Notes on Étale Cohomology

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1 Étale Maps and the Étale Site

1.1 Preliminary Concepts about Rings

Definition. An R-module M is *flat* if the functor $(-) \otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an exact functor (*a priori*, it is a right exact covariant functor).

An R-module M is *faithfully flat* if for any complex $N_1 \to N_2 \to N_3$ of R-modules, the sequence $N_1 \to N_2 \to N_3$ is exact if and only if $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact.

A ring map $R \to S$ is *flat* (resp. *faithfully flat*) if S is flat (resp. faithfully flat) as an R-module.

Proposition 1.1. *Suppose $M \in \mathrm{Mod}_R$. The following are equivalent:*

1. *M is flat over R;*
2. *for every ideal $I \subset R$ the map $I \otimes_R M \to M$ is injective;*
3. *for every finitely generated ideal $I \subset R$ the map $I \otimes_R M \to M$ is injective;*
4. *for every ideal $I \subset R$, the map $I \otimes_R M \to IM$ is an isomorphism.*
5. *for every prime ideal $\mathfrak{p}$ of R, $M_\mathfrak{p}$ is a flat $R_\mathfrak{p}$ module.*
6. *for every maximal ideal $\mathfrak{m}$ of R, $M_\mathfrak{m}$ is a flat $R_\mathfrak{m}$ module.*

We have the following properties. All of these are easy to check, noting that a tensor product is a colimit in the appropriate category.

Proposition 1.2. *Throughout we suppose $M \in \mathrm{Mod}_R$, $I, J \subset R$ are ideals of R, and $R \to S$ is a ring map.*

1. *$IM \cap JM = (I \cap J)M$.*
2. *Composition of (faithfully) flat ring maps is (faithfully) flat.*
3. *If $R \to R'$ is (faithfully) flat, and M' is a (faithfully) flat R' module, then M' is a (faithfully) flat R-module.*
4. *Suppose $R : I \to \mathrm{CRings}$ an index category, and suppose $M_i \in \mathrm{Mod}_{R_i}$ a flat R_i-module for all $i \in I$. Suppose also for each $i \to i'$ with corresponding ring-map $\varphi_{i\to i'} : R_i \to R_{i'}$, there is a $\varphi_{i\to i'}$-linear map $f_{i\to i'} : M_i \to M_{i'}$ that is functorial. Then, $\mathrm{colim}_{i\in I} M_i$ is a flat $\mathrm{colim}_{i\in I} R_i$ module.*
5. *Suppose M is (faithfully) flat over R and $R \to R'$ a ring map. Then $M \otimes_R R'$ is (faithfully) flat over R'. If $R \to R'$ is moreover faithfully flat, then $M \otimes_R R'$ is flat over R' if and only if M is flat over R.*
6. *If $R \to S$ is flat, then $R \to S$ is faithfully flat if and only if the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*
7. *If S is a localization of R at some multiplicative subset, then S is a flat R-algebra.*
8. *If M is a flat R-module, then the following are equivalent:*
   - *M is faithfully flat;*
   - *for every non-zero $N \in \mathrm{Mod}_R$, $M \otimes_R N$ is non-zero;*
   - *for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the tensor product $M \otimes_R \kappa(\mathfrak{p})$ is non-zero (here, $\kappa(\mathfrak{p}) = \mathrm{Frac}(R/\mathfrak{p})$);*
   - *for all maximal ideals $\mathfrak{m}$ of R, $M/\mathfrak{m}M$ is non-zero.*
9. *A flat local ring homomorphism is faithfully flat.*

See Section 10.39 of [Sta22] for the proofs of the above two propositions.

Definition. A ring map $R \to S$ is *finitely presented* if there exists an exact sequence $R^{\oplus m} \to R^{\oplus n} \to M \to 0$ for some $m, n \in \mathbb{N}$, i.e. $M \cong R^n/N$ where $N \subset R^n$ itself is a finite free R-module.

Definition. A ring map $R \to S$ is *standard étale* if there exist $f, g \in R[x]$ with f monic such that $S \cong (R[x]/\langle f\rangle)_g$ with f' a unit in S, and $R \to S$ is the canonical map $R \to (R[x]/\langle f\rangle)_g$.

Definition. Suppose we have a ring map $R \to S$. For any extension of R-algebras $A \to A'$ where $\ker(A \to A')^2 = 0$, consider the natural map $\mathrm{Hom}_R(S, A) \to \mathrm{Hom}_R(S, A')$. We say that $R \to S$ is *formally unramified* if this natural map is an injection, is *formally smooth* if this natural map is a surjection, and is *formally étale* if it is a bijection. Thus a formally unramified and formally smooth map is formally étale.

1.2 Étale Morphisms of Schemes

(see [Sta22] for proofs)

Lemma-Definition. A morphism $f : X \to Y$ of schemes is *flat* if for any open affines $\mathrm{Spec}(A) \subset X$ and $\mathrm{Spec}(B) \subset Y$, the corresponding ring map $B \to A$ is flat. A morphism is *faithfully flat* if it is both surjective and flat.

It is easy to check that flatness is an affine local (in the sense of Vakil) property on the target Y.

Proposition 1.3. *Suppose $f : X \to S$ is a morphism of schemes.*

1. *f is flat if and only if for every $x \in X$ the local ring map $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.*
2. *f is flat if and only if for every $S \to S'$, the pull-back functor $\mathrm{QCoh}(S') \to \mathrm{QCoh}(X \times_S S')$ induced by the map $X \times_S S' \to S'$ is an exact functor.*
3. *Composite of flat morphisms is flat.*
4. *Fibre product of two flat (resp. faithfully flat) morphisms is flat (resp. faithfully flat).*
5. *Flatness and faithful flatness are preserved by base change.*
6. *If f is flat, then for every $x \in X$ and every $s \in S$ such that $f(x) \in \overline{\{s\}}$, there exists $x' \in X$ such that $s = f(x')$ and $x \in \overline{\{x'\}}$.*
7. *If f is flat and locally of finite presentation, then it is universally open, i.e. for every $S' \to S$ the induced map $X \times_S S' \to S'$ is an open map.*
8. *If f is quasi-compact and faithfully flat (i.e. fpqc), then $T \subset S$ is open (respectively closed) iff $f^{-1}(T)$ is open (respectively closed).*

Thus, fpqc maps can be thought of as quotient maps.

Definition. A morphism $f : X \to Y$ is unramified if it is locally of finite presentation and for each $x \in X$ and $y = f(x)$, the residue field $\kappa(x)$ is a separable algebraic extension of $\kappa(y)$, and $f_x(\mathfrak{m}_{Y,y})\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ where $f_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.

A map being unramified is affine local (in the sense of Vakil) on the target Y.

Proposition 1.4. *Suppose $f : X \to Y$ is a morphism of schemes.*

1. *f is unramified if and only if it is locally of finite presentation and for any affine opens $\mathrm{Spec}(B) \subset X$ and $\mathrm{Spec}(A) \subset Y$ the induced map $f^\# : A \to B$ is formally unramified.*
2. *f is unramified if and only if it is locally of finite presentation and the diagonal map $X \to X \times_Y X$ is an open embedding.*
3. *Composite of unramified morphisms is unramified.*
4. *Base change of an unramified morphism is unramified.*
5. *Open immersions are unramified.*
6. *f is unramified if and only if it is locally of finite presentation and the diagonal map $X \to X \times_Y X$ is an open immersion.*
7. *If $f : X \to Y$ is a morphism of S-schemes where X is unramified over S and Y is locally of finite type over S, then f is unramified.*
8. *Suppose X and Y are S-schemes and $f, g : X \to Y$ morphisms over S. Suppose Y is unramified over S. Let $x \in X$ be such that $f(x) = g(x) =: y$ where the maps $f_x, g_x : \kappa(y) \to \kappa(x)$ induced by f and g are equal. Then, there exists a Zariski open neighbourhood U of x in X such that $f|_U = g|_U$.*

Lemma-Definition. Suppose $f : X \to S$ is a morphism locally of finite presentation. We say that f is *smooth* if any of the following hold:

1. for any affine opens $\mathrm{Spec}(B) \subset X$ and $\mathrm{Spec}(A) \subset S$ the induced map $f^\# : A \to B$ is formally smooth;

2. f is flat and for every S-morphism $\bar{s} : \operatorname{Spec} k \hookrightarrow S$ for k algebraically closed, the fibre $X_{\bar{s}} = X \times_S \bar{s}$ is regular

3. f is flat and all fibers $f^{-1}(s)$ are regular and remain so after extension of scalars to some perfect extension of $\kappa(s)$

We now come to the definition of an étale map.

Lemma-Definition. Suppose $f : X \to Y$ is a morphism of schemes. We say that f is *étale* if it has any of the following equivalent properties:

1. f is flat and unramified;
2. f is smooth and unramified;
3. f is flat, locally of finite presentation, and every fibre $f^{-1}(y)$ is given by the disjoint union $\bigsqcup_{i \in I} \operatorname{Spec} k_{i,y}$ where each $k_{i,y}$ is a finite separable field extension of the residue field $\kappa(y)$;
4. f is smooth and locally quasi-finite;
5. f is locally of finite presentation and for any affine opens $\operatorname{Spec}(B) \subset X$ and $\operatorname{Spec}(A) \subset S$ the induced map $f^{\#} : A \to B$ is formally étale;
6. for every $x \in X$ there is an open neighbourhood U of X around x and an open affine $V = \operatorname{Spec} A$ around $f(x)$ with $f(U) \subset V$ such that U is V-isomorphic to an open subscheme of $\operatorname{Spec}\left(A[t]/\langle f \rangle\right)_{f'}$ for some monic $f \in A[t]$ (with f' the usual derivative of f).

We have the following important properties of étale maps.

Proposition 1.5. *1. Étale morphisms are preserved under composition and base change.*

2. Being an étale morphism is a local property on both the source and the target.

3. Product of a finite family of étale morphisms is étale.

4. Suppose $g : Y \to Z$ an unramified map and $f : X \to Y$ a map such that $g \circ f$ is étale. Then, f is étale.

5. Any S-morphism between étale S-schemes is étale.

6. Étale morphisms are locally quasi-finite.

7. Open immersions are étale. Moreover, a morphism is an open immersion if and only if it is étale and universally injective.

8. A map $X \to \operatorname{Spec} k$ is étale if and only if X is the disjoint union of $\operatorname{Spec} k'$ for k' a finite separable field extension of k.

9. Étale morphisms are open.

1.3 Sites and Topoi

Suppose now $\mathcal{C}$ is a category with all fibre products.

Definition. A *Grothendieck Topology* $\mathcal{T}$ on $\mathcal{C}$ is for all $X \in \mathcal{C}$ a collection $\operatorname{Cov}_{\mathcal{T}}(X)$ of sets of morphisms $Y \to X$ with target X such that:

- for all $Y \to X$, and for all $\{U_i \to X\}_{i \in I} \in \operatorname{Cov}_{\mathcal{T}}(X)$, we have $\{U_i \times_X Y \to Y\}_{i \in I} \in \operatorname{Cov}_{\mathcal{T}}(Y)$; and
- if $\{f_i U_i \to X\}_{i \in I} \in \operatorname{Cov}_{\mathcal{T}}(X)$, and for all $i \in I$ we have $\{g_{ij} : U_{ij} \to U_i\}_{j \in J_i} \in \operatorname{Cov}_{\mathcal{T}}(X)$, then
$$\{f_i \circ g_{ij} : U_{ij} \to X : i \in I, j \in J_i\} \in \operatorname{Cov}_{\mathcal{T}}(X).$$
- if $f : X' \cong X$, then $\{f : X' \to X\} \in \operatorname{Cov}_{\mathcal{T}}(X)$.

A category $\mathcal{C}$ equipped with a Grothendieck topology $\mathcal{T}$ is called a *site*.

Here, the collection $\operatorname{Cov}_{\mathcal{T}}(X)$ should be thought of as the collection of all open covers of X.

The example that most concerns us is when $\mathcal{C}$ is the category of S-schemes or some subcategory of S-schemes, where S is some fixed base-scheme. Throughout, we use the following notation:

- Sch_S is the category of S-schemes.
- $\mathrm{Sch}_S^{\text{ét}}$ is the category of étale S-schemes, i.e. it is the subcategory of Sch_S consisting of those S-schemes U such that the structure map $U \to S$ is étale. This subcategory is a full-subcategory since any map $U \to U'$ between étale S-schemes U and U' must itself be étale (by Proposition 1.5(5)).

The following topologies are most important.

- The *Zariski topology* on Sch_S is given by

$$\mathrm{Cov}_{\mathrm{Zar}}(X) = \left\{ \{f_i : U_i \to X\}_{i \in I} : \bigcup_{i \in I} f_i(U_i) = X, f_i \text{ an open embedding} \right\}.$$

- The *étale topology* on $\mathrm{Sch}_S^{\text{ét}}$ or Sch_S is given by

$$\mathrm{Cov}_{\text{ét}}(X) = \left\{ \{f_i : U_i \to X\}_{i \in I} : \bigcup_{i \in I} f_i(U_i) = X, f_i \text{ an étale map} \right\}.$$

  The *étale site* is $\mathrm{Sch}_S^{\text{ét}}$ equipped with the above étale topology.

- The *smooth topology* on Sch_S is given by

$$\mathrm{Cov}_{\mathrm{sm}}(X) = \left\{ \{f_i : U_i \to X\}_{i \in I} : \bigcup_{i \in I} f_i(U_i) = X, f_i \text{ a smooth map} \right\}.$$

- The *fppf topology* on Sch_S is then given by

$$\mathrm{Cov}_{\mathrm{fppf}}(X) = \left\{ \{f_i : U_i \to X\}_{i \in I} : \bigcup_{i \in I} f_i(U_i) = X, f_i \text{ is flat and of finite presentation} \right\}.$$

  Here, a map is of finite presentation if it is quasi-compact, quasi-separated, and locally of finite presentation.

- The *fpqc topology* on Sch_S is then given by

$$\mathrm{Cov}_{\mathrm{fpqc}}(X) = \left\{ \{f_i : U_i \to X\}_{i \in I} : \bigcup_{i \in I} f_i(U_i) = X, f_i \text{ is flat and quasi-compact} \right\}.$$

The above topologies are listed in order of coarseness, i.e.

$$\mathrm{Cov}_{\mathrm{Zar}} \subset \mathrm{Cov}_{\text{ét}} \subset \mathrm{Cov}_{\mathrm{sm}} \subset \mathrm{Cov}_{\mathrm{fppf}} \subset \mathrm{Cov}_{\mathrm{fpqc}}.$$

This follows from the fact that for an arbitrary morphism $f : X \to S$, we have

$$f \text{ an open embedding} \implies f \text{ étale} \implies f \text{ smooth} \implies f \text{ is flat and finitely presented} \implies f \text{ is flat and quasi-compact.}$$

Now, suppose $\mathcal{C}$ is a site.

Definition. A *presheaf* $\mathcal{F}$ of sets (resp. abelian groups) on a site $\mathcal{C}$ is a contravariant functor $\mathcal{F} : \mathcal{C}^{op} \to \mathrm{Set}$ (resp. $\mathcal{F} : \mathcal{C}^{op} \to \mathrm{Ab}$).

A morphism of pre-sheaves is simply a natural transformation between the two contravariant functors.

If $f : X \to Y$ is a morphism in $\mathcal{C}$, then we denote the morphism $\mathcal{F}(f)$ of sets/abelian groups by f^*.

Definition. A presheaf $\mathcal{F}$ on a site is a *sheaf* if for all $X \in \mathcal{C}$ and for all covers $\{f_i : U_i \to X\} \in \mathrm{Cov}_{\mathcal{T}}(X)$, the following two conditions hold:

- (separated) For all $s, s' \in \mathcal{F}(X)$, if $f_i^* s = f_i^* s'$ for all $i \in I$, then $s = s'$.

- (glueability) If for each i we have $s_i \in \mathcal{F}(U_i)$ such that, writing

$$\begin{array}{ccc} U_i \times_X U_j & \xrightarrow{f'_j} & U_i \\ \downarrow{\scriptstyle f'_i} & \lrcorner & \downarrow{\scriptstyle f_i} \\ U_j & \xrightarrow{f_j} & X \end{array},$$

we have $(f'_j)^* s_i = (f'_i)^* s_j$ for all $i, j \in I$, then there exists $s \in \mathcal{F}(X)$ such that $f_i^* s = s_i$ for all $i \in I$.

The global sections functor is the covariant functor $\mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Set}$ given by $\mathcal{F} \mapsto \mathcal{F}(S)$.

The category of sheaves of sets on a site is called a *topos*.

As usual, the sheaf conditions can be summarized in the usual equalizer diagram:

$$\mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_X U_j).$$

Lemma 1.6. *A morphism of pre-sheaves $\mathcal{F} \to \mathcal{G}$ is a monomorphism in the category of pre-sheaves if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is an injective map for all $U \in \mathrm{Sch}_S^{\text{ét}}$.*

From now on, we will denote the category of sheaves of sets on the *étale site* $\mathrm{Sch}_S^{\text{ét}}$ (NOT the full category Sch_S equipped with the étale topology) by $\mathrm{Shv}_{\text{ét}}(S)$. For the subcategory of sheaves of abelian groups, we use $\mathrm{AbShv}_{\text{ét}}(S)$. One checks that $\mathrm{Shv}_{\text{ét}}(S)$ for the topologies listed above have arbitrary products, have an initial object (the empty sheaf), and have a final object that sends every object of $\mathrm{Sch}_S^{\text{ét}}$ to a singleton set.

Remark 1.7. While most of the results that follow are direct consequences for a topos on a site, we stick to the étale site for concreteness. For reference, chapter 1 of [Tam94] and chapter 2-3 of [Mil80] develop the theory of sheaves on a site in complete generality.

However, at some point we will need to use different topologies on Sch_S, for instance the fppf site or the Zariski site (the usual topology). This will be true especially during discussions of comparison theorems.

1.4 Examples of Étale Sheaves

We now describe some important classes of examples of sheaves on $\mathrm{Sch}_S^{\text{ét}}$. All of these examples come from [Con].

Example 1.8. For any $X \in \mathrm{Sch}_S$, consider the functor $\underline{X} : (\mathrm{Sch}_S^{\text{ét}})^{op} \to \mathrm{Set}$ given by

$$\underline{X}(T) = \mathrm{Hom}_S(T, X)$$

where $\mathrm{Hom}_S(T, X)$ denotes the homset in the category Sch_S. Then, $\underline{X} \in \mathrm{Shv}_{\text{ét}}(S)$. Yoneda's lemma implies that the functor $\mathrm{Sch}_S^{\text{ét}} \to \mathrm{Shv}_S^{\text{ét}}$ given by $X \mapsto \underline{X} = \mathrm{Hom}_S(\cdot, X)$ is fully faithful; sheaves in the essential image of this functor are called *representable*.

