Problem Set 3

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Throughout, p is going to denote a prime number.

1. (*Proposition 3.9*) Let k be a field not of characteristic 2. For $\alpha, \beta \in k^*$, prove that the following are equivalent.

   (a) $(\alpha, \beta)_k$ is isomorphic as k-algebras to $(1, -1)_k \cong M_2(k)$.

   (b) $(\alpha, \beta)_k$ is not a division algebra.

   (c) $(\alpha, \beta)_k$ is an isotropic quaternary regular quadratic space.

   (d) $(\alpha, \beta)_k^0$ is an isotropic quaternary regular quadratic space.

   (e) $\langle \alpha \rangle \perp \langle \beta \rangle$ represents 1.

   Hint: When is $x \in (\alpha, \beta)_k$ invertible?

2. (*Example 4.3*) Check that the p-adic valuation $|.|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ given by $|p^k \frac{a}{b}| = p^{-k}$ (where $a, b \in \mathbb{Z}, b \neq 0, ab \not\equiv 0 \pmod{p}$) is a non-archimedean valuation.

3. Prove that $\mathbb{Z}_p := \{|x|_p \leq 1 : x \in \mathbb{Q}_p\}$ is a ring with unique maximal ideal $p\mathbb{Z}_p$.

4. Let $a \in \mathbb{Z}_p^*$. Using Hensel's Lemma or otherwise, prove the following statements:

   (a) Suppose p is odd. Then $a \in (\mathbb{Z}_p^*)^2$ if and only if the reduction $\overline{a} \in \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$ satisfies $\overline{a} \in (\mathbb{F}_p^*)^2$.

   (b) Suppose $p = 2$. Show that $a \in (\mathbb{Z}_2^*)^2$ if and only if $a \equiv 1 \pmod{8}$.

5. (*Lemma 4.44*) Suppose p is odd, and $V \cong \langle \epsilon_1 \rangle \perp \cdots \perp \langle \epsilon_n \rangle$ for units $\epsilon_i \in \mathbb{Z}_p^*$. If $n \geq 3$, show that V is isotropic.

6. (*Exercise 4.48*) Suppose U and V are regular quadratic spaces over a non-archimedean local field, and let $r := \dim V - \dim U \in \{0, 1, 2\}$. Show that there exists an isometry $U \hookrightarrow V$ if and only if

$$V \cong \begin{cases} U & \text{if } r = 0, \\ U \perp \langle \mathrm{disc} U \cdot \mathrm{disc} V \rangle & \text{if } r = 1, \\ U \perp H & \text{if } r = 2. \end{cases}$$

   Here, H is the hyperbolic plane.

7. Using Hilbert reciprocity over $\mathbb{Q}$, prove quadratic reciprocity.

8. Show that the spaces $\langle 1 \rangle \perp \langle 1 \rangle \perp \langle 1 \rangle \perp \langle 1 \rangle$ and $\langle b \rangle \perp \langle b \rangle \perp \langle b \rangle \perp \langle b \rangle$ are isomorphic over $\mathbb{Q}$ for all $b \in \mathbb{Q}^*$.

9. Consider the three quadratic forms

   $f = x_1^2 + x_2^2 + 16x_3^2 - x_4^2$, $\quad g = 3x_1^2 + 7x_2^2 - 4x_3x_4$, $\quad$ and $\quad h = 2x_1^2 + 2x_2^2 + 5x_3^2 - 16x_4^2 - 2x_2x_3 - 2x_1x_3$.

   Which of these forms are isotropic over $\mathbb{Q}$? Are any of these isomorphic to each other?

10. Suppose V and W are regular quadratic spaces over $\mathbb{Q}$ with $V_\infty \cong W_\infty$. Suppose that there exists a finite prime p_0 such that $V_p \cong W_p$ for all primes $p \neq p_0$. Can you conclude that $V \cong W$?