Problem Set 1

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Throughout, k is going to denote a field of characteristic not 2.

1. (*Exercise 1.9*) Any quadratic space over k admits an orthogonal basis.

2. (*Exercise 1.15*) Suppose V is a regular quadratic space. Let W be a subspace. Show that the following are equivalent:

   (a) W is regular;

   (b) $W^\perp$ is regular;

   (c) $W \cap W^\perp = \{0\}$;

   (d) $V = W \oplus W^\perp$.

3. (*Exercise 1.16*) Suppose V is any quadratic space. Consider the subspace

$$\mathrm{rad}(V) := \{v \in V : B(x, v) = 0 \quad \text{for all } x \in V\}.$$

   Let V' be *any* subspace of V such that $V = V' \oplus \mathrm{rad}(V)$. Show that V' is regular, and that

$$V = \mathrm{rad}(V) \perp V'.$$

   Hence, show that every non-zero quadratic form $f \in k[x_1, ..., x_n]$ is of the form

$$f = h(a_{11}x_1 + \cdots + a_{1n}x_n, ..., a_{r1}x_1 + \cdots + a_{rn}x_n)$$

   for some $1 \le r \le n$, some regular quadratic form $h \in k[y_1, ..., y_r]$, and some $A = (a_{ij}) \in GL_n(k)$.

4. (*Exercise 1.18*) Let V be a regular quadratic space over k and $a \in k$. Then V represents a if and only if $\langle -a \rangle \perp V$ is isotropic.

5. (*Exercise 1.30*) Suppose V is a regular quadratic space over k such that we have a decomposition

$$V = H_1 \perp \cdots \perp H_r \perp V'$$

   where V' is either 0 or anisotropic, and $0 \le r \le \frac{1}{2} \dim V$. Show, using Witt's extension theorem, that this r does not depend on the above decomposition of V. Hence, show that $r = \mathrm{ind}V$.

6. (*Section 1.5*) Let us prove Proposition 1.30:

   Suppose Q_1 and Q_2 are two regular quadratic forms on the same vector space V over k. If $O(V, Q_1) = O(V, Q_2)$, then there exists $\lambda \in k^*$ such that $Q_2 = \lambda Q_1$.

   We prove this in the following steps.

   (a) Prove the following lemma: Suppose W is a regular quadratic space over k, and fix $v \in W$ with $Q_W(v) \neq 0$. If $\ell : W \to k$ is a linear functional such that $B(w, v)\ell(w) = 0$ for all $w \in W$, then $\ell = 0$.

   (b) Let $v \in V$ be such that $Q_1(v) \neq 0$. Show that $Q_1(v)B_2(v, w) = Q_2(v)B_1(v, w)$ for all $w \in V$.

   (c) Using (b), prove the proposition.