Problem Set 5

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Throughout, unless otherwise specified, we fix a positive definite quadratic space V over $\mathbb{Q}$ and a full lattice L in V.

1. By computing the first few coefficients of both sides, show that $E_6 E_8 = E_{14}$.

2. Notice that $E_6^2 \in M_{12}(SL_2(\mathbb{Z}), \mathbb{1})$. This vector space is spanned by E_{12} and Δ. Find $a, b \in \mathbb{C}$ such that $E_6^2 = aE_{12} + b\Delta$.

3. (*Theorem 0.3*) By studying the modular form $\theta \in M_{1/2}(\Gamma_1(4), \mathbb{1})$ given by

$$\theta(z) := \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z},$$

prove Jacobi's Theorem.

Theorem. *The number of integer solutions (x, y) to $x^2 + y^2 = n$ for a given positive integer n is given by*

$$4 \left(\sum_{d|n, d \equiv 1 \pmod 4} 1 - \sum_{d|n, d \equiv 3 \pmod 4} 1 \right).$$

4. We prove the following famous theorem of Lagrange.

Theorem (Lagrange's Four Square Theorem)**.** *Every positive integer can be written as the sum of four squares.*

We will in fact go beyond, by providing a formula!

(a) Consider the quadratic form $f := x^2 + y^2 + z^2 + w^2$. Evaluate the first few terms of the Fourier expansion of Θ_f.

(b) Recall the modular form $F \in M_2(\Gamma_0(2))$ given by

$$F(z) = E_2(z) - 2E_2(2z).$$

Compute its first few Fourier coefficients.

(c) Show that Θ_f and F form a basis for $M_2(\Gamma_0(4))$.

(d) Show that $G(z) := F(2z)$ also lies inside $M_2(\Gamma_0(4))$.

(e) By studying the first few Fourier coefficients of G, F, and Θ_f, find a formula for

$$r(n) := \#\{(x, y, z, w) \in \mathbb{Z}^4 : n = x^2 + y^2 + z^2 + w^2\}.$$

(f) Using the formula obtained, show that $r(n) > 0$ for all $n \geq 0$ thus proving Lagrange's theorem.

(g) Can you write down an explicit formula for the number of ways a prime number p can be written as the sum of four squares? How about for squares of prime numbers? How about for odd square-free integers?