The global sections functor $\mathcal{F} \mapsto \mathcal{F}(S)$ is naturally isomorphic to the functor $\mathcal{F} \to \mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(S)}(\underline{S}, \mathcal{F})$.

Example 1.9. If S' is a scheme and there is a morphism $j : S' \to S$ is étale, then we can define the *pull-back* functor $j^* : \mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Shv}_{\text{ét}}(S')$ which sends any $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(S)$ to the étale sheaf $j^*\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(S')$ defined by $j^*\mathcal{F}(X \to S') = \mathcal{F}(X \to S' \xrightarrow{j} S)$ (i.e. any $X \in \mathrm{Sch}_{S'}$ is considered as a scheme over S by composing the structure map with j). If the étale map $j : S' \to S$ is understood, then we usually denote $j^*\mathcal{F}$ by $\mathcal{F}|_{S'}$.

Example 1.10. If $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_{\text{ét}}(S)$, the presheaf

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) : U \mapsto \mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is in fact a sheaf of sets, called the *hom-sheaf.* If $\mathcal{F}, \mathcal{G} \in \mathrm{AbShv}_{\text{ét}}(S)$ instead, then one can check that

$$U \mapsto \mathrm{Hom}_{\mathrm{AbShv}_{\text{ét}}(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is a sheaf of abelian groups. This is also denoted by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, where the context should make it clear which hom-sheaf is being referred to.

Example 1.11. Suppose $f : S \to S'$ is an arbitrary map of schemes. The *push-forward* functor $f_* : \mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Shv}_{\text{ét}}(S')$ takes a sheaf $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(S)$ and sends it to the sheaf $(f_*\mathcal{F}) : U \mapsto \mathcal{F}(S' \times_S U)$ for all étale S-schemes U.

Whenever $f : S' \to S$ is étale, one checks that (f_*, f^*) is an adjoint pair, i.e.

$$\mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(S')}(f^*\mathcal{F}, \mathcal{F}') \cong \mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(S)}(\mathcal{F}, f_*\mathcal{F}').$$

This adjunction holds true more generally, but for f not étale, the definition of the push-forward functor is more involved (see below).

Example 1.12. As an example of the previous adjunction, we compute the pull-back of a representable sheaf. Suppose $j : S' \to S$ is an étale map, and suppose $X \in \mathrm{Sch}_S^{\text{ét}}$ is arbtirary. For any $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(S')$, we have

$$\mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(S')}(j^*\underline{X}, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(S)}(\underline{X}, j_*\mathcal{F}) = (j_*\mathcal{F})(X) = \mathcal{F}(X \times_S S') = \mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(S')}(\underline{X \times_S S'}, \mathcal{F}).$$

Yoneda's lemma implies that we have a natural isomorphism $j^*\underline{X} \cong \underline{X \times_S S'}$.

Example 1.13. If $S = \mathrm{Spec}\, k$ where k is a separably closed field, then étale k-schemes are disjoint unions of copies of S. Thus, $\mathrm{Sch}_S^{\text{ét}}$ is equivalent to the category of sets via the global-sections functor.

Example 1.14. Consider the pre-sheaves $\mathbb{G}_a = \mathbb{G}_{a,S} : (\mathrm{Sch}_S^{\text{ét}})^{op} \to \mathrm{Ab}$ given by $\mathbb{G}_a(U) = \Gamma(U, \mathcal{O}_U)$ viewed as an additive group, and $\mathbb{G}_m = \mathbb{G}_{m,S} : (\mathrm{Sch}_S^{\text{ét}})^{op} \to \mathrm{Ab}$ given by $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^*$ (the group of units). On morphisms, both $\mathbb{G}_m$ and $\mathbb{G}_a$ sends $f : U \to U'$ to the map of abelian groups induced by the map $f^\# : \mathcal{O}_{U'} \to \mathcal{O}_U$. By the usual glueing properties of ordinary sheaves, one checks that $\mathbb{G}_a$ and $\mathbb{G}_m$ are in fact sheaves. In fact, one can show that

$$\mathbb{G}_{a,S} = S \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\left(\mathbb{Z}[t]\right) \quad \text{and} \quad \mathbb{G}_{m,S} = S \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\left(\mathbb{Z}[t, t^{-1}]\right).$$

In fact, these sheaves are representable, specifically by $\mathbb{G}_{a,S} \cong S \times_{\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[t]$ and $\mathbb{G}_{m,S} \cong S \times_{\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[t, t^{-1}]$.

One may also view the representable sheaf $\mathbb{G}_{a,S}$ as a sheaf of rings, in which case we denote this by $\mathcal{O}_{S,\text{ét}}$.

1.5 Galois Modules and the Étale Topos over $\mathrm{Spec}\, k$

Throughout this subsection, we suppose that $S = \mathrm{Spec}\, k$ for some field k. We fix a choice of separable algebraic closure $\bar{k}$ of k. Throughout, denote by G_k the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$. This section will describe the equivalence between $\mathrm{Shv}_{\text{ét}}(\mathrm{Spec}\, k)$ and the category of (left) discrete G_k-sets. As a consequence, Galois cohomology for $\bar{k}/k$ computes sheaf cohomology on $\mathrm{AbShv}_{\text{ét}}(\mathrm{Spec}\, k)$.

Definition. Suppose G is a profinite group (i.e. a compact Hausdorff totally disconnected topological group). A *(left) discrete G-set* is a set M with a group action G such that, when M is equipped with the discrete topology, the G-action is continuous. Concretely, M is a discrete G-set if and only if for each $m \in M$ the stabilizer subgroup $\mathrm{Stab}_G(m) = \{g \in G : gm = m\}$ is an open (and thus also a closed finite index) subgroup of G.

The category of discrete G-sets will be denoted by $G\mathrm{Set}^{disc}$.

If M is an abelian group that is also a discrete G-set and the G-action is compatible with the group structure on M in the obvious way, then we say that M is a *discrete G-module.*

Recall that G_k is a profinite group, where a basis of open neighbourhoods of the identity is given by the collection of $\mathrm{Gal}(K/k)$ for K/k a finite Galois extension. A discrete G_k-set is often called a *discrete Galois set.*

Remark 1.15. The abelian category of discrete G-modules has enough injectives, and so we can find derived functors for left-exact functors. For an explicit construction of injectives, see Conrad's notes [Con].

Remark 1.16. Recall that X is étale over $\mathrm{Spec}\, k$ if and only if X is the disjoint union of various $\mathrm{Spec}\, k'$ for k' a finite separable extensions of k. Writing $X = \bigsqcup_i X_i$ (with $X_i = \mathrm{Spec}\, k_i$, k_i/k finite separable), we see that the collection $\{X_i\}$ is an étale cover of $\mathrm{Spec}\, k$, and so the two sheaf axioms imply that

$$\mathcal{F}(X) = \prod_{i \in I} \mathcal{F}(X_i) \cong \prod_{i \in I} \mathcal{F}(\mathrm{Spec}\, k_i)$$

for every $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(\mathrm{Spec}\, k)$. It thus suffices to understand the properties of sheaves on $\mathrm{Spec}\, k_i \to \mathrm{Spec}\, k$.

Theorem 1.17. *We have a natural equivalence of categories*

$$\mathrm{Sch}_{\mathrm{Spec}\, k}^{\text{ét}} \xleftrightarrow{\sim} \mathrm{Shv}_{\text{ét}}(\mathrm{Spec}\, k) \xleftrightarrow{\sim} G_k\mathrm{Set}^{disc}.$$

We omit a detailed proof; instead, we describe what each of the functors are without checking that the functors do indeed induce equivalences. For details see [Con]. Throughout, we will fix a choice of separable algebraic closure $\bar{k}$ of k.

$\mathrm{Sch}_{\mathrm{Spec}\, k}^{\text{ét}} \to G_k\mathrm{Set}^{disc}$

This is simply given by $X \mapsto X(\bar{k})$.

$\mathrm{Sch}_{\mathrm{Spec}\, k}^{\text{ét}} \to \mathrm{Shv}_{\text{ét}}(\mathrm{Spec}\, k)$

This is simply the functor of points $X \mapsto \underline{X}$ described previously.

$\mathrm{Shv}_{\text{ét}}(\operatorname{Spec} k) \to G_k\mathrm{Set}^{disc}$

Fix a sheaf $\mathcal{F}$. If $k' \hookrightarrow k''$ is any inclusion of finite separable extensions of k, then we have a corresponding morphism $\operatorname{Spec} k'' \to \operatorname{Spec} k'$ in $\mathrm{Sch}_{\operatorname{Spec} k}$. This gives a natural map $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'')$. Notice thus that $\mathcal{F} \circ \operatorname{Spec}$ gives a covariant functor from the category of finite separable extensions of k to Set.

The action of the Galois group $\mathrm{Gal}(k''/k')$ on k'' induces an action of $\mathrm{Gal}(k''/k')$ on $\operatorname{Spec} k''$, and thus we have an action of $\mathrm{Gal}(k''/k')$ on $\mathcal{F}(\operatorname{Spec} k'')$. Moreover, as $k' \hookrightarrow k''$ is $\mathrm{Gal}(k''/k')$-invariant, it follows that $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'')$ is $\mathrm{Gal}(k''/k')$-invariant.

Lemma 1.18. *For any $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(\operatorname{Spec} k)$ and for any finite Galois extension k''/k' (where k''/k is a finite separable extension), the natural map $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'')$ is in fact injective and induces a natural bijection*

$$\mathcal{F}(\operatorname{Spec} k') \xrightarrow{\sim} \mathcal{F}(\operatorname{Spec} k'')^{\mathrm{Gal}(k''/k')}.$$

Proof. Notice that we have an isomorphism of k'' algebras

$$k'' \otimes_{k'} k'' \xrightarrow{\sim} \prod_{g \in \mathrm{Gal}(k''/k')} k'', \quad x \otimes y \mapsto (x \cdot g(y))_{g \in \mathrm{Gal}(k''/k')}.$$

Here, $k'' \otimes_{k'} k''$ is a k'' algebra via the left tensor factor, whereas the product $\prod_{\mathrm{Gal}(k''/k')} k''$ is a k'' algebra via the diagonal inclusion. Now, $\operatorname{Spec} k'' \to \operatorname{Spec} k'$ is an étale cover, and so the sheaf axioms imply that

$$\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'') \rightrightarrows \mathcal{F}(\operatorname{Spec} k'' \times_{\operatorname{Spec} k'} \operatorname{Spec} k'')$$

is exact, where the top and bottom arrows are respectively the identification of k'' with the left and right tensor factors respectively $k'' \otimes_{k'} k''$. Exactness automatically implies that $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'')$ is injective. Moreover, from the explicit isomorphism of algebras $k'' \otimes_{k'} k'' \xrightarrow{\sim} \prod_{g \in \mathrm{Gal}(k''/k')} k''$, $\quad x \otimes y \mapsto (x \cdot g(y))_{g \in \mathrm{Gal}(k''/k')}$, we then see that in

$$\operatorname{Spec} k'' \rightrightarrows \bigsqcup_{g \in \mathrm{Gal}(k''/k')} \mathcal{F}(\operatorname{Spec} k'')$$

the top arrow corresponds to the diagonal inclusion $s \mapsto (s, ..., s)$, whereas the bottom arrow is $s \mapsto (gs)_{g \in G}$. Thus, we see from the exactness of

$$\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'') \rightrightarrows \bigsqcup_{g \in \mathrm{Gal}(k''/k')} \mathcal{F}(\operatorname{Spec} k'')$$

that $s \in \mathcal{F}(\operatorname{Spec} k'')^{\mathrm{Gal}(k''/k')}$ if and only if $s \in \ker\Big(\mathcal{F}(\operatorname{Spec} k'') \rightrightarrows \bigsqcup_{g \in \mathrm{Gal}(k''/k')} \mathcal{F}(\operatorname{Spec} k'')\Big)$ if and only if $s \in \mathcal{F}(\operatorname{Spec} k')$. □

Now let Σ be the set of all finite Galois extensions of k inside $\bar{k}$ (where recall that we fix a choice of $\bar{k}$). The previous lemma implies that for every $k' \to k''$ in Σ we have a corresponding inclusion $\mathcal{F}(\operatorname{Spec} k') \cong \mathcal{F}(\operatorname{Spec} k'')^{\mathrm{Gal}(k''/k')} \subset \mathcal{F}(\operatorname{Spec} k'')$. This in particular implies that the map $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'')$ is compatible with Galois actions via the surjection $\mathrm{Gal}(k''/k) \to \mathrm{Gal}(k'/k)$. Hence, we see that

$$M_{\mathcal{F}} := \mathrm{colim}_{k' \in \Sigma} \mathcal{F}(\operatorname{Spec} k')$$

has a natural structure of a discrete left $\mathrm{Gal}(\bar{k}/k)$-set, and moreover by unravelling the definition of $M_{\mathcal{F}}$ as a filtered colimit, we see that the natural map $\mathcal{F}(\operatorname{Spec} k') \to M_{\mathcal{F}}$ is an isomorphism onto $M_{\mathcal{F}}^{\mathrm{Gal}(\bar{k}/k')}$ as $\mathrm{Gal}(k'/k)$-sets. This gives us the functor

$$\mathrm{Shv}_{\text{ét}}(\operatorname{Spec} k) \to G_k\mathrm{Set}^{disc}, \quad \mathcal{F} \mapsto M_{\mathcal{F}}.$$

$G_k\mathrm{Set}^{disc} \to \mathrm{Shv}_{\text{ét}}(\operatorname{Spec} k)$

Suppose M is a discrete G_k-set. For any finite separable extension k' over k define

$$\mathcal{F}_M(\operatorname{Spec} k') = \left\{ (m_i)_i \in \prod_{i : k' \hookrightarrow \bar{k}} M \;\middle|\; m_{g \circ i} = g(m_i) \text{ for all } g \in \mathrm{Gal}(\bar{k}/k) \right\}.$$

Here, i runs through all finitely many k-embeddings of k' into $\bar{k}$. Here, note that $m_i \in M^{\mathrm{Gal}(\bar{k}/i(k'))}$ since $g \in \mathrm{Gal}(\bar{k}/i(k'))$ if and only if $g \circ i = i$. Moreover, for any $k' \in \Sigma$ (for Σ the set of all finite Galois extensions of

k inside $\bar{k}$ as above), we have the canonical inclusion $k' \hookrightarrow \bar{k}$ arising from membership in Σ, and this inclusion defines the bijection

$$\mathcal{F}_M(\operatorname{Spec} k') \cong M^{\operatorname{Gal}(\bar{k}/k')}.$$

For any k-embedding $j : k' \hookrightarrow k''$ of finite separable extensions of k, define the map

$$\mathcal{F}_M(j) : \mathcal{F}_M(\operatorname{Spec} k') \to \mathcal{F}_M(\operatorname{Spec} k''), \quad \mathcal{F}_m(j)\big((m_{i'})_{i'}\big) = (m_{i''\circ j})_{i''}$$

where we note that every $i' : k' \hookrightarrow \bar{k}$ over k is of the form $i'' \circ j$ for some $i'' : k'' \hookrightarrow \bar{k}$ over k.

For an arbitrary étale $\operatorname{Spec} k$-scheme we then have

$$\mathcal{F}_M(\bigsqcup \operatorname{Spec} k_\alpha) = \prod \mathcal{F}_M(\operatorname{Spec} k_\alpha),$$

and similarly for arbitrary morphisms in $\mathrm{Sch}^{\text{ét}}_{\operatorname{Spec} k}$. In this way we have a pre-sheaf $\mathcal{F}_M$, which can be checked to be a sheaf.

1.6 Sheafification, Extensions-by-Initials, and Pull-Backs

Definition. Suppose we have a pre-sheaf $\mathcal{F} : (\mathrm{Sch}^{\text{ét}}_S)^{op} \to \mathrm{Set}$. The *sheafification of* $\mathcal{F}$ is a sheaf $\mathcal{F}^{sh} \in \mathrm{Shv}_{\text{ét}}(S)$ equipped with a pre-sheaf morphism $\mathcal{F} \to \mathcal{F}^{sh}$ with the universal property that any pre-sheaf morphism $\mathcal{F} \to \mathcal{G}$ with $\mathcal{G} \in \mathrm{Shv}_{\text{ét}}(S)$ factors through to a unique map $\mathcal{F}^{sh} \to \mathcal{G}$.

Remark 1.19. While the exact construction doesn't matter and is in fact not helpful, we still present it here for completeness. We describe a $-^+$ construction on pre-sheaves that takes arbitrary pre-sheaves to separated pre-sheaves (i.e. those that only satisfy the separated axiom of sheaves), and that takes separated pre-sheaves to sheaves. Thus, given an arbitrary pre-sheaf, applying the $-^+$ construction once yields a separated pre-sheaf that satisfies a corresponding universal property, and applying $-^+$ again yields a sheaf.

For any $U \in \mathrm{Sch}^{\text{ét}}_S$ and any étale cover $\mathcal{U} = \{U_i\}_{i\in I}$ of U, let $H^0(\mathcal{U}, \mathcal{F})$ be the set of I-tuples $(s_i) \in \prod_{i\in I} \mathcal{F}(U_i)$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j, where $U_{ij} := U_i \times_U U_j$. A refinement $\mathcal{U}'$ of $\mathcal{U}$ defines a map $H^0(\mathcal{U}, \mathcal{F}) \to H^0(\mathcal{U}', \mathcal{F})$. Taking the colimit yields

$$\mathcal{F}^+(U) := \operatorname{colim}_{\mathcal{U}\in \mathrm{Cov}_{\text{ét}}(U)} H^0(\mathcal{U}, \mathcal{F}).$$

Proposition 1.20. *Suppose $\mathcal{F}$ is a pre-sheaf with sheafification $\mathcal{F}^{sh}$.*

1. *For any $s \in \mathcal{F}^{sh}(U)$ there exists an étale cover $\{U_i\}$ of U and an element $s_i \in \mathcal{F}(U_i)$ mapping (via $\mathcal{F} \to \mathcal{F}^{sh}$) to $s|_{U_i} \in \mathcal{F}^{sh}(U_i)$.*

2. *For any $s, t \in \mathcal{F}(U)$ which are mapped to the same element in $\mathcal{F}^{sh}(U)$, there exists an étale cover $\{U_i\}$ of U such that $s|_{U_i} = t|_{U_i}$ in $\mathcal{F}(U_i)$ for all i.*

3. *If $\mathcal{F} \to \mathcal{G}$ is a monomorphism of pre-sheaves, then the sheafification $\mathcal{F}^{sh} \to \mathcal{G}^{sh}$ is also a monomorphism in the category of* $\mathrm{Shv}_{\text{ét}}(S)$.

