Problem Set 2

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Throughout, k is going to denote a field of characteristic not 2.

1. (*Corollary 1.25.1*) Suppose σ can be expressed as a product of n-reflections. Then, σ can be expressed as a product of n reflections with the first (or last) symmetry chosen arbitrarily.

2. Let V be any regular quadratic space with $\dim V \geq 2$, and suppose $\alpha \in Q(V)$. Consider the set
$$Q^{-1}(a) = \{v \in V : Q(v) = a\}.$$
Since $SO(V)$ preserves Q, we know that there is a group action of $SO(V)$ on $Q^{-1}(a)$. Show that this group action is transitive. (*Hint: Witt*)

3. (*Exercise 2.11*) We prove the following result of Chevallay for the special case of $d = 2$.

   Theorem (Chevalley (1935))**.** *Let $n, d \in \mathbb{N}$ be such that $n > d$. Then, every polynomial of total degree d in n variables has a non-trivial zero (i.e. a zero not in $\mathbb{F}_q^n \setminus \{(0, ..., 0)\}$).*

   We will prove this in the following steps:

   (a) Show that $k^* \subseteq Q(V)$ whenever V is regular with $\dim V \geq 2$.

   (b) Show that any regular quadratic space over a finite field of dimension $n \geq 3$ is always isotropic.

   (c) Finish of the proof for the $d = 2$ case of Chevallay's theorem.

4. (*Exercise 3.7*) Suppose A and B are quaternion algebras. If $\varphi : A \to B$ is an algebra isomorphism, show that φ maps scalars to scalars, maps pure quaternions to pure quaternions, and commutes with conjugation, norms, and traces.

5. (*Exercise 3.11*) Using Proposition 3.8, classify all quaternion algebras up to isomorphism for $k = \mathbb{R}$, $k = \mathbb{C}$, and for k a finite field.

6. (*Exercise 3.39*) Show that a regular ternary quadratic space V over a field k is isotropic if and only if its Hasse algebra SV is Brauer equivalent to the quaternion algebra $(-1, -1)_k$.