Problem Set 5

Kush Singhal

31 July 2023 - 7 August 2023

Throughout, p is going to denote a prime number. Unless otherwise mentioned, all lattices will be assumed to be regular.

1. (*Corollary 7.9.3*) Suppose L is unimodular. Show that $Q(L) \supseteq \mathbb{Z}_p^*$ whenever $\text{rank}L \geq 2$, and that $Q(L) = \mathbb{Z}_p$ for $\text{rank}L \geq 3$.

2. (*Proposition 7.11 and Exercise 7.12*) Go through the proof of Proposition 7.11 as given in the lecture notes. Then, by proving that every element of $O(L)$ is the product of at most 2 reflections in $O(L)$ whenever $\text{rank}L = 2$, modify the proof of Proposition 7.11 to prove the following stronger statement:

   Theorem. *If L is a regular lattice of rank $n \geq 2$, then every element of $O(L)$ is the product of at most $2n - 2$ reflections in $O(L)$.*

3. (*Exercise 7.15*) Show the following equivalences, where $u \in \{1, 3, 5, 7\}$ can be arbitrary.

   $$\langle 1\rangle \perp \langle 1\rangle \cong \langle 5\rangle \perp \langle 5\rangle\,; \quad \langle 1\rangle \perp \langle 2\rangle \cong \langle 3\rangle \perp \langle 6\rangle\,; \quad \langle 1\rangle \perp \langle 4\rangle \cong \langle 5\rangle \perp \langle 20\rangle\,; \quad \langle u\rangle \perp \langle\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\rangle \cong \langle u\rangle \perp \langle 1\rangle \perp \langle -1\rangle\,;$$

   $$\langle u\rangle \perp \langle\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)\rangle \cong \langle 3u\rangle \perp \langle -u\rangle \perp \langle -u\rangle\,; \qquad \langle\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)\rangle \perp \langle\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)\rangle \cong \langle\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\rangle \perp \langle\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\rangle\,.$$

   (as an added challenge, try not to use Theorem 7.16).

4. For $p \in \{2, 3, 5\}$ (and any other prime of your choice), compute the Jordan decomposition of the following quadratic forms. Which of these lie in the same $\mathbb{Z}_p$-class?

   $$\begin{aligned} f_1(x_1, x_2, x_3) &= 2x_1^2 + 2x_2^2 + 2x_1x_2 - 45x_3^2, \\ f_2(x_1, x_2, x_3) &= x_1x_2 + x_2x_3 + x_3x_1 \\ f_3(x_1, x_2, x_3) &= 2x_1x_2 + 2x_2x_3 + 2x_3x_1 - x_1^2 - x_2^2 - x_3^2, \\ f_4(x_1, x_2, x_3) &= 5x_1^2 + 13x_2^2 + 11x_3^2 + 2x_2x_3 + 2x_3x_1 + 16x_1x_2 \end{aligned}$$

   Can you compute the set $f_i(\mathbb{Z}_p^3)$?

5. Over which p are the following quadratic forms $\mathbb{Z}_p$-equivalent?

   $$x_1^2 + 2x_2^2 + 6x_3^2 + 6x_2x_3 \quad \text{and} \quad 2x_1^2 + 3x_2^2 + 5x_3^2.$$

6. (*Exercise 8.11*) Show that the lattices $\langle 1\rangle \perp \langle 11\rangle$ and $\left\langle \left(\begin{smallmatrix} 3 & -1 \\ -1 & 4 \end{smallmatrix}\right) \right\rangle$ are isomorphic over $\mathbb{Z}_p$ for all primes p, and yet they are not in the same class over $\mathbb{Z}$.

7. (*Exercise 8.12*) Show that $L = \langle 1\rangle \perp \langle 11\rangle$ represents 3 p-adically for all primes p, but does not represent 3 over $\mathbb{Q}$.

8. Let V be the rational quadratic space with quadratic form $x^2 + y^2 + z^2$. Construct a full $\mathbb{Z}$-lattice L in V such that $L_2 \cong \left\langle \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) \right\rangle \perp \langle 12\rangle$, $L_3 \cong \langle 2\rangle \perp \langle 3\rangle \perp \langle 6\rangle$, and $L_p \cong \langle 1\rangle \perp \langle 1\rangle \perp \langle 1\rangle$ for all $p \geq 5$.

9. Compute class representatives of the genus of the quadratic form $x^2 + 14y^2$.