By sheafification, we can construct the image sheaf of a scheme map, the quotient of a sheaf by an equivalence relation, etc. Everything carries over to $\mathrm{AbShv}_{\text{ét}}(S)$ as expected. We give two more constructions that require sheafification

Definition. Suppose $j : U \to S$ is an étale morphism. Define the functor $j^{\mathrm{Set}}_! : \mathrm{Shv}_{\text{ét}}(U) \to \mathrm{Shv}_{\text{ét}}(S)$ by taking $\mathcal{F}$ to the sheafification $j^{\mathrm{Set}}_!\mathcal{F}$ of the pre-sheaf

$$V \mapsto \bigsqcup_{f\in Hom_S(V,U)} \mathcal{F}(V \xrightarrow{f} U).$$

This functor is called the *extension-by-$\emptyset$* functor[1].

Definition. Suppose $j : U \to S$ is an étale morphism. Define the functor $j^{\mathrm{Ab}}_! : \mathrm{AbShv}_{\text{ét}}(U) \to \mathrm{AbShv}_{\text{ét}}(S)$ by taking $\mathcal{F}$ to the sheafification $j^{\mathrm{Ab}}_!\mathcal{F}$ of the pre-sheaf

$$V \mapsto \bigoplus_{f\in Hom_S(V,U)} \mathcal{F}(V \xrightarrow{f} U).$$

This functor is called the *extension-by-0* functor.

[1] $j^{\mathrm{Set}}_!$ is sometimes read as the 'shriek-pushforward' by j.

Remark 1.21. This is in fact the same construction categorically speaking, since both disjoint unions $\bigsqcup$ in Set and direct sums $\bigoplus$ in Ab are co-products. However, since the forgetful functor Ab $\to$ Set does not take co-products (direct sums) to co-products (disjoint unions), it follows that $j_!^{Ab}$ and $j_!^{\mathrm{Set}}$ are not compatible with the forgetful functor $\mathrm{AbShv}_{\text{ét}}(-) \to \mathrm{Shv}_{\text{ét}}(-)$.

Proposition 1.22. *For any $j : U \to S$ étale, the functor $j_!^{\mathrm{Set}}$ (resp. $j_!^{\mathrm{Ab}}$) is left-adjoint to $j^*\mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Shv}_{\text{ét}}(U)$ (resp. $j^*\mathrm{AbShv}_{\text{ét}}(S) \to \mathrm{AbShv}_{\text{ét}}(U)$).*

Proof Sketch. We prove for $j_!^{\mathrm{Set}}$, since the proof for $j_!^{\mathrm{Ab}}$ follows similarly. Given any $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(U)$ and any $\mathcal{G} \in \mathrm{Shv}_{\text{ét}}(S)$, we need to find a bijection of sets

$$\mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(S)}(j_!\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(U)}(\mathcal{F}, j^*\mathcal{G})$$

where for notational simplicity we write $j_!$ for $j_!^{\mathrm{Set}}$. Suppose $\varphi : j_!\mathcal{F} \to \mathcal{G}$ is a sheaf morphism. Then, we have a map $\hat{\varphi} : \mathcal{F} \to j^*\mathcal{G}$ which, for any $(V \xrightarrow{g} U) \in \mathrm{Sch}_U^{\text{ét}}$ is given by the composition

$$\mathcal{F}(V \xrightarrow{g} U) \hookrightarrow \bigsqcup_{g' \in Hom_S(V,U)} \mathcal{F}(V \xrightarrow{g'} U) \to (j_!\mathcal{F})(V \xrightarrow{j \circ g} S) \xrightarrow{\varphi_V} \mathcal{G}(V \xrightarrow{j \circ g} S) = (j^*\mathcal{G})(V).$$

On the other hand, given a map $\psi : \mathcal{F} \to j^*\mathcal{G}$, notice that we have a pre-sheaf morphism defined by

$$\bigsqcup_{f \in Hom_S(S',U)} \mathcal{F}\left(S' \xrightarrow{f} U\right) \xrightarrow{\bigsqcup_f \psi\left(S' \xrightarrow{f} U\right)} \bigcup_{f \in Hom_S(S',U)} (j^*\mathcal{G})\left(S' \xrightarrow{f} U\right) \subseteq \mathcal{G}(S')$$

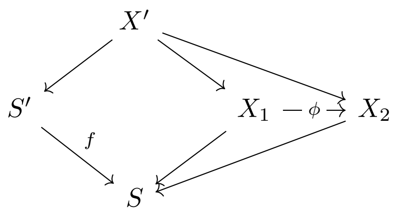
for any $S' \in \mathrm{Sch}_S^{\text{ét}}$. This factors into a sheaf morphism $\tilde{\psi} : j_!\mathcal{F} \to \mathcal{G}$. It is now straightforward to see that $\hat{}$ and $\tilde{}$ are inverse maps to each other. $\square$

Now, given an arbitrary map $f : S' \to S$ of schemes, we construct the pull-back functor $f^* : \mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Shv}_{\text{ét}}(S')$. For arbitrary scheme morphisms the construction is more delicate, though one can easily see that if f is étale the construction is the same as the one done previously.

Fix $X' \in \mathrm{Sch}_{S'}^{\text{ét}}$. Consider the subcategory $\mathcal{I}(X')$ of $\mathrm{Sch}_S^{\text{ét}}$ whose objects are étale S-schemes X equipped with a map $X' \to X$ such that

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

commutes, and morphisms between two objects $(X_1, X' \to X_1)$ and $(X_2, X' \to X_2)$ for $X_1, X_2 \in \mathrm{Sch}_S^{\text{ét}}$ are maps $\phi : X_1 \to X_2$ such that



commutes. This category is co-filtered since for any two $X_1, X_2 \in \mathrm{Sch}_S^{\text{ét}}$ equipped with maps $X' \to X_1, X' \to X_2$, the étale S-scheme $X_1 \times_S X_2$ equipped with the canonically induced map $X' \to X_1 \times_S X_2$ belongs to the category $\mathcal{I}(X')$ as well. Thus, given an arbitrary $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(S)$, we can define the filtered colimit

$$(f^{-1}\mathcal{F})(X') := \operatorname*{colim}_{X \in \mathcal{I}(X')} \mathcal{F}(X),$$

in the category of sets. This gives a pre-sheaf $f^{-1}\mathcal{F} : \mathrm{Sch}_{\text{ét}}(S') \to \mathrm{Set}$. Since filtered colimits in sets commute with finite limits, it follows that the functor $\mathcal{F} \mapsto f^{-1}\mathcal{F}$ from pre-sheaves on S to pre-sheaves on S' commutes with with finite limits as well. Sheafifying the above construction, we get a functor $f^* : \mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Shv}_{\text{ét}}(S')$ that commutes with all finite limits.

Remark 1.23. If $f : S' \to S$ is étale, then notice that the category $\mathcal{I}(X')$ has an initial object $X' \to S \in \mathcal{I}(X')$, and so the colimit taken above degenerates to $(f^{-1}\mathcal{F})(X') = \mathcal{F}(X' \to S)$ which is already a sheaf. We thus see that the given definition of pull-backs coincides with the previous definition of pull-backs by étale morphisms.

Remark 1.24. The same construction can be made to define the pull-back functor $f^* : \mathrm{AbShv}_{\text{ét}}(S) \to \mathrm{AbShv}_{\text{ét}}(S')$. Notice as before that these two pull-backs are not compatible with the forgetful functor, since limits and colimits in the categories Set and Ab are different.

Example 1.25. Suppose $k \to k'$ is an extension of fields with compatible choices of separable closures $\bar{k}$ and $\bar{k}'$. Let $f : \operatorname{Spec} k' \to \operatorname{Spec} k$ be the induced map. Under the equivalence of categories between Galois-sets and étale sheaves of sets, one can check that the functor $f^* : \mathrm{Shv}_{\text{ét}}(\operatorname{Spec} k) \to \mathrm{Shv}_{\text{ét}}(\operatorname{Spec} k')$ corresponds to the functor $G_k\mathrm{Set}^{disc} \to G_{k'}\mathrm{Set}^{disc}$ given by composing the action map with the continuous map $\mathrm{Gal}(\bar{k}'/k') \to \mathrm{Gal}(\bar{k}/k)$.

Proposition 1.26. *For an arbitrary morphism $f : S' \to S$ of schemes, the pull-back $f^* : \mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Shv}_{\text{ét}}(S')$ constructed above is left-adjoint to the push-forward $f_* : \mathrm{Shv}_{\text{ét}}(S') \to \mathrm{Shv}_{\text{ét}}(S)$.*

Proof Sketch. As usual we define the maps going either way, but do not check all details. Fix $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(S)$ and $\mathcal{F}' \in \mathrm{Shv}_{\text{ét}}(S')$; we want to construct a natural bijection

$$\mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(S')}(f^*\mathcal{F}, \mathcal{F}') \cong \mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(S)}(\mathcal{F}, f_*\mathcal{F}').$$

Suppose $\alpha : f^*\mathcal{F} \to \mathcal{F}'$ is a sheaf-homomorphism. Then, define $\beta^\alpha : \mathcal{F} \to f_*\mathcal{F}'$ as follows. For any $X \in \mathrm{Sch}_S^{\text{ét}}$, consider $X' = X \times_S X' \in \mathrm{Sch}_{S'}^{\text{ét}}$. The cartesian square defining X' implies that X is in the subcategory $\mathcal{I}(X')$ of $\mathrm{Sch}_S^{\text{ét}}$, and so we have a well-defined map $\mathcal{F}(X) \to \operatorname{colim}_{Y \in \mathcal{I}(X')} \mathcal{F}(Y') = (f^{-1}\mathcal{F})(X \times_S S')$. Composing this with the canonical map induced by sheafification, we get a map $\mathcal{F}(X) \to f^*\mathcal{F}(X \times_S S')$, and so we define $\beta_X^\alpha : \mathcal{F}(X) \to f_*\mathcal{F}'(X) = \mathcal{F}'(X \times_S S')$ by composing the map $\mathcal{F}(X) \to f^*\mathcal{F}(X \times_S S')$ with $\alpha_{X \times_S S'}$.

On the other hand, suppose $\beta : \mathcal{F} \to f_*\mathcal{F}'$ is a sheaf morphism. Fix $X' \in \mathrm{Sch}_{S'}^{\text{ét}}$. For each $X \in \mathcal{I}(X')$, we have a map $\beta_X : \mathcal{F}(X) \to \mathcal{F}'(X \times_S S')$. Since $X' \to X$ and $X' \to S'$, we have an induced map $X' \to X \times_S S'$ so that $X \times_S S' \in \mathcal{I}(X')$, and we thus get a map $\mathcal{F}'(X \times_S S') \to \mathcal{F}'(X')$. Pre-composing with β_X, we have a map $\mathcal{F}(X) \to \mathcal{F}'(X')$ for all $X \in \mathcal{I}(X')$, and so we get a map from the colimit $(f^{-1}\mathcal{F})(X') \to \mathcal{F}'(X')$. This defines a pre-sheaf morphism $f^{-1}\mathcal{F} \to \mathcal{F}'$, which then factors into a sheaf morphism $\alpha^\beta : f^*\mathcal{F} \to \mathcal{F}'$.

Unravelling definitions, one checks immediately that $\alpha^{\beta^\alpha} = \alpha$ and $\beta^{\alpha^\beta} = \beta$ (on the nose). ☐

1.7 Category Theoretic Properties of the Étale Topoi

Proposition 1.27. *Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf-map between sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_{\text{ét}}(S)$.*

1. *φ is a monomorphism in the category $\mathrm{Shv}_{\text{ét}}(S)$ if and only if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all $U \in \mathrm{Sch}_S^{\text{ét}}$.*

2. *φ is an epimorphism in the category $\mathrm{Shv}_{\text{ét}}(S)$ if and only if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is étale locally surjective, i.e. for all $U \in \mathrm{Sch}_S^{\text{ét}}$ and for all $t \in \mathcal{G}(U)$, there exists an étale cover $\{U_i\} \in \mathrm{Cov}_{\text{ét}}(U)$ and for each i there exists $s_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t|_{U_i}$.*

3. *φ is an epic monomorphism in the category $\mathrm{Shv}_{\text{ét}}(S)$ if and only if it is an isomorphism.*

Similar results hold for $\mathrm{AbShv}_{\text{ét}}(S)$. In particular, $\mathrm{AbShv}_{\text{ét}}(S)$ is an abelian category.

Proof. We prove part (1). The other two results follow as in ordinary sheaf theory. It is easy to see that if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective, then for any $\psi : \mathcal{F}' \to \mathcal{G}$ we have at most one map $\psi' : \mathcal{F}' \to \mathcal{F}$ such that $\psi = \varphi \circ \psi'$ since there is at most one map ψ'_U satisfying $\psi_U = \varphi_U \circ \psi'_U$.

Now suppose φ is a monomorphism, and suppose for some $U \in \mathrm{Sch}_S^{\text{ét}}$ we have $x_1, x_2 \in \mathcal{F}(U)$ such that $\varphi_U(x_1) = \varphi_U(x_2)$. For $i = 1, 2$, we define two maps $\psi^{(i)} : \underline{U} \to \mathcal{F}$. Indeed, for any $V \in \mathrm{Sch}_S^{\text{ét}}$, we define a map $\psi_V^{(i)} : \underline{U}(V) = Hom_S(V, U) \to \mathcal{F}(V)$. For all $f \in \underline{U}(V)$, define $\psi_V^{(i)}(f) = \mathcal{F}(f)(x_i)$ (where note that for $f : V \to U$, the contravariant functor $\mathcal{F}$ gives $\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$). For any $g : W \to V$ we have

$$\begin{array}{ccc}
\underline{U}(V) & \xrightarrow{\psi_V^{(i)}} & \mathcal{F}(V) \\
\Big\downarrow{\scriptstyle (-)\circ g} & \begin{array}{ccc} f & \longmapsto & \mathcal{F}(f)(x_i) \\ \downarrow & & \downarrow \\ f \circ g & \longmapsto & \mathcal{F}(f \circ g)(x_i) \end{array} & \Big\downarrow{\scriptstyle \mathcal{F}(g)} \\
\underline{U}(W) & \xrightarrow{\psi_W^{(i)}} & \mathcal{F}(W)
\end{array}$$

where notice that $\mathcal{F}(f \circ g)(x_i) = \mathcal{F}(g)\big(\mathcal{F}(f)(x_i)\big)$. Hence $\psi^{(i)} : \underline{U} \to \mathcal{F}$ is a sheaf morphism. For any $V \in \mathrm{Sch}_S^{\text{ét}}$, we have

$$\varphi_V \circ \psi_V^{(i)}(f) = \varphi_V\big(\mathcal{F}(f)(x_i)\big) = \varphi_U(x_i)$$

for every $f \in \underline{U}(V)$, and so $\varphi_V \circ \psi_V^{(1)} = \varphi_V \circ \psi_V^{(2)}$. Since φ_V is a monomorphism, it follows that $\psi_V^{(1)} = \psi_V^{(2)}$ and so $x_1 = x_2$. Therefore φ_U is injective for all U, as required. □

Part (1) immediately implies the following.

Corollary 1.27.1. *For any* $U \in \mathrm{Sch}_S^{\text{ét}}$*, the functor* $\mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Set}, \mathcal{F} \mapsto \mathcal{F}(U)$ *is left-exact. The restriction of this functor to abelian sheaves* $\mathrm{AbShv}_{\text{ét}}(S) \to \mathrm{Ab}$ *is also left-exact.*

While $\mathrm{AbShv}_{\text{ét}}(S)$ is an abelian category, the category $\mathrm{Shv}_{\text{ét}}(S)$ is not. Thus, one needs to define exactness slightly differently.

Definition. Suppose $F : C \to C'$ is a covariant functor between two categories that admit finite limits and colimits. The functor F is *left exact* if it commutes with finite limits, and it is *right exact* if it commutes with finite colimits. It is exact if it is both left exact and right exact.

Remark 1.28. In a general category, it is useful to keep the following facts in mind when thinking about exactness:

- Right adjoints commute with all limits and are thus always left-exact.
- Left adjoints commute with all colimits and are thus always right-exact.
- Colimits always commute with colimits, and similarly limits always commute with limits.
- In Set (and other 'nice' enough categories), filtered colimits commute with finite limits.

As an application of the last statement, we have the following result.

Proposition 1.29. *Sheafification is an exact functor from pre-sheaves of sets (resp. abelian groups) on* $\mathrm{Sch}_S^{\text{ét}}$ *to sheaves of sets (resp. abelian groups) on* $\mathrm{Sch}_S^{\text{ét}}$*.*

Example 1.30. Suppose $f : S' \to S$ is arbitrary. Since (f^*, f_*) is an adjunction, it follows that f^* is right exact while f_* is left exact.

We claim that f^* is also left exact. If f is étale, this follows from the adjunction

$$\mathrm{Shv}_{\text{ét}}(S) \xrightarrow{\;f^*\;} \mathrm{Shv}_{\text{ét}}(S'), \qquad f_*,\ f_!^{\mathrm{Set}} : \mathrm{Shv}_{\text{ét}}(S') \to \mathrm{Shv}_{\text{ét}}(S), \qquad f_!^{\mathrm{Set}} \dashv f^* \dashv f_*$$

where recall the notation that for two categories $\mathcal{C}$ and $\mathcal{D}$ with functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we write

$$\mathcal{C} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{D} \quad (F \dashv G)$$

to denote that (F, G) is an adjunction, i.e. F is left-adjoint to G, or equivalently, that G is right-adjoint to F. In general, for an arbitrary $f : S' \to S$, notice that the functor f^{-1} on pre-sheaves is defined by taking a filtered colimit of sets at each point. Sheafification is done similarly, since the $(-)^+$ construction is defined by filtered colimits at each point. Hence, in all, $f^*\mathcal{F}$ for $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(S)$ is constructed by defining $f^*\mathcal{F}(X)$ for all $X' \in \mathrm{Sch}_{S'}^{\text{ét}}$ by taking a filtered colimit. However, all limits are computed pointwise, i.e. if $\mathcal{F}_i$ is a diagram of pre-sheaves or of sheaves then $(\lim \mathcal{F}_i)(X) = \lim(\mathcal{F}_i(X))$. Since finite limits commute with filtered colimits in Set (and in fact in Ab as well), it follows that f^* commutes with all finite limits. Therefore f^* is left-exact, and hence is an exact functor.

