Problem Set 4

Kush Singhal

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Throughout, R is going to denote a principal ideal domain, with field of fractions K. Throughout, V is going to denote a quadratic space over K.

1. Using Proposition 6.5, prove the following.

   (a) (*Corollary 6.5.1*) Suppose U is a subspace of V, and L is an R-submodule of U. Then L is a lattice in U if and only if L is a lattice in V.

   (b) (*Corollary 6.5.2*) $L \cap M$ is a lattice in V whenever L and M are lattices in V.

   (c) (*Corollary 6.5.3*) $L + M$ is a lattice in V whenever L and M are lattices in V. Moreover, $L + M$ is full whenever at least one of L or M is full.

   A harder (possibly impossible) question: can you try to prove the above three results *without* appealing to Proposition 6.5, at least in the special case of $R = \mathbb{Z}_p$ or $R = \mathbb{Z}$?

2. (*Lemma 6.10*) Suppose $L = Rx_1 + \cdots + Rx_n$ is a full lattice of V. Let $x \in V \setminus \{0\}$ be arbitrary, and write $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$. Show that
$$\mathfrak{c}_x = \bigcap_{1 \le i \le n, \alpha_i \neq 0} (\alpha_i^{-1} R).$$

3. (*Exercise 6.13*) Suppose $x_1, ..., x_n$ is a basis for V, and let L be a full lattice. Then, $x_1, ..., x_n$ is a basis for L if and only if x_i is a maximal vector of L for all $1 \le i \le n$.

4. (*Exercise 6.18*) If $\dim V$ is odd, show that $\mathrm{cls} L = \mathrm{cls}^+ L$.

5. (*Exercise 6.19*) For any lattice L in any quadratic space V, set
$$\mathrm{rad}(L) = \{x \in L, B(x, v) = 0 \ \forall \ v \in L\}.$$
   Recall also the subspace $\mathrm{rad}(V)$ from Question 3 of Problem Set 1.

   (a) Show that $\mathrm{rad}(L)$ is a lattice in V.

   (b) Show that $\mathrm{rad}(KL) = K\mathrm{rad}(L)$ and that $\mathrm{rad} L = L \cap \mathrm{rad}(KL)$.

   (c) Show that L is a regular lattice if and only if $\mathrm{rad}(L) = \{0\}$.

   (d) Show $\mathrm{rad}(L \perp M) = \mathrm{rad}(L) \perp \mathrm{rad}(M)$.

   (e) Given any lattice L in V, show that there exists a regular lattice M in V such that $L = M \perp \mathrm{rad}(L)$.

6. (*Lemma 6.24*) Suppose M is a non-zero lattice in the regular quadratic space V. Construct a lattice J on V such that $L := M \perp J$ is a full lattice in V with the same scale and norm as M.

7. Consider the PID $R = \mathbb{C}[x]$, and consider the regular ternary quadratic form Q over R defined by
$$Q(f, g, h) := (1 + x)f^2 + (x^2 + 1)g^2 + h^2 - xgh.$$
   Compute the discriminant, scale, and norm of this ternary quadratic form.

8. Suppose $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$ with valuation $|.|_p$, for some prime p. Suppose V is a regular quadatic space over $\mathbb{Q}_p$ and L a full $\mathbb{Z}_p$-lattice in V. Show the following.

   (a) If $x, y \in L$ are such that $|B(x,y)|_p$ is the largest, then $B(L,L) = B(x,y)\mathbb{Z}_p = \mathfrak{s}L$.

   (b) If $x \in L$ is such that $|Q(x)|_p$ is the largest, then $\mathfrak{n}L = Q(x)\mathbb{Z}_p$.