$$

where the degree p part $\frac{1}{p!}P_p(\mathcal{V})$ lives in $A^p(X) \otimes \mathbb{Q}$.

We have a final (formal) class attached to a vector bundle.

Definition. Suppose $\mathcal{V}$ is a locally free sheaf of rank r on X with Chern roots $x_1, ..., x_r$. Then, the *Todd Class* of $\mathcal{V}$ is

$$Todd(\mathcal{V}) = \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}}.$$

Using the previous properties of Chern classes, we get for instance that

$$Todd(\mathcal{V}) = Todd(\mathcal{V}_1)Todd(\mathcal{V}_2)$$

for any short exact sequence $0 \to \mathcal{V}_1 \to \mathcal{V} \to \mathcal{V}_2 \to 0$.

We can now state Grothendieck-Riemann-Roch in extreme (but still not utmost) generality. Here, recall that

$$T_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X).$$

Theorem 5.30 (Grothendieck-Riemann-Roch)**.** *Suppose X and Y are locally of finite type over S, and $f : X \to Y$ a proper smooth S-morphism. Let $\mathcal{V}$ be a locally free sheaf of rank r on X. If $R^i f_* \mathcal{V}$ are locally free sheaves on Y of finite rank as well, then*

$$f_*\big(Todd(T_{X/Y}) \cdot ch(\mathcal{V})\big) = \sum_{i=0}^{r} (-1)^i ch(R^i f_* \mathcal{V}).$$

We can simplify the above statement if $Y = \operatorname{Spec} k$ for k a field. To do this, we need to introduce some new notation. If $f : X \to \operatorname{Spec} k$ is proper, then we have proper pushforward

$$p_* : CH_0(X) \to CH_0(\operatorname{Spec} k).$$

However, $CH_0(\operatorname{Spec} k) = \mathbb{Z} \cdot [\operatorname{Spec} k]$, and so by sending $[\operatorname{Spec} k]$ to 1 we get a morphism

$$\deg : CH_0(X) \to \mathbb{Z},$$

the *degree map*. Of course, various properties of the degree map follow immediately from the corresponding properties of proper pushforward. This degree map coincides with the usual notion of the degree of a vector bundle over curves.

Lemma 5.31. *Suppose C is a proper irreducible curve over k, and $\mathcal{V}$ a locally free $\mathcal{O}_X$-module of constant rank on C. Then,*

$$\deg\big(c_1(\mathcal{V}) \cap [C]\big)$$

coincides with the usual notion of degree.

Now, notice that $R^i f_* \mathcal{V}$ (for $f : X \to \operatorname{Spec} k$ proper) is precisely $H^i(X, \mathcal{V})$. Here, we recall that a coherent sheaf on $\operatorname{Spec} k$ is the same as a finite dimensional k-vector space. Now, $A^i(\operatorname{Spec} k) = 0$ for all $i \geq 0$, and so we see that

$$ch(R^i f_* \mathcal{V}) = \dim_k H^i(X, \mathcal{V}).$$

Definition. For a coherent sheaf $\mathcal{E}$ on X, where X is a scheme over k, the *Euler characteristic* of $\mathcal{E}$ is

$$\chi(X, \mathcal{E}) := \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{E}).$$

We can thus reformulate the Grothendieck-Riemann-Roch theorem as follows.

Theorem 5.32. *Suppose X is smooth, proper, and locally of finite type over k. Let $\mathcal{V}$ be a locally free sheaf of rank r on X. Then*

$$\deg\big(Todd(T_{X/Y}) \cdot ch(\mathcal{V})\big) = \chi(X, \mathcal{V}).$$

Here, by $\deg\big(Todd(T_{X/Y}) \cdot ch(\mathcal{V})\big)$ we really mean the degree of the degree 0 part of the cycle $Todd(T_{X/Y}) \cdot ch(\mathcal{V}) \cap [X]$.

6 Non-Abelian Hodge Theory

6.1 Higgs Sheaves

Fix a morphism $\pi : X \to Y$. Suppose $\theta : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/Y}$ is some $\mathcal{O}_X$-linear map, for $\mathcal{E}$ a $\mathcal{O}_X$-module. Then, as with connections, the above induces a map

$$\theta_p : \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^p_{X/Y} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{p+1}_{X/Y}$$

given by

$$\theta(e \otimes \omega) = (-1)^p (\theta e) \wedge \omega$$

on local sections e and ω of $\mathcal{E}$ and $\Omega_{X/Y}$ respectively. Write $\wedge^p \theta$ for the p-fold composition

$$\mathcal{E} \mapsto \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^p_{X/Y}.$$

Definition. Let $\mathcal{E}$ be a sheaf on X. A *Higgs field on $\mathcal{E}$ (relative to Y)* is an $\mathcal{O}_X$-linear map $\theta : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/Y}$ such that $\wedge^2 \theta = 0$.

A *Higgs sheaf on X relative to Y* is a sheaf equipped with a Higgs field.

Remark 6.1. This is highly reminiscent of the definition of an integrable connection. In fact, these are related if one writes down the correct generalization of a de Rham complex and a connection. This is done in [Lan14].

Let $\theta : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/Y}$ be a $\mathcal{O}_X$-linear map. Now, suppose v is a local section of $T_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{O}_X)$ over an open $U \subset X$, then $\theta_v := (1 \circ v) \circ \theta$ yields a $\mathcal{O}_U$-linear map $\mathcal{E}|_U \to \mathcal{E}|_U$. Hence, the data of a $\mathcal{O}_X$-linear map $\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/Y}$ is the same as the data of an $\mathcal{O}_X$-linear map $T_{X/Y} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$. A computation shows that the condition $\wedge^2 \theta = 0$ corresponds to the condition that $\theta_v \circ \theta_w = \theta_w \circ \theta_v$ for any local sections v, w of $T_{X/Y}$. Hence, the data of a Higgs field is the same as the data of a $\mathcal{O}_X$-linear map $T_{X/Y} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ such that the image sheaf is a *commutative* $\mathcal{O}_X$-subalgebra of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$.

Of course, a Higgs field induces a complex structure on $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/Y}$, where the differential maps θ are all $\mathcal{O}_X$-linear (unlike in the case of connections).

6.2 Classical Non-Abelian Hodge Theory: A Summary

Suppose X is a smooth projective $\mathbb{C}$-variety of dimension n, so that by Serre's GAGA it is a Kähler manifold of $\mathbb{C}$-dimension n.

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