Similarly, we see that $f^* : \mathrm{AbShv}_{\text{ét}}(S) \to \mathrm{AbShv}_{\text{ét}}(S')$ is an exact functor whereas $f_* : \mathrm{AbShv}_{\text{ét}}(S') \to \mathrm{AbShv}_{\text{ét}}(S)$ is only left-exact.

Remark 1.31. In general, one should not expect the push-forward to be right-exact.

We do not prove the following theorem; see [Con].

Theorem 1.32 (Topological Invariance of the Étale Site)**.** *Suppose $f : S' \to S$ is a universal homeomorphism (i.e. for any base change $T \to S$, the induced map $S' \times_S T \to T$ is a homeomorphism; or equivalently, it is affine, universally closed, universally injective and surjective). Then,*

1. *the categories* $\mathrm{Shv}_{\text{ét}}(S')$ *and* $\mathrm{Shv}_{\text{ét}}(S)$ *are equivalent, where the functors* $f_* : \mathrm{Shv}_{\text{ét}}(S') \to \mathrm{Shv}_{\text{ét}}(S)$ *and* $f^* : \mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Shv}_{\text{ét}}(S')$ *are inverse equivalences.*
2. *the sites* $\mathrm{Sch}_S^{\text{ét}}$ *and* $\mathrm{Sch}_{S'}^{\text{ét}}$ *are equivalent via the functor* $\mathrm{Sch}_S^{\text{ét}} \to \mathrm{Sch}_{S'}^{\text{ét}}, X \mapsto X \times_S S'$*.*

1.8 Locally Constant and Constructible Sheaves

Definition. Suppose A is an arbitrary set. The *constant sheaf on* A is a sheaf in $\mathrm{Shv}_{\text{ét}}(S)$ isomorphic to the sheaf $\underline{A}_S$ given by the sheafification of the pre-sheaf $(\mathrm{Sch}_S^{\text{ét}})^{op} \to \mathrm{Set}, U \mapsto A$. Equivalently, the constant sheaf on A is the sheaf represented by $\sqcup_{a \in A} S$.

Remark 1.33. Notice that if $f : S' \to S$ is an arbitrary map of schemes, then we have a natural isomorphism

$$f^* \underline{A}_S \cong \underline{A}_{S'}.$$

For this reason, it is usual to write $\underline{A}$ for the sheaf $\underline{A}_S$.

Remark 1.34. If S is connected, there is a natural bijection

$$\underline{A}_S(S) \cong \mathrm{Hom}_S\left(S, \bigcup_{A \in S} S\right) \cong A,$$

natural in both S and A.

Remark 1.35. The constant sheaves functor $\mathrm{Set} \to \mathrm{Shv}_{\text{ét}}(S), A \mapsto \underline{A}_S$ is left-adjoint to the global sections functor $\mathcal{F} \mapsto \mathcal{F}(S)$. Indeed, if we have a morphism $\underline{A}_S \to \mathcal{F}$ then we automatically have a map $A = \underline{A}_S(S) \to \mathcal{F}(S)$. On the other hand, suppose we are only given a map $s : A \to \mathcal{F}(S)$. Then, we have a map $A \to \mathcal{F}(U)$ for all étale $U \to S$ by simply composing s with the map $\mathcal{F}(U \to S) : \mathcal{F}(S) \to \mathcal{F}(U)$. We thus have a pre-sheaf morphism which uniquely induces a canonical sheaf morphism $\underline{A}_S \to \mathcal{F}$.

In particular, we see that the constant sheaves functor is right exact while the global sections functor is left exact.

Definition. A sheaf $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(S)$ is *locally constant* if there exists an étale cover $\{S_i \to S\}_{i \in I}$ such that each $\mathcal{F}|_{S_i}$ is a constant sheaf. If in addition each of the sets $\mathcal{F}(S_i)$ (the set corresponding to $\mathcal{F}|_{S_i}$, i.e. $\mathcal{F}|_{S_i} = \underline{\mathcal{F}(S_i)}_{S_i}$) is finite, then $\mathcal{F}$ is said to be *locally constant constructible* (abbreviated *lcc*). Let the (full sub-)category of lcc sheaves be denoted by $\mathrm{Shv}_{\text{ét}}^{lcc}(S)$.

Theorem 1.36 (Classification of LCC Sheaves)**.** *The functor* $\underline{(-)}_S : \mathrm{Sch}_S^{\text{ét}} \to \mathrm{Shv}_{\text{ét}}(S), X \mapsto \underline{X}_S$ *restricts to an equivalence of categories between finite étale* S*-schemes and lcc sheaves on* $S_{\text{ét}}$*. In other words, an étale sheaf* $\mathcal{F}$ *on* S *is lcc if and only if there exists a finite étale* S*-scheme* X *such that* $\mathcal{F} \cong \underline{X}_S$*.*

For the proof, see [Con, Theorem 1.1.7.2].

We can extend this theorem slightly in case $\mathcal{F}$ is an abelian étale sheaf (c.f. [Tam94, II.9.2.3] and [Tam94, II.9.3.3]).

Proposition 1.37. *If* $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(S)$ *is LCC, then there exists a unique commutative étale group scheme* $G \in \mathrm{Sch}_S^{\text{ét}}$ *such that* $\mathcal{F} = \underline{G}_S$*. If in addition* $\mathcal{F}$ *has finite stalks, then* $G \to S$ *is finite. The sheaf* $\mathcal{F}$ *is constructible if and only if* $G \to X$ *is finitely presented.*

We now discuss a corollary to the classification of LCC sheaves in the special case of $\mathrm{Spec}\, R$ for R a Dedekind domain.

Definition. Suppose K is the field of fractions of a Dedekind domain R. Let $\mathfrak{p}$ be a non-zero prime of R; then $\mathrm{Gal}(\bar{K}_\mathfrak{p}/K_\mathfrak{p})$ embeds into $\mathrm{Gal}(\bar{K}/K)$. The inertia subgroup $I_\mathfrak{p}$ of $\mathrm{Gal}(\bar{K}/K)$ is the subgroup $\mathrm{Gal}(\bar{K}_\mathfrak{p}/K_\mathfrak{p}^{unr})$ of $\mathrm{Gal}(\bar{K}_\mathfrak{f}/K_\mathfrak{f}) \subset \mathrm{Gal}(\bar{K}/K)$ corresponding to the maximal unramified extension of $K_\mathfrak{p}$.

A Galois action $\mathrm{Gal}(\bar{K}/K)$ on a set S is said to be *unramified at* $\mathfrak{p}$ if the inertia subgroup $I_\mathfrak{p}$ acts trivially on S. In this case, we have an obvious induced action of $\mathrm{Gal}(\bar{K}/K)/I_\mathfrak{p}$ on the set S. Restricting to the decomposition group $G_\mathfrak{p}$ of $\mathfrak{p}$, we then get an action of $G_\mathfrak{p}/I_\mathfrak{p} = \mathrm{Gal}(\bar{K}_\mathfrak{p}/K_\mathfrak{p})$ on S.

Proposition 1.38. *Suppose* K *is the field of fractions of a Dedekind domain* R*, and suppose* $i : \mathrm{Spec}\, K \to \mathrm{Spec}\, R$ *the map corresponding to the inclusion* $R \hookrightarrow K$*. Then, the functor*

$$\mathrm{Shv}_{\text{ét}}^{lcc}(\mathrm{Spec}\, R) \to \mathrm{Shv}_{\text{ét}}^{lcc}(\mathrm{Spec}\, K), \quad \mathcal{F} \mapsto i^* \mathcal{F}$$

is a fully faithful functor (i.e. is a bijection on hom-sets) with essential image equal to the category of those finite discrete $\mathrm{Gal}(\bar{K}/K)$*-sets that are unramified at all closed points of* $\mathrm{Spec}\, R$*.*

In particular, if $\mathcal{G}$ *is an lcc sheaf on* $\mathrm{Sch}_{\mathrm{Spec}\, K}^{\text{ét}}$*, then its associated continuous representation of* $\mathrm{Gal}(\bar{K}/K)$ *on a finite set is unramified at all places of* R *if and only if* $\mathcal{G}$ *extends to an lcc sheaf over* $\mathrm{Sch}_{\mathrm{Spec}\, R}^{\text{ét}}$*.*

Proof. That the functor is fully faithful is clear; it suffices to compute its essential image. By the classification theorem, an étale lcc sheaf on $\operatorname{Spec} R$ is of the form $\underline{X}$ for some finite étale $\operatorname{Spec} R$-scheme X. Then, the pull-back along i is of the form $\underline{X}_K$ where $X_K = X \times_{\operatorname{Spec} R} \operatorname{Spec} K$ is the base change to K; it is obvious that X_K is a finite étale $\operatorname{Spec} K$-scheme. The Galois action corresponding to a finite étale K-scheme X_K is precisely the action of $G_K := \operatorname{Gal}(\bar{K}/K)$ on $X_K(\operatorname{Spec} \bar{K}) = \operatorname{Hom}_{\operatorname{Spec} K}(\operatorname{Spec} \bar{K}, X_K)$. Hence, we need to show that given a finite étale K-scheme X_K the natural action of G_K on $X_K(\operatorname{Spec} \bar{K})$ is unramified at all places of R if and only if we can find a finite étale R-scheme X (functorial in X_K) such that $X_K = X \times_{\operatorname{Spec} R} \operatorname{Spec} K$.

Now, X_K is a finite étale K-scheme if and only if $X_K \cong \bigsqcup_{j=1}^{r} \operatorname{Spec} K_j$ for some finite separable extensions K_j/K. Then, we have

$$X_K(\operatorname{Spec} \bar{K}) = \bigsqcup_{j=1}^{r} \operatorname{Hom}_{\operatorname{Spec} K}(\operatorname{Spec} \bar{K}, \operatorname{Spec} K_j) = \bigsqcup_{j=1}^{r} \operatorname{Hom}_K(K_j, \bar{K}).$$

Notice that G_K is unramified on $X_K(\operatorname{Spec} \bar{K})$ if and only if the restriction of G_K is unramified on each $\operatorname{Hom}_K(K_j, \bar{K})$. Thus, it suffices to consider a single finite separable extension L/K. In this case, we claim that we can choose X to be $\operatorname{Spec} R_L$ where R_L is the integral closure of R in L. It thus suffices to show that $\operatorname{Spec} R_L \to \operatorname{Spec} R$ is étale if and only if the G_K action on $\operatorname{Hom}_K(L, \bar{K})$ is unramified everywhere. However, since R_L is the integral closure of R, we know that R_L is flat (in fact, free) over R. On the other hand, the kernel of the action map of G_K on $\operatorname{Hom}_K(L, \bar{K})$ is precisely the Galois group of the Galois closure of L over K. Hence, we need to show that $\operatorname{Spec} R_L \to \operatorname{Spec} R$ is unramified if and only if the Galois closure of L is contained in the maximal unramified extension of K. It is easily seen that both of these statements are equivalent to asking that L/K be an everywhere unramified extension (i.e. every non-zero prime ideal of R is unramified in R_L). □

We finish this section with some statements about constructible sheaves. We do not prove anything here since it is not clear to me that this is actually useful just yet; for proofs, see [Con].

Definition. Suppose S is a Noetherian topological space. A *stratification* of S is a finite set $\{S_i\}$ of pair-wise disjoint non-empty subsets S_i that are locally closed in S, satisfy $\bigcup_i S_i = S$, and the closure of each S_i is a union of S_js. The S_is are the *strata* of the stratification.

If S is a Noetherian scheme, then a sheaf $\mathcal{F} \in \operatorname{Shv}_{\text{ét}}(S)$ is *constructible* if there exists a stratification $\{S_i\}$ of the underlying Zariski topological space of S such that the restriction $\mathcal{F}|_{S_i}$ to each stratum is lcc.

Proposition 1.39 (Local Nature of Constructibility)**.** *Suppose S is a Noetherian scheme with étale cover $\{U_i\}$. If $\mathcal{F} \in \operatorname{Shv}_{\text{ét}}(S)$ is a sheaf such that $\mathcal{F}|_{U_i}$ is a constructible sheaf on U_i for all i, then $\mathcal{F}$ is constructible.*

Constructible sheaves are important due to the following result, which allows one to generalize statements about constructible sheaves to statements about arbitrary sheaves.

Proposition 1.40. *Suppose S is a Noetherian scheme.*

1. *Every sheaf in $\operatorname{Shv}_{\text{ét}}(S)$ is the filtered direct limit of its constructible subsheaves, and subsheaves of constructible sheaves are constructible.*

2. *A sheaf in $\operatorname{AbShv}_{\text{ét}}(S)$ such that each of its sections is locally killed by a non-zero integer is the filtered direct limit of its constructible abelian subsheaves.*

3. *A sheaf in $\operatorname{AbShv}_{\text{ét}}(S)$ is Noetherian (i.e. its subobjects satisfy the ascending chain condition) if and only if it is constructible, provided that it is torsion.*

We now give some standard examples of constructible sheaves.

Example 1.41. If $X \to S$ is a quasi-compact étale map to a Noetherian scheme S, then $\underline{X}_S$ is constructible.

Example 1.42. Using Noetherian induction, one can show that the pull-back, image under a map, and finite limits of constructible sheaves are constructible.

Also, if $j : U \to S$ is a quasi-compact étale map to a Noetherian scheme S, then $j_!$ sends constructible sheaves to constructible sheaves.

1.9 Stalks

For a k-point $x \in S(\operatorname{Spec} k)$ (i.e. $x : \operatorname{Spec} k \to S$ is a map) and an étale sheaf $\mathcal{F} \in \operatorname{Shv}_{\text{ét}}(S)$, denote by $\mathcal{F}_x$ the pull-back $x^*\mathcal{F} \in \operatorname{Shv}_{\text{ét}}(\operatorname{Spec} k)$.

Definition. A *geometric point of* $\mathrm{Sch}_S^{\text{ét}}$ *(or of S)* is a map $\bar{s} : \operatorname{Spec} k \to S$ (i.e. a k-point of S) for a separably closed field k. In this case, the category $\mathrm{Shv}_{\text{ét}}(\operatorname{Spec} k)$ is naturally equivalent to Set, and so we may consider $\mathcal{F}_{\bar{s}}$ to be a set. A geometric point thus induces an exact functor $\mathcal{F} \mapsto \mathcal{F}_{\bar{s}}$ from $\mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Set}$, called the *fibre functor* or *stalk functor* at $\bar{s}$. Since pull-backs have a right-adjoint (namely the push-forward), the stalk functor commutes with arbitrary colimits.

Two geometric points $\bar{s}$ and $\bar{s}'$ of S are equivalent when their physical image points in S are the same, i.e. their associated fibre functors are (non-canonically) isomorphic to each other.

An *étale neighbourhood* of a geometric point $\bar{s} \in S(\operatorname{Spec} k)$ is an étale map $U \to S$ equipped with a map $u : \operatorname{Spec} k \to U$ whose composite with $U \to S$ is $\bar{s}$ (i.e. $\bar{s}$ factors through $U \to S$).

Lemma 1.43. *Let $\mathcal{F}$ be an arbitrary pre-sheaf on S. Suppose $\bar{s} : \operatorname{Spec} k \to S$ (k separably closed) is a geometric point whose image lies in the underlying set of S (i.e. is an honest point of S), then the natural map*

$$\left(\operatorname*{colim}_{(U,u) \text{ étale nbd of } \bar{s}} \mathcal{F}(U) \right) \to (\mathcal{F}^{sh})_{\bar{s}} = \left(\bar{s}^* \mathcal{F}^{sh} \right)(\operatorname{Spec} k)$$

is a bijection, the colimit (in sets) taken over the category of étale neighbourhoods of $\bar{s}$.

Proof. Note first that, due to the sheafification-forgetful adjunction, if $\mathcal{G}$ is a pre-sheaf on $\operatorname{Spec} k$ with k separably closed, then $(\mathcal{G})^{sh}(\operatorname{Spec} k) = \mathcal{G}(\operatorname{Spec} k)$. In particular, $(\bar{s}^*\mathcal{F})(\operatorname{Spec} k) = \bar{s}^{-1}\mathcal{F}(\operatorname{Spec} k)$ where now $\bar{s}^{-1}\mathcal{F}$ is only a pre-sheaf. The lemma then follows immediately from the construction of $\bar{s}^{-1}\mathcal{F}$. ☐

Corollary 1.43.1. *Keep notation as above. Then, two elements $a, b \in \mathcal{F}(S)$ are equal in some étale neighbourhood of $\bar{s}$ if and only if their images in $\mathcal{F}_{\bar{s}}$ coincide.*

While the lemma and its corollary are trivial, they immediately yield the following result just as in ordinary sheaf theory.

Proposition 1.44. *Suppose Σ is set of geometric points whose images cover S set-theoretically.*

- *A map $\varphi : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Shv}_{\text{ét}}(S)$ is epic (resp. monic, resp. an isomorphism) if and only if the pull-back $\varphi_s : \mathcal{F}_s \to \mathcal{G}_s$ is surjective (resp. injective, resp. bijective) for all $s \in \Sigma$.*
- *Two maps $\varphi, \psi : \mathcal{F} \to \mathcal{F}$ in $\mathrm{Shv}_{\text{ét}}(S)$ are equal if and only if their pull-backs φ_s and ψ_s are equal for all $s \in \Sigma$.*

Hence, just as in ordinary sheaf theory, one may work with étale sheaves stalk-locally. See [Tam94, Chapter 2, Theorem (5.6)] for a proof.

Proposition 1.45. *Fix a geometric point $\bar{s} : \operatorname{Spec} \bar{k} \to S$.*

1. *The functor $\mathcal{F} \mapsto \mathcal{F}_{\bar{s}}$ is exact and commutes with arbitrary colimits.*
2. *If $v : \operatorname{Spec} \bar{k} \to \operatorname{Spec} \bar{k}$ is an S-morphism of geometric points $\bar{s} : \operatorname{Spec} \bar{k} \to S, \bar{s}' : \operatorname{Spec} \bar{k} \to S$, then $\mathcal{F}_{\bar{s}} \cong \mathcal{F}_{\bar{s}'}$.*
3. *If $f : S \to S'$, then $\bar{s}$ is naturally a geometric point of S', and moreover $(f^*\mathcal{F})_{\bar{s}} \cong \mathcal{F}_{f \circ \bar{s}}$ for all sheaves $\mathcal{F}$ on $\mathrm{Sch}_{S'}^{\text{ét}}$.*

See [Tam94, Chapter 2, Proposition (5.2)] for a proof.

1.10 Henselization

We follow [Mil80] and [Tam94] in this section, though the definition is taken from [FK88]. For proofs see [Mil80].

Recall that a ring homomorphism $A \to B$ is *finite* if B is finitely generated as an A-module. A local ring homomorphism $A \to B$ is quasi-finite if $A/\mathfrak{m}_A \to B/(\mathfrak{m}_A B)$ is finite.

Definition. A local ring A is *Henselian* if every local quasi-finite homomorphism from A to a localization of a finitely generated A-algebra is finite.

A Henselian ring is *strictly Henselian* if the residue class field $A/\mathfrak{m}$ is separably algebraically closed.

This definition is slightly hard to work with. The following characterizations are far more useful.

Definition. We say that a local ring $(A, \mathfrak{m})$ *satisfies Hensel's Lemma* if, for every monic polynomial $P \in A[x]$ such that upon reducing modulo $\mathfrak{m}$ we have $P(x) \equiv \bar{Q}(x)\bar{R}(x)$ for some relatively prime monic polynomials $\bar{Q}, \bar{R} \in (A/\mathfrak{m})[x]$, there exist monic polynomials $Q, R \in A[x]$ such that $P = QR$ in $A[x]$ and modulo $\mathfrak{m}$ we have $Q \equiv \bar{Q}$ and $R \equiv \bar{R}$.

Proposition 1.46. *For a local ring $(A, \mathfrak{m})$, the following are equivalent.*

- *A is Henselian.*
- *A satisfies Hensel's lemma.*
- *Any finite A-algebra is a product of local rings (in such a case, a finite A-algebra B is of the form $B = \prod_{\mathfrak{m}_i} B_{\mathfrak{m}_i}$ where the product ranges over all maximal ideals $\mathfrak{m}_i$ of B.)*
- *If $f : X \to \operatorname{Spec} A$ is quasi-finite and separated, then $X \cong \bigsqcup_{i=0}^n X_i$ where $\mathfrak{m} \notin f(Y_0)$, $f|_{X_i} : X_i \to \operatorname{Spec} A$ is finite, and X_i is the spec of a local ring.*
- *If $f : X \to \operatorname{Spec} A$ is étale and there is a point $x \in X$ such that $f(x) = \mathfrak{m}$ and $\kappa(x) = \kappa(\mathfrak{m}) \cong A/\mathfrak{m}$, then f has a section $s : \operatorname{Spec} A \to X$.*
- *Suppose $f_1, ..., f_r \in A[x_1, ..., x_r]$ are such that there exists an $a = (a_1, ..., a_r) \in (A/\mathfrak{m})^r$ such that $\bar{f}_i(a) = 0$ for all $1 \leq i \leq r$, and $\det\left(\frac{\partial f_i}{\partial x_j}(a)\right) \neq 0$, then there exists $b \in A^n$ such that $\bar{b} = a$ and $f_i(b) = 0$ for $1 \leq i \leq r$.*

Corollary 1.46.1. *Any (topologically-)complete local ring is Henselian.*

Corollary 1.46.2. *If A is Henselian then so is any finite local A-algebra B and any quotient A/I of A.*

Corollary 1.46.3. *If $f : Y \to X$ is an étale map of schemes such that $\kappa(f(y)) = \kappa(y)$ for some $y \in Y$, then the map $\hat{\mathcal{O}}_{X,f(y)} \to \hat{\mathcal{O}}_{Y,y}$ between the completions of the local rings is an isomorphism.*

Proposition 1.47. *If A is Henselian, then the functor $B \mapsto B \otimes_A (A/\mathfrak{m})$ induces an equivalence between the category of finite étale A algebras B and the category of finite étale $(A/\mathfrak{m})$-algebras.*

Proposition 1.48. *Let A be a local ring. The following are equivalent.*

1. *A is strictly-Henselian.*
2. *A is Henselian and all finite étale A-algebras are trivial, i.e. isomorphic to A^n for some n.*
3. *If $A \to B$ is a local homomorphism from A to the localization B of an étale A-algebra, then $A \to B$ is an isomorphism.*
4. *If $X \to S = \operatorname{Spec} A$ is étale and if x is a point in the closed fibre $X_{\mathfrak{m}}$, there exists a section $u : \operatorname{Spec} A \to X$ such that $u(\mathfrak{m}) = x$.*

Corollary 1.48.1. *A strictly Henselian ring has no finite étale extensions apart from itself.*

The following generalizes the previous proposition.

Proposition 1.49. *Suppose X is a proper scheme over a Henselian local ring A, and let X_0 be the closed fibre of X. The functor $Y \mapsto Y \times_X X_0$ induces an equivalence between the category of schemes Y finite and étale over X, and the category of schemes finite and étale over X_0.*

Definition. The *Henselization* A^h of a local ring A is a Henselian (Noetherian) local ring A^h (with maximal ideal $\mathfrak{m}^h$ such that $A^h/\mathfrak{m}^h \cong A/\mathfrak{m}$) equipped with an inclusion $A \hookrightarrow A^h$ such that any map $A \to B$ for B Henselian factors through $A \hookrightarrow A^h$.

We can similarly define the *strict-Henselization* A^{sh} of A.

Remark 1.50. The Henselization of a local ring always exists, and can be thought of as the intersection of all Henselian local subrings B in $\hat{A}$ (the $\mathfrak{m}_A$-adic completion of A) such that $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.

Remark 1.51. Notice that the inclusion $A \hookrightarrow A^{sh}$ into the strict-henselization factors as $A \hookrightarrow A^h \hookrightarrow A^{sh}$.

Example 1.52. Suppose $(A, \mathfrak{m})$ is a normal Noetherian local ring with field of fractions K. Let $\bar{K}$ be separable closure of K, and let B be the integral closure of A in $\bar{K}$. Let $\mathfrak{n}$ be some maximal ideal of B containing $\mathfrak{m}B$. The decomposition group $D = D_{\mathfrak{n}}$ is

$$D = \{\sigma \in \operatorname{Gal}(\bar{K}/K) : \sigma\mathfrak{n} = \mathfrak{n}\}.$$

The ring B^D of elements of B fixed by D is the integral closure of A in the fixed field $\bar{K}^D$, and $\mathfrak{n}^D$ is a prime ideal in B^D. Since $A \subset K$, we have $A \subset B^D$. It can be checked (c.f. [Mil80]) that the localization $(B^D)_{\mathfrak{n}^D}$ is precisely the Henselization of A.

If instead we use the inertia subgroup

$$I = I_{\mathfrak{n}} = \{\sigma \in \operatorname{Gal}(\bar{K}/K) : \sigma x = x \quad \forall x \in \mathfrak{n}\}$$

of $\bar{K}/K$, rather than the decomposition group D, then $(B^I)_{\mathfrak{n}^I}$ is the strict Henselization A^{sh} of A.

Example 1.53. If k is a field and A the localization of $k[x_1, ..., x_n]$ at the ideal $\langle x_1, ..., x_n \rangle$, then A^h is the integral closure of A in $k[[x_1, ..., x_n]]$, i.e. it is the ring of formal power series $P \in k[[x_1, ..., x_n]]$ such that P is algebraic over A.

Example 1.54. Both Henselization and strict-Henselization commutes with quotients, i.e. $(A/\mathfrak{m})^h \cong A^h/(\mathfrak{m}A^h)$ and $(A/\mathfrak{m})^{sh} \cong A^{sh}/(\mathfrak{m}A^{sh})$

Proposition 1.55. *The henselization A^h as well as the strict-henselization A^{sh} of a local ring A are flat over A.*

2 Cohomology

2.1 Constructing the Étale Fundamental Group

Throughout, we let S be an arbitrary connected scheme and $\bar{s}$ a geometric point of S.

Lemma 2.1 (Rigidity of Pointed Étale Covers)**.** *Suppose S' is a connected étale S-scheme, and choose a geometric point $\bar{s}'$ of S' above $\bar{s}$ where WLOG $\bar{s}'$ and $\bar{s}$ have the same residue field k. Suppose S'' is a separated étale S-scheme. If $f, g : S' \to S''$ are two S-maps such that $f(\bar{s}') = g(\bar{s}')$ in $S''(k)$, then $f = g$.*

Remark 2.2. When $S'' = S'$, this is analogous to the statement that deck transformations of a connected covering are uniquely determined by the image of a single point.

Proof. Since S'' is separated, the diagonal map $S'' \to S'' \times_S S''$ is a closed immersion. However, S'' is étale over S and so $S'' \to S'' \times_S S''$ is an étale map, and in particular, is an open map. Thus the diagonal $S'' \hookrightarrow S'' \times_S S''$ is a connected component of $S'' \times_S S''$. Since S' is connected and the image of S' under $(f, g) : S' \to S'' \times_S S''$ intersects the diagonal at the geometric point $\bar{s}'$, the image of S' under (f, g) must be contained in the diagonal. Hence $f = g$. □

Suppose S' is a connected finite étale S-scheme. Since S and S' are both connected, the degree of the map $S' \to S$ is constant and so it makes sense to talk about the *degree* of S' over S. If $\deg(S'/S) = n$, then there are exactly n geometric points in the fibre over $\bar{s}$. The rigidity lemma then implies that $\#\mathrm{Aut}(S'/S) \leq n$ where $\mathrm{Aut}(S'/S)$ is the set of S-isomorphisms from $S' \to S'$; moreover, the S-automorphism of S' is uniquely determined by where it sends a fixed geometric point of S'.

Definition. A finite étale map $S' \to S$ between connected non-empty schemes is *Galois* or a (finite) *Galois covering* if $\#\mathrm{Aut}(S'/S) = \deg(S'/S)$. In other words, S' is Galois over S if and only if for every geometric point $\bar{s}'$ of S' lying above a geometric point $\bar{s}$ of S, there exists an automorphism $f \in \mathrm{Aut}(S'/S)$ such that $f(\bar{s}) = \bar{s}'$.

In this case, the *Galois group* of S' is defined to be the opposite group $\mathrm{Gal}(S'/S) := \mathrm{Aut}(S'/S)^{op}$. The Galois group thus acts simply transitively on geometric fibres.

Example 2.3. Suppose R is a Dedekind domain with fraction field K, and suppose K' is a finite separable extension of K with R' the integral closure of R in K'. Then, one can check that $\operatorname{Spec} R' \to \operatorname{Spec} R$ is a finite Galois covering if and only if K'/K is a Galois extension of fields *and* R' is everywhere unramified over R. Notice that R' being everywhere unramified over R is required for the map $\operatorname{Spec} R' \to \operatorname{Spec} R$ to be an étale map.

Since $\mathrm{Aut}(S'/S)$ acts on S' on the left, it follows that $\mathrm{Gal}(S'/S)$ acts on S' from the right.

Lemma 2.4. *Suppose $S' \to S$ is a finite Galois covering with Galois group G, then the action map $S' \times G \to S' \times_S S'$ given by $(s', g) \mapsto (s', s'g)$ is an isomorphism of finite étale S'-schemes.*

Conversely, if there is a group G acting on S' over S on the right such that the action map $S' \times G \to S' \times_S S'$ given by $(s', g) \mapsto (s', s'g)$ is an isomorphism of finite étale S'-schemes, then $S' \to S$ is a finite Galois covering with Galois group G.

Remark 2.5. This is a generalization of the usual isomorphism

$$L \otimes_K L \cong \prod_{g \in \mathrm{Gal}(L/K)} L, \quad x \otimes y \mapsto (x \cdot g(y))_{g \in \mathrm{Gal}(L/K)}$$

for L/K a finite Galois extension from Galois theory.

Proof. We use the structure map $\mathrm{proj}_1 : S' \times_S S' \to S'$ to be the structure map of $S' \times_S S'$ over S'. Let $|G| = n$. Since $S' \to S$ is a finite étale map of degree n, base changing implies that $S' \times_S S' \to S'$ is also a finite étale map of degree n. Notice that $S' \times G$ is the finite disjoint union of S's indexed by G, and so $S' \times G$ is also a finite étale S'-scheme of degree n over S'. In particular, it follows that the action map $S' \times G \to S' \times_S S'$, being a scheme-map of finite étale S'-schemes, is finite étale as well. The rigidity lemma implies that the fibre over a geometric point of $S' \times_S S'$ is a singleton, and so the action map is a finite étale map of degree 1. Hence the action map must be an isomorphism.

For the converse, note that G is a subgroup of $\mathrm{Gal}(S'/S)$, and the requirement that $S' \times G \to S' \times_S S'$ being an isomorphism implies that, looking at cardinalities of fibres, we have $|\mathrm{Gal}(S'/S)| \geq |G| = \deg(S'/S)$. However $\deg(S'/S) \geq |\mathrm{Gal}(S'/S)|$, and hence S' is Galois over S. □

Lemma 2.6. *Suppose S', S'' are connected finite Galois covers of S, and suppose $\pi : S'' \to S'$ is an S-map (such a map is unique by rigidity, noting that finite schemes are separated). Then, π induces a surjective map*

$$\pi^{\#} : \mathrm{Aut}(S''/S) \twoheadrightarrow \mathrm{Aut}(S'/S)$$

which for any $f'' \in \mathrm{Aut}(S''/S)$ satisfies

$$\pi \circ f'' = (\pi^{\#} f'') \circ \pi.$$

Proof. Fix an arbitrary geometric point $\bar{s}''$ of S'' lying above $\bar{s}$. Then π sends $\bar{s}''$ to some geometric point $\bar{s}'$ of S' lying above $\bar{s}$. We need to show that for any $f'' \in \mathrm{Aut}(S''/S)$ we can find a unique $f' \in \mathrm{Aut}(S'/S)$ such that

$$\begin{array}{ccc} S'' & \xrightarrow{f''} & S'' \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ S' & \xrightarrow{f'} & S' \end{array} .$$

Indeed, notice that $\pi \circ f''(\bar{s}'') \in S'_{\bar{s}'}$. Since S' is Galois over S, we can find a unique $f' \in \mathrm{Aut}(S'/S)$ such that $f'(\bar{s}') = \pi \circ f''(\bar{s}'') \in S'_{\bar{s}'}$. Thus $(f' \circ \pi)(\bar{s}'') = (\pi \circ f'')(\bar{s}'')$ where $f' \circ \pi, \pi \circ f''$ are S-maps from S'' to S'. The rigidity lemma then implies that $f' \circ \pi = \pi \circ f''$.

Hence, fixing a map $\pi : S'' \to S'$ in the category of connected finite étale S-schemes, there is an induced injective map

$$\mathrm{Aut}(S''/S) \to \mathrm{Aut}(S'/S).$$

Suppose now that $f' \in \mathrm{Aut}(S'/S)$. Then $f'(\bar{s}')$ is a geometric point of S' over $\bar{s}$. Since π is surjective (as π is finite étale and thus both closed and open, and S' is connected), we can lift $\bar{s}'$ to a geometric point of S'', and the same argument as above gives a (not necessarily unique) automorphism $f'' \in \mathrm{Aut}(S''/S)$. $\square$

Hence, for any map $\pi : S'' \to S'$ of connected finite Galois covers of S, we have a surjective map

$$\mathrm{Gal}(S''/S) \twoheadrightarrow \mathrm{Gal}(S'/S).$$

Moreover, the proof shows that this surjective map is canonical when we work with pointed connected finite Galois covers of S, i.e. we fix a geometric point of S and require compatibility with lifts of this geometric point.

Definition. The *étale fundamental group* of S based at $\bar{s}$ is the profinite group

$$\pi_1^{\text{ét}}(S, \bar{s}) := \lim_{(S', \bar{s}')} \mathrm{Gal}(S'/S)$$

where the limit is taken over all connected finite Galois covers $S' \to S$ with a fixed geometric point $\bar{s}'$ over $\bar{s}$.

When no confusion is to be caused, we drop the 'ét' in the superscript and simply write π_1 for the étale fundamental group.

The surjectivity of the maps $\mathrm{Gal}(S''/S) \to \mathrm{Gal}(S'/S)$ implies that $\pi_1(S, \bar{s}) \to \mathrm{Gal}(S'/S)$ is surjective for all pointed connected finite Galois covers $(S', \bar{s}') \to (S, \bar{s})$.

There is an equivalent way to think about the étale fundamental group (see [Mil80, Chapter 1, Section 5]). As before suppose S is a connected scheme and $\bar{s} : \mathrm{Spec}\, k \to S$ ($k = \bar{k}$) a geometric point of S. Consider the category $\mathrm{Sch}_S^{\text{ét,fin}}$ of finite étale S-schemes (these are necessarily surjective for S connected, since étale maps are open whereas finite maps are closed). Consider the functor $F : \mathrm{Sch}_S^{\text{ét,fin}} \to \mathrm{Sets}$ given by $F = \mathrm{Hom}_S(\bar{s}, -)$. In other words, F is the functor of geometric points of a scheme. This is a covariant functor.

It is a fact that this functor is *strictly pro-representable*, i.e. there exists a filtered diagram $(X_i, \phi_{ij})_{i \in I}$ in $\mathrm{Sch}_S^{\text{ét,fin}}$ in which the transition maps $\phi_{ij} : X_j \to X_i$ (for $i \to j$) are epimorphisms, and there exist elements $\bar{x}_i \in F(X_i) = \mathrm{Hom}_S(\mathrm{Spec}\, k, X_i)$ such that $\bar{x}_i = \phi_{ij} \circ \bar{x}_j$ and such that the natural transformation

$$\alpha : \mathrm{colim}_{i \in I} \mathrm{Hom}_S(X_i, -) \Rightarrow F, \quad \alpha_Y : \mathrm{colim}_{i \in I} \mathrm{Hom}_S(X_i, Y) \to F(Y) = \mathrm{Hom}_S(\mathrm{Spec}\, k, Y), \quad \alpha_Y(g : X_i \to Y) = g \circ f_i.$$

induced by the f_i is a natural isomorphism. This system of X_i and transition maps ϕ_{ij} plays the role of a 'universal covering space' for S. One sees that

$$\pi^{\text{ét}}(S, \bar{s}) = \lim_{i \in I} \mathrm{Gal}(X_i/S).$$

2.2 Some Properties of $\pi_1^{\text{ét}}$

Proposition 2.7. *The étale fundamental group $\pi_1^{\text{ét}}$ defines a covariant functor from the category of pointed connected schemes to the category of pro-finite groups.*

Proof. Suppose $f : (T, \bar{t}) \to (S, \bar{s})$ is a morphism of connected pointed schemes. For any pointed connected Galois cover $(S', \bar{s}') \to (S, \bar{s})$ of degree n with $k(\bar{s}') = k(\bar{s})$, the fibre product $S'_T := S' \times_S T$ is a degree n finite étale T-scheme with a canonical point $\bar{t}' = \bar{t} \times_{\bar{s}} \bar{s}'$. Since S'_T is finite étale over T, there are only finitely many connected components each of which is thus both open and closed in T. Suppose T' is the connected component containing $\bar{t}'$. Then $T' \to T$ is a connected finite étale cover.

Now, notice that $\mathrm{Gal}(S'/S)$ acts transitively on geometric fibres of $S' \to S$, and so via pull-back must also act transitively on geometric fibres of $S'_T \to T$. Let $H \subseteq \mathrm{Gal}(S'/S)$ be the stabilizer of T' over T (here $\mathrm{Gal}(S'/S)$ acts on the connected components of S'_T over T via pull-back). By connectedness of T', any $g \in \mathrm{Gal}(S'/S)$ that sends $\bar{t}'$ to a geometric point in T' must send all of T' to T', and so $g \in H$. Transitivity of $\mathrm{Gal}(S'/S)$ on geometric fibres then implies that H itself is transitive on geometric fibres of $T' \to T$. Hence $T' \to T$ is a connected Galois cover with Galois group H. The continuous composite

$$\pi_1(T, \bar{t}) \twoheadrightarrow \mathrm{Gal}(T'/T) \cong H \subset \mathrm{Gal}(S'/S)$$

is compatible with the connecting maps $\mathrm{Gal}(S''/S) \to \mathrm{Gal}(S'/S)$ induced by maps $S'' \to S'$ of connected Galois covers of S. Passing to the limit we get a continuous map

$$\pi_1(f) : \pi_1(T, \bar{t}) \to \pi_1(S, \bar{s}).$$

It is moreover easy to see that π_1 respects composition in f. □

Example 2.8. Suppose $S = \mathrm{Spec}\, k$ for a field k, and pick a separable closure $\bar{s} : \mathrm{Spec}\, \bar{k} \to \mathrm{Spec}\, k$. Then, connected finite Galois covers of k are precisely Specs of finite Galois extensions k'/k. Moreover, via Spec, we have a canonical isomorphism between $\mathrm{Gal}(\mathrm{Spec}\, k'/\mathrm{Spec}\, k) = \mathrm{Aut}(\mathrm{Spec}\, k'/\mathrm{Spec}\, k)^{op}$ and $\mathrm{Gal}(k'/k)$. Since $\mathrm{Gal}(\bar{k}/k)$ is the profinite group defined by taking the limit over all finite Galois extensions k', we then see that

$$\pi_1(S, \bar{s}) \cong \mathrm{Gal}(\bar{k}/k)$$

canonically. Now suppose k' is an arbitrary field extension of k with a choice of separable closure $\bar{k}'$, then we have the following commutative diagram

$$\begin{array}{ccc} \pi_1(\mathrm{Spec}\, k, \bar{s}) & \xleftarrow{\pi_1(\mathrm{Spec}\,(k \to k'))} & \pi_1(\mathrm{Spec}\, k', \bar{s}') \\ \| & & \| \\ \mathrm{Gal}(\bar{k}/k) & \longleftarrow & \mathrm{Gal}(\bar{k}'/k') \end{array}$$

where the bottom row is the usual canonical map coming from Galois theory.

Proposition 2.9 (Connectivity Criterion via $\pi_1^{\text{ét}}$)**.** *Suppose $f : X \to Y$ is a map of connected schemes. Pick a geometric point $\bar{x}$ of X and define $\bar{y} = f(\bar{x})$. The map $\pi_1(f) : \pi_1(X, \bar{x}) \to \pi_1(Y, \bar{y})$ is surjective if and only if $X \times_Y Y'$ is connected for all connected finite étale covers $Y' \to Y$.*

Proof. Notice that the image of $\pi_1(X, \bar{x})$ under $\pi_1(f)$ is closed, since profinite groups are always compact and Hausdorff. Thus, $\pi_1(f)$ is surjective if and only if it is dense, i.e. if and only if for all connected pointed finite étale covers $(Y', \bar{y}')$ of $(Y, \bar{y})$ the composite

$$\pi_1(X, \bar{x}) \to \pi_1(Y, \bar{y}) \twoheadrightarrow \mathrm{Gal}(Y'/Y)$$

is surjective. However, from the proof of the covariant functoriality of π_1, we know that the above composite factors as

$$\pi_1(X, \bar{x}) \twoheadrightarrow \mathrm{Gal}(X'/X) \hookrightarrow \mathrm{Gal}(Y'/y)$$

where X' is the connected component of $X \times_Y Y'$ containing the geometric point $\bar{x}' = (\bar{x}, \bar{y}')$. Hence, we see that $\pi_1(f)$ is surjective if and only if for all connected pointed finite étale covers $(Y', \bar{y}')$ the finite groups $\mathrm{Gal}(X'/X)$ and $\mathrm{Gal}(Y'/Y)$ have the same order. The order of the Galois groups is the degree of the covering, and so $\pi_1(f)$ is surjective if and only if $\deg(X'/X) = \deg(Y'/Y)$. However, notice that X' is an open subscheme of $X \times_Y Y'$ and so

$$\deg(X'/X) \leq \deg(X \times_Y Y'/X) = \deg(Y'/Y)$$

with equality if and only if $X' = X \times_Y Y'$, i.e. $X \times_Y Y'$ is connected. The result follows. □

Example 2.10. Suppose X is an irreducible normal scheme with function field K. Fixing a separable closure $\bar{K}$, we have a geometric point $\bar{x} : \operatorname{Spec} \bar{K} \to \operatorname{Spec} K \hookrightarrow X$. By functoriality of π_1, the inclusion of the generic point $\operatorname{Spec} K \hookrightarrow X$ yields the continuous map

$$\operatorname{Gal}(\bar{K}/K) = \pi_1(\operatorname{Spec} K, \bar{x}) \to \pi_1(X, \bar{x}).$$

It is a fact that $X' \times_{\operatorname{Spec} X} \operatorname{Spec} K$ is connected for all connected finite étale X-schemes X' (see [Con]). Thus the connectivity criterion implies that $\pi_1(f)$ is surjective. This has two consequences:

1. It turns out that the kernel of $\pi_1(f)$ is the Galois group $\operatorname{Gal}(\bar{K}/L)$ where L is the maximal subextension of $\bar{K}/K$ such that the normalization of X in each finite subextension of L is finite étale over X. For instance, if $X = \operatorname{Spec} R$ for R a Dedekind domain, then L/K is the maximal extension that is everywhere unramified over R (unramified in the usual number theoretic sense).

2. If X is irreducible and normal and $U \subset X$ is a Zariski open containing the geometric point $\bar{x}$, then $\pi_1(U, \bar{x}) \to \pi_1(X, \bar{x})$ is surjective.

Theorem 2.11 (Grothendieck)**.** *If S is connected, then there is a canonical equivalence of categories between the category $\operatorname{FSch}_{\text{ét}}(S)$ of finite étale morphisms $S' \to S$ and the category $\pi_1(S, \bar{s})\mathrm{FSet}$ of finite discrete (left) $\pi_1(S, \bar{s})$-sets, given by $S' \mapsto S'(\bar{s})$. Moreover, under this equivalence connected covers correspond to the finite sets with transitive $\pi_1(S, \bar{s})$-action.*

This equivalence is also functorial in $(S, \bar{s})$, i.e. if $f : (T, \bar{t}) \to (S, \bar{s})$ where T is also connected, then the natural equality of sets $(T \times_S S')(\bar{t}) = S'(\bar{s})$ respects the $\pi_1(T, \bar{t})$-actions, where $\pi_1(T, \bar{t})$ acts on $S'(\bar{s})$ through $\pi_1(f)$.

Remark 2.12. This is a vast generalization of the equivalence of the category of finite étale k-algebras and the category of discrete $\operatorname{Gal}(\bar{k}, k)$ sets.

As a corollary of this theorem of Grothendieck and the classification of lcc sheaves, we have the following equivalence.

Corollary 2.12.1. *Suppose S is a connected scheme with a geometric point $\bar{s}$. The fibre-functor $\mathcal{F} \mapsto \mathcal{F}_{\bar{s}} = \bar{s}^*\mathcal{F}$ sets up an equivalence between*

- *the category $\operatorname{Shv}_{\text{ét}}^{lcc}(S)$ of lcc sheaves of sets on $\operatorname{Sch}_S^{\text{ét}}$, and the category $\pi_1(S, \bar{s})\mathrm{FSet}$ of finite discrete left $\pi_1(S, \bar{s})$-sets; and*
- *the abelian category $\operatorname{AbShv}_{\text{ét}}^{lcc}(S)$ of lcc sheaves of abelian groups on $\operatorname{Sch}_S^{\text{ét}}$, and the category $\pi_1(S, \bar{s})\mathrm{FMod}$ of finite discrete left $\pi_1(S, \bar{s})$-modules.*

This equivalence is moreover functorial in $(S, \bar{s})$.

Example 2.13. Let X be a connected scheme. Let K be some subextension of $\bar{k}$ over k, and suppose $x : \operatorname{Spec} K \to X$ be a K-point. Pick also a geometric point $\bar{x} : \operatorname{Spec} \bar{k} \to X$ over x. If $\mathcal{F} \in \operatorname{Shv}_{\text{ét}}(X)$, then we have $\mathcal{F}_x := x^*\mathcal{F} \in \operatorname{Shv}_{\text{ét}}(\operatorname{Spec} K)$. The choice of $\bar{x}$ gives us a canonical equivalence of categories between $\operatorname{Shv}_{\text{ét}}(\operatorname{Spec} K)$ and $\operatorname{Gal}(\bar{k}/K)$ sets, by sending $\mathcal{F}_x$ to $\mathcal{F}_x(\bar{x})$.

However, if $\mathcal{F}$ is moreover lcc, then the pointed map $(x, \bar{x}) \to (X, \bar{x})$ induces via functoriality of π_1 a continuous map of profinite groups $\operatorname{Gal}(\bar{k}/K) = \pi_1^{\text{ét}}(\operatorname{Spec} K, \bar{x}) \to \pi_1(X, \bar{x})$. Thus, the finite discrete $\operatorname{Gal}(\bar{k}/K)$-set $\mathcal{F}_x(\bar{x})$ is canonically constructed from the finite discrete $\pi_1(X, \bar{x})$-set $\mathcal{F}(\bar{x})$ via $\operatorname{Gal}(\bar{k}/K) \to \pi_1(X, \bar{x})$. We recover the equivalence of categories between $\operatorname{Shv}_{\text{ét}}(\operatorname{Spec} K)$ and discrete $\operatorname{Gal}(\bar{k}/K)$ sets

Proposition 2.14. *Suppose X is a connected smooth algebraic curve over a perfect field k, and let K be the field of rational functions on X. Fix a separable closure $\bar{K}$ of K and write $\bar{x} : \operatorname{Spec} \bar{K} \to \operatorname{Spec} K$ corresponding to the inclusion $K \subset \bar{K}$. Then, $\pi_1^{\text{ét}}(X, \bar{x})$ is the Galois group of the maximal unramified subextension of $\bar{K}/K$.*

See [Tam94, p. II.9.2.4] for the proof.

2.3 Étale Cohomology Groups

Recall that the category $\operatorname{AbShv}_{\text{ét}}(S)$ is an abelian category, and the global-sections functor $\operatorname{AbShv}_{\text{ét}}(S) \to \mathrm{Ab}, \mathcal{F} \mapsto \mathcal{F}(S)$ is left-exact.

Lemma 2.15. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are abelian categories, and $F : \mathcal{C} \to \mathcal{D}$ is a functor with an exact left-adjoint $G : \mathcal{D} \to \mathcal{C}$. Then, F preserves injectives.*

Proof. Let I be an injective in $\mathcal{C}$. Notice that FI is an injective if and only if $\mathrm{Hom}_{\mathcal{D}}(-, FI)$ is an exact functor. However, by the (G, F)-adjunction, $\mathrm{Hom}_{\mathcal{D}}(-, FI)$ is an exact functor if and only if $\mathrm{Hom}_{\mathcal{C}}(G(-), I)$ is an exact functor. Since G is assumed to be exact and I is an injective, it follows that $\mathrm{Hom}_{\mathcal{C}}(G(-), I)$ is indeed exact. Therefore FI is an injective. □

Lemma 2.16. *The abelian category* $\mathrm{AbShv}_{\text{ét}}(S)$ *has enough injectives.*

Proof. We follow the proof in [Mil80].For any field k with $k = \bar{k}$, the category $\mathrm{AbShv}_{\text{ét}}(\operatorname{Spec} k)$ is *equivalent* to the category Ab via the global-sections functor; in particular, $\mathrm{AbShv}_{\text{ét}}(\operatorname{Spec} k)$ has enough injectives for all k with $k = \bar{k}$. Now, consider an arbitrary $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(S)$. Then, for all geometric points $\bar{x} : \operatorname{Spec} k \to S$ of S, we have $\bar{x}^*\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(\operatorname{Spec} k)$, and so there exists an injective $\mathcal{I}_{\bar{x}} \in \mathrm{AbShv}_{\text{ét}}(\operatorname{Spec} k)$ and a monomorphism $\bar{x}^*\mathcal{F} \hookrightarrow \mathcal{I}_{\bar{x}}$. Since $\bar{x}^*$ is an exact functor and is left-adjoint to $\bar{x}_*$, it follows that $\bar{x}_*\mathcal{I}_{\bar{x}}$ is an injective in $\mathrm{AbShv}_{\text{ét}}(S)$.

Define the sheaves

$$\mathcal{F}^* := \prod_{x \in S} (\bar{x})_* (\bar{x}^*\mathcal{F}) \quad \text{and} \quad \mathcal{I} := \prod_{x \in S} (\bar{x})_* \mathcal{I}_{\bar{x}}$$

where for each $x \in S$ we pick a geometric point $\bar{x} : \operatorname{Spec} \overline{\kappa(x)} \to \operatorname{Spec} \kappa(x) \xrightarrow{x} S$. Since $\bar{x}^*\mathcal{F} \hookrightarrow \mathcal{I}_{\bar{x}}$ is an injection, and since $\S_*$ is left-exact, we have that $\mathcal{F}^* \hookrightarrow \mathcal{I}$. Also, by looking at stalks, we see that $\mathcal{F} \hookrightarrow \mathcal{F}^*$. Hence, we have found a monomorphism $\mathcal{F} \hookrightarrow \mathcal{I}$. By the previous argument, we know that all the $\bar{x}_*\mathcal{I}_{\bar{x}}$ are injectives. Since the product of injectives is an injective, it then follows that $\mathcal{I}$ is an injective. □

Recall that push-forwards are left-exact as well.

Definition. *Étale cohomology* $H^\bullet_{\text{ét}}(S, \cdot)$ on $\mathrm{AbShv}_{\text{ét}}(S)$ is the (classical) right derived δ-functor of the left-exact global-sections functor $\mathcal{F} \mapsto \mathcal{F}(S)$.

Given a morphism $f : X \to S$ of schemes, the *higher direct images* $R^\bullet f_*$ is the (classical) right derived δ-functor of $f_* : \mathrm{AbShv}_{\text{ét}}(X) \to \mathrm{AbShv}_{\text{ét}}(S)$.

Remark 2.17. More generally, we can consider the total derived functors

$$\mathrm{R}\Gamma(S, -) : D^+\mathrm{AbShv}_{\text{ét}}(S) \to D^+\mathrm{Ab}$$

and

$$\mathrm{R}f_* : D^+\mathrm{AbShv}_{\text{ét}}(X) \to D^+\mathrm{AbShv}_{\text{ét}}(S).$$

Example 2.18. Consider abelian sheaves on $\mathrm{Sch}^{\text{ét}}_{\operatorname{Spec} k}$. Writing $G = \mathrm{Gal}(\bar{k}/k)$, we know that $\mathrm{AbShv}_{\text{ét}}(\operatorname{Spec} k)$ is equivalent to the category of all G-modules.

If $\mathcal{F}$ corresponds to a module M under the correspondence given in Section 1.5, notice that $\Gamma(\operatorname{Spec} k, \mathcal{F}) = M^G$ is the module of fixed points of M under G. Their right derived functors must then coincide under the categorical equivalence described in Section 1.5. However, the right derived functor of Γ is precisely étale cohomology, whereas the right derived functor of $M \mapsto M^G$ is precisely Galois cohomology. Hence Galois cohomology computes sheaf cohomology, and vice versa.

In particular, we can restate Hilbert's Theorem 90 as follows

Proposition 2.19 (Hilbert's Theorem 90)**.** $H^1_{\text{ét}}(\operatorname{Spec} K, \mathbb{G}_m) = 0$.

The following is a generalization of Hilbert's Theorem 90, as a result of *Tsen's Theorem.*

Proposition 2.20. *Let K be a function field in one variable over an algebraically closed constant field. Then* $H^q(\operatorname{Spec} K, \mathbb{G}_m) = 0$ *for all* $q \geq 1$.

The following lemma is an immediate consequence of the exactness of pull-backs.

Lemma-Definition. Given a morphism $f : X \to S$ of schemes, there is a unique natural transformation $f^* : H^\bullet_{\text{ét}}(S, -) \to H^\bullet_{\text{ét}}(X, f^*(-))$ extending the canonical map in degree 0, called the *cohomological pull-back.* The cohomological pull-back behaves well with compositions via uniqueness and naturality, and by the fact that usual pull-backs in degree 0 behave well.

Since push-forwards and the global sections functor carry injectives to injectives, we also have the following result as a corollary to Grothendieck's spectral sequence.

Proposition 2.21 (Leray Spectral Sequence)**.** *Suppose $f : X \to S$ and $h : S \to S'$ are maps of schemes. Then, we have rightward oriented spectral sequences*

$$H^p_{\text{ét}}(S, R^q f_*(-)) \Rightarrow H^{p+q}_{\text{ét}}(X, -), \quad \textit{and } R^p h_* \circ R^q f_* \Rightarrow R^{p+q}(h \circ f)_*.$$

In the language of derived categories, we in fact have equalities of derived functors

$$\mathrm{R}\Gamma(S, -) \circ \mathrm{R}f_* = \mathrm{R}\Gamma(X, -) : D^+\mathrm{AbShv}_{\text{ét}}(X) \to D^+\mathrm{Ab} \quad \textit{and} \quad \mathrm{R}h_* \circ \mathrm{R}f_* = \mathrm{R}(h \circ f)_* : D^+\mathrm{AbShv}_{\text{ét}}(X) \to D^+\mathrm{AbShv}_{\text{ét}}(S').$$

Proposition 2.22. *Pull-backs by étale maps send injectives to injectives.*

Proof. Let $j : U \to S$ be an étale map. Since j^* is right-adjoint to $j_!$, Lemma 2.15 implies that it is sufficient to prove that $j_!$ is exact. It is already right-exact since it is a left adjoint. By construction of $j_!$, one can check that it is left exact. □

Corollary 2.22.1. *Suppose $j : U \to S$ is étale. For all $n \geq 0$, the functor $H^\bullet_{\text{ét}}(U, j^*(-)) : \mathrm{AbShv}_{\text{ét}}(S) \to \mathrm{Ab}$ is (canonically equivalent to) the classical right derived functor of the left-exact functor $\mathrm{AbShv}_{\text{ét}}(S) \to \mathrm{Ab}, \mathcal{F} \mapsto \mathcal{F}(U)$.*

Proof. Notice that $H^n_{\text{ét}}(U, j^*(-))$ is effaceable for all $n \geq 1$, since for any $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(S)$ we can find an injective sheaf $\mathcal{I}$ on S such that $\mathcal{F} \hookrightarrow \mathcal{I}$, and since $j^*\mathcal{I}$ is injective it follows that $H^\bullet_{\text{ét}}(U, j^*\mathcal{I}) = 0$. Thus, $H^\bullet_{\text{ét}}(U, j^*(-))$ is also a universal δ-functor by Proposition A.4. Since we have

$$H^0_{\text{ét}}(U, j^*\mathcal{F}) = \Gamma(U, j^*\mathcal{F}) = \mathcal{F}(U)$$

for all $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(S)$, it then follows from the universal property of universal δ-functors and of right derived functors that $H^\bullet_{\text{ét}}(U, j^*\mathcal{F})$ is the right derived functor of the left-exact functor $\mathcal{F} \to \mathcal{F}(U)$. □

Remark 2.23. In fact, more is true. We have equalities of total derived functors

$$\mathrm{R}\Gamma(U, -) \circ j^* = \mathrm{R}\Gamma_S(U, -) : D^+\mathrm{AbShv}_{\text{ét}}(S) \to D^+\mathrm{Ab},$$

where we denote by $\Gamma_S(U, -) : \mathrm{AbShv}_{\text{ét}}(S) \to \mathrm{Ab}$ the left-exact functor $\mathcal{F} \mapsto \mathcal{F}(U)$. This is a direct corollary of the Grothendieck Spectral Sequence Theorem for derived categories, noting that $j^* : \mathrm{AbShv}_{\text{ét}}(S) \to \mathrm{AbShv}_{\text{ét}}(U)$ is an exact functor and so $\mathrm{R}j^* = j^*$.

Definition. Given $U \in \mathrm{Sch}^{\text{ét}}_S$, we denote by $H^\bullet_{\text{ét}}(U, -)$ the functor $H^\bullet_{\text{ét}}(U, j^*(-))$ (or equivalently, the classical right derived functor of $\mathcal{F} \mapsto \mathcal{F}(U)$).

Proposition 2.24. *For any morphism $f : X \to S$ of schemes, and any $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(X)$, and for any $n \geq 0$, the sheafification of the pre-sheaf*

$$(\mathrm{Sch}^{\text{ét}}_S)^{op} \to \mathrm{Ab}, \quad U \mapsto H^n_{\text{ét}}(X \times_S U, \mathcal{F})$$

is $R^n f_* \mathcal{F}$.

Proof. Consider any injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$ in $\mathrm{AbShv}_{\text{ét}}(X)$. By definition, $R^n f_*\mathcal{F}$ is the n'th cohomology group of the complex of sheaves $f_*\mathcal{I}^\bullet$ in $\mathrm{AbShv}_{\text{ét}}(S)$. On the other hand, by the previous corollary we know that the n'th cohomology group of the complex of groups $\mathcal{I}^\bullet(U \times_S X)$ is precisely $H^n_{\text{ét}}(U \times_S X, \mathcal{F})$. Since sheafification is exact, it commutes with the formation of cohomology, and so the sheafification of $U \mapsto H^n_{\text{ét}}(U \times_S X, \mathcal{F}) = H^n(\mathcal{I}^\bullet(U \times_S X))$ coincides with the n'th cohomology group of the sheafification of $U \mapsto \mathcal{I}^\bullet(U \times_S X) = f_*\mathcal{I}^\bullet(U)$, which is precisely $U \mapsto R^n f_*\mathcal{F}(U)$. The result follows. □

2.4 Čech Cohomology

Let $\mathcal{U} = \{f_i : U_i \to S\}_{i \in I}$ be an étale cover for X. For any $p+1$-tuple $(i_0, ..., i_p)$ of indices in I, write

$$U_{i_0} \times_S \cdots \times_S U_{i_p} =: U_{i_0, i_1, \ldots, i_p}.$$

Suppose now $\mathcal{P} : (\mathrm{Sch}^{\text{ét}}_S)^{op} \to \mathrm{Ab}$ is a pre-sheaf on $\mathrm{Sch}^{\text{ét}}_S$. The canonical projection

$$U_{i_0, \ldots, i_p} \to U_{i_0, \ldots, \hat{i}_j, \ldots, i_p}$$

(where the hat denotes omission) induces a restriction morphism $\mathcal{P}(U_{i_0,\ldots,\hat{i}_j,\ldots,i_p}) \to \mathcal{P}(U_{i_0,\ldots,i_j,\ldots,i_p})$. Write this morphism as $\mathrm{res}_j^{i_0,\ldots,i_p}$. We can now define a complex $C^\bullet(\mathcal{U},\mathcal{P})$ given by

$$C^p(\mathcal{U},\mathcal{P}) = \prod_{i_0,\ldots,i_p\in I} \mathcal{P}(U_{i_0,\ldots,i_p})$$

and coboundary maps $d^p : C^p(\mathcal{U},\mathcal{P}) \to C^{p+1}(\mathcal{U},\mathcal{P})$ which takes an element $s = (s_{i_0,\ldots,i_p}) \in C^p(\mathcal{U},\mathcal{P})$ to

$$(d^p s)_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \mathrm{res}_j^{i_0,\ldots,i_j,\ldots,i_{p+1}}(s_{i_0,\ldots,\hat{i}_j,\ldots,i_{p+1}}).$$

Definition. The cohomology groups of the complex $C^\bullet(\mathcal{U},\mathcal{P})$ given above are called the *Čech cohomology groups* $\check{H}^p(\mathcal{U},\mathcal{P})$ of the pre-sheaf $\mathcal{P}$ with respect to the covering $\mathcal{U}$ of X.

If $\mathcal{P}$ is a sheaf, then notice that

$$\check{H}^0(\mathcal{U},\mathcal{P}) = \ker\left(\prod_i \mathcal{P}(U_i) \to \prod_{i,j} \mathcal{P}(U_{ij}), (s_i)_i \mapsto (s_i - s_j)_{i,j}\right) \cong \mathcal{P}(S)$$

by the glueability axiom of sheaves. If $\mathcal{P}$ is not a sheaf, then we only have a canonical injection $\mathcal{P}(S) \hookrightarrow \check{H}^0(\mathcal{U},\mathcal{P})$.

A second étale covering $\mathcal{V} = \{g_j : V_j \to S\}_{j\in J}$ is a *refinement* of $\mathcal{U}$ if there is a map $\tau : J \to I$ such that for each $j \in J$, $g_j : V_j \to S$ factors through $f_{\tau j} : U_{\tau j} \to S$. This map τ then induces maps $\tau^p : C^p(\mathcal{U},\mathcal{P}) \to C^p(\mathcal{V},\mathcal{P})$ which sends $s = (s_{i_0,\ldots,i_p}) \in C^p(\mathcal{U},\mathcal{P})$ to the p-cochain $\big((\tau^p s)_{j_0,\ldots,j_p}\big)$ where $(\tau^p s)_{j_0,\ldots,j_p}$ is the image of $s_{\tau j_0,\ldots,\tau j_p}$ under the map $\mathcal{P}\big(V_{j_0} \times_S \cdots \times_S V_{j_p} \to U_{\tau j_0} \times_S \cdots \times_S U_{\tau j_p}\big)$.

The map $\tau^\bullet : C^\bullet(\mathcal{U},\mathcal{P}) \to C^\bullet(\mathcal{V},\mathcal{P})$ is in fact a chain map, and so we get an induced map on cohomology

$$\rho_{\mathcal{V},\mathcal{U},\tau} : \check{H}^p(\mathcal{U},\mathcal{P}) \to \check{H}^p(\mathcal{V},\mathcal{P}).$$

It can be checked (see [Mil80, Lemma 2.1] or [Tam94, Lemma 2.2.7]) that this map $\rho_{\mathcal{V},\mathcal{U},\tau}$ is in fact independent of τ, and so for any refinement $\mathcal{V}$ of $\mathcal{U}$ we get a natural map on Čech cohomology

$$\rho_{\mathcal{V},\mathcal{U}} : \check{H}^p(\mathcal{U},\mathcal{P}) \to \check{H}^p(\mathcal{V},\mathcal{P}).$$

Definition. The *Čech (étale) cohomology groups* of the pre-sheaf $\mathcal{P}$ over S is the colimit

$$\check{H}^p_{\text{ét}}(S,\mathcal{P}) = \operatorname*{colim}_{\mathcal{U}\in\mathrm{Cov}_{\text{ét}}(S)} \check{H}^p(\mathcal{U},\mathcal{P}).$$

Remark 2.25. Suppose now $S' \to S$ is étale. Then, we can define the Čech complex $C^\bullet(\mathcal{U}/S',\mathcal{P})$ for any open cover $\mathcal{U}$ of S' in the obvious way, and this is very clearly equal to the Čech complex $C^\bullet(\mathcal{U},\mathcal{P}|_{S'})$. It follows that $\check{H}^p_{\text{ét}}(S',\mathcal{P}|_{S'})$ is the same as $\check{H}^p_{\text{ét}}(S',\mathcal{P})$.

We give a more homological-algebraic interpretation of Čech cohomology. Fix an étale cover $\mathcal{U} = \{U_i \to S'\}_{i\in I}$ of S', where $S' \to S$ is an étale S-scheme. We define the (covariant) functor $\check{H}^0(\mathcal{U}/S',-)$ from abelian pre-sheaves on $\mathrm{Sch}_S^{\text{ét}}$ to Ab which to each pre-sheaf $\mathcal{F}$ assigns

$$\begin{aligned}\check{H}^0(\mathcal{U}/S',\mathcal{F}) &:= \ker\left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_{S'} U_j)\right) \\ &= \left\{(s_i) \in \prod_i \mathcal{F}(U_i) : s_i|_{U_i\times_{S'}U_j} = s_j|_{U_i\times_{S'}U_j}\right\}.\end{aligned}$$

This is the usual 0'th Čech cohomology group.

Proposition 2.26. *The Čech cohomology groups $\check{H}^*(\mathcal{U}/S',\mathcal{F})$ defined above are the right-derived functors (in the category of pre-sheaves on $\mathrm{Sch}_S^{\text{ét}}$) of $\check{H}^0(\mathcal{U}/S',\mathcal{F})$.*

Proof. We follow the proof given in [Tam94]. It suffices to show that $\check{H}^*(\mathcal{U}/S',-)$ as defined above is a universal δ-functor. That $\check{H}^*(\mathcal{U}/S',\mathcal{F})$ is a δ-functor is obvious, since for any exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}" \to 0$ of abelian pre-sheaves we get the obvious exact sequence of Čech complexes

$$0 \to C^\bullet(\mathcal{U}/S',\mathcal{F}') \to C^\bullet(\mathcal{U}/S',\mathcal{F}) \to C^\bullet(\mathcal{U}/S',\mathcal{F}'') \to 0.$$

Since $\check{H}^*(\mathcal{U}/S', -)$ is the usual cohomology of complexes, we get natural connecting morphisms $\delta : \check{H}^p(\mathcal{U}/S', \mathcal{F}'') \to \check{H}^{p+1}(\mathcal{U}/S', \mathcal{F}')$ making all the required diagrams commute. Hence $\check{H}^*(\mathcal{U}/S', -)$ is a δ-functor.

We now show that each $\check{H}^p(\mathcal{U}/S', -)$ is effaceable for all $p \geq 1$. In other words, it suffices to show that the cochain complex $C^\bullet(\mathcal{U}/S', \mathcal{I})$ is an exact cochain complex for any injective abelian pre-sheaf $\mathcal{I}$. For any $U \in \mathrm{Sch}_S^{\text{ét}}$, define the abelian pre-sheaf $Z_U : (\mathrm{Sch}_S^{\text{ét}})^{op} \to \mathrm{Ab}$ by

$$Z_U(X) = \bigoplus_{\mathrm{Hom}_S(X,U)} \mathbb{Z}$$

on étale schemes X, i.e. $Z_U(X)$ is the free abelian group generated by the set $\mathrm{Hom}_S(X, U)$. For $f : X \to Y$ the natural map $(-) \circ f : \mathrm{Hom}_S(Y, U) \to \mathrm{Hom}_S(X, U)$ induces a natural map $Z_U(Y) \to Z_U(X)$. By a Yoneda Lemma style argument, one can show that

$$\mathrm{Hom}(Z_U, \mathcal{F}) \cong \mathcal{F}(U)$$

for all abelian pre-sheaves $\mathcal{F}$. It follows that

$$\begin{aligned} C^q(\mathcal{U}/S', \mathcal{I}) = \prod_{i_0,\ldots,i_q} \mathcal{I}(U_{i_0} \times_{S'} \cdots \times_{S'} U_{i_q}) &\cong \prod_{i_0,\ldots,i_q} \mathrm{Hom}(Z_{U_{i_0} \times_{S'} \cdots \times_{S'} U_{i_q}}, \mathcal{I}) \\ &\cong \mathrm{Hom}\left(\bigoplus_{i_0,\ldots,i_q} Z_{U_{i_0} \times_{S'} \cdots \times_{S'} U_{i_q}}, \mathcal{I} \right). \end{aligned}$$

Since $\mathcal{I}$ is injective, the cochain complex $C^\bullet(\mathcal{U}/S', \mathcal{I})$ will be exact if the complex

$$\bigoplus_i Z_{U_i} \leftarrow \bigoplus_{i,j} Z_{U_i \times_{S'} U_j} \leftarrow \bigoplus_{i,j,k} Z_{U_i \times_{S'} U_j \times_{S'} U_k} \leftarrow \cdots$$

is exact. By definition, since we are in the category of pre-sheaves, the above complex is exact if and only if the complex

$$\bigoplus_i Z_{U_i}(X) \leftarrow \bigoplus_{i,j} Z_{U_i \times_{S'} U_j}(X) \leftarrow \bigoplus_{i,j,k} Z_{U_i \times_{S'} U_j \times_{S'} U_k}(X) \leftarrow \cdots$$

is exact for all étale S-schemes X. Fix $f \in \mathrm{Hom}(X, S')$. Define $\mathrm{Hom}_f(X, U_i)$ to be the set of morphisms $g : X \to U_i$ such that the composition $X \xrightarrow{g} U_i \to U$ is f. Then we have the partitions

$$\mathrm{Hom}(X, U_i) = \bigsqcup_{f \in \mathrm{Hom}(X,S')} \mathrm{Hom}_f(X, U_i),$$

which extend

$$\mathrm{Hom}(X, U_{i_0} \times_{S'} \cdots \times_{S'} U_{i_q}) = \bigsqcup_{f \in \mathrm{Hom}(X,S')} \mathrm{Hom}_f(X, U_{i_0} \times_{S'} \cdots \times_{S'} U_{i_q}) = \bigsqcup_{f \in \mathrm{Hom}(X,S')} \prod_{j=0}^{q} \mathrm{Hom}_f(X, U_{i_j}).$$

Writing $S(f) = \bigsqcup_i \mathrm{Hom}_f(X, U_i)$, it then follows that

$$\begin{aligned} \bigoplus_{i_0,\ldots,i_q} Z_{U_{i_0} \times_{S'} \cdots \times_{S'} U_{i_q}}(X) &= \mathbb{Z}\left(\bigsqcup_{i_0,\ldots,i_q} \mathrm{Hom}(U_{i_0} \times_{S'} \cdots \times_{S'} U_{i_q}, X) \right) \\ &\cong \mathbb{Z}\left(\bigsqcup_{f \in \mathrm{Hom}(X,S')} \bigsqcup_{i_0,\ldots,i_q} \prod_{j=0}^{q} \mathrm{Hom}_f(X, U_{i_j}) \right) \\ &= \bigoplus_{f \in \mathrm{Hom}(X,S')} \mathbb{Z}\left(\prod_{j=0}^{q} S(f) \right). \end{aligned}$$

Hence the complex

$$\bigoplus_i Z_{U_i}(X) \leftarrow \bigoplus_{i,j} Z_{U_i \times_{S'} U_j}(X) \leftarrow \bigoplus_{i,j,k} Z_{U_i \times_{S'} U_j \times_{S'} U_k}(X) \leftarrow \cdots$$

is exact if and only if for each $f \in \mathrm{Hom}(X, S')$ the simplicial complex

$$\mathbb{Z}S(f) \leftarrow \mathbb{Z}S(f)^2 \leftarrow \mathbb{Z}S(f)^3 \leftarrow \cdots$$

is exact. A simple computation shows that this is indeed the case, finishing the proof. □

Remark 2.27. If $\mathcal{V}$ is any refinement of $\mathcal{U}$, then the obvious map $\check{H}^0(\mathcal{U}/S', \mathcal{F}) \to \check{H}^0(\mathcal{V}/S', \mathcal{F})$ induces, by universality of delta functors, a unique morphism $\check{H}^*(\mathcal{U}/S', \mathcal{F}) \to \check{H}^*(\mathcal{V}/S', \mathcal{F})$ of δ-functors. By uniqueness, it follows that this morphism is precisely the refinement morphism $\rho_{\mathcal{V},\mathcal{U}} : \check{H}^*(\mathcal{U}, \mathcal{P}) \to \check{H}^*(\mathcal{V}, \mathcal{P})$ described above.

We omit the proof of the following theorem (see [Tam94, Theorem 2.2.6]).

Theorem 2.28. *The functor $\mathcal{F} \mapsto \check{H}^0_{\text{ét}}(S', \mathcal{F})$ is an additive left-exact functor, whose right-derived functors are precisely the Čech cohomology groups $\check{H}^p_{\text{ét}}(S', \mathcal{F})$.*

2.5 Relationship Between Étale Cohomology and Čech Cohomology

Čech cohomology is fundamentally a cohomology theory on the category of *pre-sheaves*, whereas étale cohomology is a cohomology theory on the category of *sheaves*. Thus, while they are closely related, the fact that epimorphisms behave differently in the category of sheaves versus the category of pre-sheaves complicates things.

Proposition 2.29. *The inclusion functor ι from $\text{AbShv}_{\text{ét}}(S)$ to the category of pre-sheaves on $\text{Sch}^{\text{ét}}_S$ is a left-exact additive functor whose right-derived functors $R^\bullet\iota$ are given by*

$$\big((R^n\iota)\mathcal{F}\big)(X) = H^n_{\text{ét}}(X, \mathcal{F}) \quad \forall X \in \text{Sch}^{\text{ét}}_S.$$

Proof. Since pull-backs by étale maps send injective sheaves to injective sheaves, each of the pre-sheaves $X \mapsto H^n_{\text{ét}}(X, -)$ is eraseable. Since $H^\bullet_{\text{ét}}(X, -)$ is a δ-functor for all $X \in \text{Sch}^{\text{ét}}_S$, it follows that the collection of maps from sheaves to pre-sheaves taking $\mathcal{F}$ to the pre-sheaf $X \mapsto H^n_{\text{ét}}(X, \mathcal{F})$ is a δ-functor, and thus universal. Since they agree at the $n = 0$ piece, it follows that $R^n\iota = (X \mapsto H^n_{\text{ét}}(X, -))$. □

Definition. Let $\underline{H}^n_{\text{ét}}$ denote the right-derived functors of $R^n\iota$. The proposition then states that $\underline{H}^n_{\text{ét}}(\mathcal{F})$ is the pre-sheaf sending $X \in \text{Sch}^{\text{ét}}_S$ to $H^n_{\text{ét}}(X, \mathcal{F})$.

Proposition 2.30. *The sheafification of $\underline{H}^n_{\text{ét}}$ is 0 for all $n \geq 1$.*

Proof. Consider the inclusion functor $\iota : \text{AbShv}_{\text{ét}}(S) \to \text{AbPreShv}_{\text{ét}}(S)$ and the sheafification functor $(-)^{sh} : \text{AbPreShv}_{\text{ét}}(S) \to \text{AbShv}_{\text{ét}}(S)$; their composition $\iota(-)^{sh} : \text{AbShv}_{\text{ét}}(S) \to \text{AbShv}_{\text{ét}}(S)$ is the identity functor I on $\text{AbShv}_{\text{ét}}(S)$. Since sheafification is an exact functor, all of its derived functors vanish, and in particular every pre-sheaf is $(-)^{sh}$-acyclic. Hence, we have a spectral sequence

$$E_2^{pq} = R^q(-)^{sh}\big(\underline{H}^p(\mathcal{F})\big) \Rightarrow R^{p+q}I(\mathcal{F})$$

for every sheaf $\mathcal{F}$. Since $R^q(-)^{sh} = 0$ for $q \geq 1$, we have $E_2^{pq} = 0$ for all $q > 0$. It follows from Proposition A.12 that all the edge morphisms $R^pI(\mathcal{F}) \to E_2^{p,0} = \underline{H}^p(\mathcal{F})^{sh}$ are isomorphisms. However, I is exact and so $R^pI = 0$ for all $p \geq 1$. Hence $\underline{H}^p(\mathcal{F})^{sh} = 0$ for all $p \geq 1$, as required. □

Note that the inclusion functor $\iota : \text{AbShv}_{\text{ét}}(S) \to \text{AbPreShv}_{\text{ét}}(S)$ sends injective objects to injective objects. As a consequence of the Grothendieck spectral sequence applied to the functors $\iota : \text{AbShv}_{\text{ét}}(S) \to \text{AbPreShv}_{\text{ét}}(S)$ and $\check{H}^0(\mathcal{U}, -) : \text{AbPreShv}_{\text{ét}}(S) \to \text{Ab}$ (resp. $\check{H}^0_{\text{ét}}(S, -)$), noting that their composition is precisely the global-sections functor, we have the following result.

Theorem 2.31 (Spectral sequences for Čech cohomology)**.** *We have cohomological spectral sequences*

$$\check{H}^q(\mathcal{U}, \underline{H}^p_{\text{ét}}(\mathcal{F})) \Rightarrow H^{p+q}_{\text{ét}}(S, \mathcal{F}) \quad \forall \text{ étale covers } \mathcal{U} \text{ of } S, \qquad \text{and} \qquad \check{H}^q_{\text{ét}}(S, \underline{H}^p_{\text{ét}}(\mathcal{F})) \Rightarrow H^{p+q}_{\text{ét}}(S, \mathcal{F})$$

that are moreover functorial in $\mathcal{F}$.

Remark 2.32. By abstract universal δ-functor nonsense, if one can show that the Čech cohomology functors when restricted to sheaves give rise to a δ-functor (the obstruction here is the exactness of the long sequence of cohomology groups attached to a short exact sequence of sheaves), then Čech cohomology and the usual étale cohomology would coincide. This occurs with added conditions on S.

By looking at the edge maps, we get the following results. All of these follow by appealing to Proposition A.12 after showing that $E_2^{pq} = 0$ for some values of p. See [Tam94] for the explicit proofs.

Proposition 2.33. *Suppose $\mathcal{U}$ is an étale covering of S, and $\mathcal{F} \in \text{AbShv}_{\text{ét}}(S)$ such that $H^q(U_{i_0} \times_S \cdots \times_S U_{i_r}, \mathcal{F}) = 0$ for all $q \geq 0$ and all multi-indices $(i_0, ..., i_r)$. Then, the edge morphisms $\check{H}^q(\mathcal{U}, \mathcal{F}) \to H^q_{\text{ét}}(S, \mathcal{F})$ are isomorphisms for all p.*

Proposition 2.34. *For any $\mathcal{F} \in \text{AbShv}_{\text{ét}}(S)$, the edge morphisms $\check{H}^q(\mathcal{U}, \mathcal{F}) \to H^q_{\text{ét}}(S, \mathcal{F})$ are isomorphisms for $q = 0, 1$ and is an injection for $q = 2$.*

More generally, if $\check{H}^q_{\text{ét}}(S, \underline{H}^p(\mathcal{F})) = 0$ for $0 < q < n$, then the edge map $\check{H}^m_{\text{ét}}(S, \mathcal{F}) \to H^m_{\text{ét}}(S, \mathcal{F})$ is an isomorphism for all $m \leq n$ and an injection for $m = n + 1$.

There are some hard theorems that describe sufficient conditions for when étale and Čech cohomology coincide. We state them without proof below.

Theorem 2.35 (Artin). *If S is a scheme such that every finite subset of points of S is contained in an open affine subscheme, then we have canonical isomorphisms $H^\bullet_{\text{ét}}(S,-) \cong \check{H}^\bullet_{\text{ét}}(S,-)$.*

For instance, this holds if S is quasi-projective over a Noetherian ring.

Theorem 2.36 (Cartan's Lemma). *Let $\mathcal{F}$ be a sheaf of abelian groups on a scheme S. Let $\mathcal{K}$ be a class of étale S-schemes U such that*

- *if $U, V \in \mathcal{K}$ then $U \times_S V \in \mathcal{K}$; and*
- *every étale S-scheme X has an étale S-covering $\{U_i \to X\}$ such that $U_i \in \mathcal{K}$.*

If $\check{H}^i(U,\mathcal{F}) = 0$ for all $U \in \mathcal{K}$ and all $i \geq 1$, then the canonical edge map

$$\check{H}^i_{\text{ét}}(S,\mathcal{F}) \to H^i_{\text{ét}}(S,\mathcal{F})$$

is an isomorphism for all i.

2.6 Flabby Sheaves

Definition. A sheaf $\mathcal{F} \in \text{AbShv}_{\text{ét}}(S)$ is *flabby* if $H^n(U,\mathcal{F}) = 0$ for all $n > 0$ and all $U \in \text{Sch}^{\text{ét}}_S$.

Proposition 2.37. *Suppose $\mathcal{F} \in \text{AbShv}_{\text{ét}}(S)$. The following are equivalent:*

1. *$\mathcal{F}$ is flabby;*
2. *$\check{H}^q(\mathcal{U}/U,\mathcal{F}) = 0$ for all $q > 0$, for all $U \in \text{Sch}^{\text{ét}}_S$, and for all étale coverings $\mathcal{U}$ of U;*
3. *$\check{H}^q(U,\mathcal{F}) = 0$ for all $U \in \text{Sch}^{\text{ét}}_S$.*

Proof. Since $\mathcal{F}$ is flabby, $\underline{H}^n(\mathcal{F}) = 0$ for all $n \geq 1$, and so Proposition 2.33 implies that $\check{H}^q(\mathcal{U}/U,\mathcal{F}) = 0$ for all $q > 0$, all $U \in \text{Sch}^{\text{ét}}_S$ and all étale coverings $\mathcal{U}$ of U. Thus (1) $\implies$ (2). The implication (2) $\implies$ (3) follows immediately by taking colimits. Now assume that (3) holds.

We have $H^1_{\text{ét}}(U,\mathcal{F}) \cong \check{H}^1_{\text{ét}}(U,\mathcal{F}) = 0$ for all $U \in \text{Sch}^{\text{ét}}_S$. We now proceed by induction using the spectral sequence for Čech cohomology. Suppose $\underline{H}^i_{\text{ét}}(\mathcal{F}) = 0$ for all $i < n$. Then we have $\check{H}^q_{\text{ét}}(U,\underline{H}^p_{\text{ét}}(\mathcal{F})) = 0$ for all $p < n$ and all q, and in particular for $p + q \leq n$ whenever $q \geq 1$. The spectral sequence then implies that $\check{H}^n_{\text{ét}}(U,\mathcal{F}) = 0$ for all $U \in \text{Sch}^{\text{ét}}_S$. ☐

We have some obvious properties of flabby sheaves.

Proposition 2.38.
1. *Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence in the category $\text{AbShv}_{\text{ét}}(S)$. If $\mathcal{F}'$ is flabby, then this sequence is also exact in $\text{AbPreShv}_{\text{ét}}(S)$.*
2. *If $\mathcal{F}'$ and $\mathcal{F}$ are flabby in the exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, then so is $\mathcal{F}''$.*
3. *If the direct sum $\mathcal{F} \oplus \mathcal{G}$ of abelian sheaves is flabby, so is $\mathcal{F}$ and $\mathcal{G}$.*
4. *Injective abelian sheaves are flabby.*
5. *Pull-backs by étale morphisms send flabby sheaves to flabby sheaves.*

2.7 Cohomology with Supports

Suppose X is a scheme, $i : Z \hookrightarrow X$ a closed immersion, and $j : U \hookrightarrow X$ an open immersion such that X is the disjoint union of $i(Z)$ and $j(U)$. We have the functor $\mathcal{F} \mapsto \ker(\mathcal{F}(X) \to \mathcal{F}(U))$ from $\text{AbShv}_{\text{ét}}(X)$ to Ab. This is obviously an additive covariant functor. It is also left-exact, since the functors $\Gamma(X,-)$ and $\Gamma(U,-)$ are left-exact.

Definition. The right derived functors $H^\bullet_Z(X,-)$ of the functor $\mathcal{F} \mapsto \ker(\mathcal{F}(X) \to \mathcal{F}(U))$ are called the *(étale) cohomology groups with support in Z.*

Remark 2.39. The functors $H^\bullet_Z(X,-)$ are contravariant in (X,U).

The proof of the following result is omitted (c.f. [Mil80, Chapter 3, Proposition 1.25])

Proposition 2.40. *For any sheaf $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(X)$ there is a long exact sequence*

$$0 \to H^0_Z(X,\mathcal{F}) \to \mathcal{F}(X) \to \mathcal{F}(U) \to H^1_Z(X,\mathcal{F}) \to H^1_{\text{ét}}(X,\mathcal{F}) \to H^1_{\text{ét}}(U,\mathcal{F}) \to H^2_Z(X,\mathcal{F}) \to \cdots .$$

Proposition 2.41 (Excision)**.** *Let $Z \subset X$ and $Z' \subset X'$ be closed subschemes, and let $f : X' \to X$ be an étale morphism such that $f|_{Z'} : Z' \to Z$ induces an isomorphism, and such that $f(X' \setminus Z') \subset X \setminus Z$. Then $H^p_Z(X,\mathcal{F}) \cong H^p_{Z'}(X', f^*\mathcal{F})$ for all $p \geq 0$ and all sheaves $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(X)$.*

Proof. Note that f^* is exact and preserves injectives. Since both sides are right-derived functors of their $p = 0$ component, it suffices to show that $H^0_Z(X,\mathcal{F}) \cong H^0_{Z'}(X', f^*\mathcal{F})$. Setting $U = X \setminus Z$ and $U' = X' \setminus Z'$, we have an exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0_Z(X,\mathcal{F}) & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{F}(U) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0_{Z'}(X', f^*\mathcal{F}) & \longrightarrow & f^*\mathcal{F}(X') & \longrightarrow & f^*\mathcal{F}(U'). \end{array}$$

Let $s \in H^0_Z(X,\mathcal{F})$ map to zero in $H^0_{Z'}(X', f^*\mathcal{F})$. Taking $s \in \mathcal{F}(X)$, we see that $s|_U = 0$ since $s \in H^0_Z(X,\mathcal{F})$ and that $s|_{X'} = 0$ since it maps to zero in $H^0_{Z'}(X', f^*\mathcal{F})$. Since $U \to X$ and $f : X' \to X$ covers X (as $U = X \setminus Z$, and $Z = f(Z') \subset f(X')$), it follows that $s = 0$ in $\mathcal{F}(X)$. Thus the left vertical map is injective.

On the other hand, suppose $s' \in H^0_{Z'}(X', f^*\mathcal{F})$, which we again regard as an element of $f^*\mathcal{F}(X')$. Then, $s'|_{U'} = 0$, and so the sections $s' \in f^*\mathcal{F}(X') = \mathcal{F}(X')$ and $0 \in \mathcal{F}(U)$ agree on $X' \times_X U \subset U'$. Thus these sections glue to give a section $s \in \Gamma(X,\mathcal{F})$. Since $s|_U = 0$, we have $s \in H^0_Z(X,\mathcal{F})$. □

Now suppose X is a separated variety, and fix an embedding $j : X \hookrightarrow \bar{X}$ making X an open subvariety of a complete variety $\bar{X}$.

Definition. The *cohomology groups with compact support* are given by

$$H^p_{\text{ét},c}(X,\mathcal{F}) := H^p_{\text{ét}}(\bar{X}, j_!\mathcal{F}).$$

Remark 2.42. If $\mathcal{F}$ is a torsion sheaf, then $H^p_{\text{ét},c}(X,\mathcal{F})$ is independent of the choice of $\bar{X}$.

Set

$$\Gamma_c(X,\mathcal{F}) := \bigcup_Z \ker\big(\mathcal{F}(X) \to \mathcal{F}(X \setminus Z)\big)$$

where Z runs through all complete subvarieties of X. We then have the following proposition, whose proof is omitted (see [Mil80, Chapter 3, Proposition 1.29]).

Proposition 2.43. *Suppose X and $\bar{X}$ are as above, we consider any $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(X)$.*

1. $H^0_{\text{ét},c}(X,\mathcal{F}) = \Gamma_c(X,\mathcal{F})$*;*

2. *The functors $H^p_{\text{ét},c}(X,-)$ form a δ-functor.*

3. *For any complete subvariety Z of X, there is a canonical morphism of δ-functors $H^p_Z(X,-) \to H^p_c(X,-)$.*

2.8 Torsors

Definition. For S a scheme and G a group object in $\mathrm{Shv}_{\text{ét}}(S)$, a *left G-torsor* is an object $\mathcal{F}$ in $\mathrm{Shv}_{\text{ét}}(S)$ that has non-empty stalks and is equipped with a left G-action such that the canonical map $G \times \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ defined by $(g,s) \mapsto (gs,s)$ is an isomorphism. The notion of *right G-torsors* is defined similarly.

The trivial left G-torsor is the sheaf G with the G-action defined by left multiplication.

Example 2.44. If S is connected and X a connected Galois cover of S with Galois group G, then the representable sheaf X is a right torsor for the S-group $S \times G$.

Remark 2.45. Note that a choice of element in $\mathcal{F}(U)$ whenever $\mathcal{F}(U)$ is non-empty induces an isomorphism $\mathcal{F}|_U \cong G|_U$ as left G-torsors. Since $\mathcal{F}$ has non-empty stalks and is thus non-empty étale locally, it follows that $\mathcal{F}$ and G are étale locally isomorphic. In particular, if G is LCC then so is $\mathcal{F}$.

We have the following theorem, which we do not prove.

Theorem 2.46. *Let G be an abelian sheaf on $\mathrm{Sch}_S^{\text{ét}}$. The set of isomorphism classes of left G-torsors in $\mathrm{AbShv}_{\text{ét}}(S)$ naturally forms a group with group operation products, identity the trivial left G-torsor, and inverses given by twisting the action of G via the inverse homomorphism $G \to G, x \mapsto -x$.*

Moreover, this group of (isomorphism classes of) left G-torsors in $\mathrm{AbShv}_{\text{ét}}(S)$ is canonically isomorphic to $H^1_{\text{ét}}(X, G)$, and this isomorphism is bifunctorial in (X, G).

Remark 2.47. This result follows by showing that the group of classes of left G-torsors is isomorphic to $\check{H}^1_{\text{ét}}(X, G)$. The essential idea in constructing this isomorphism is to notice that local trivializations of a left G-torsor on an open cover correspond to Čech 1-cocycles on this open cover, and that different local trivializations differ by a Čech 1-coboundary.

Example 2.48. An *étale line bundle* on S is an invertible $\mathcal{O}_{S,\text{ét}}$-module on S, i.e. it is an (étale-)locally free $\mathcal{O}_{S,\text{ét}}$-module sheaf of rank 1. The set of isomorphism classes of étale line bundles is denoted $\mathrm{Pic}_{\text{ét}}(S)$. Just as for Zariski-sheaves, $\mathrm{Pic}_{\text{ét}}(S)$ is a group under tensor products. Notice that an étale line bundle can be equally well thought of as a $\mathbb{G}_{m,S}$-torsor. Hence,

$$\mathrm{Pic}_{\text{ét}}(S) \cong H^1_{\text{ét}}(S, \mathbb{G}_m).$$

Using the bifunctoriality of the above isomorphism, we can give a very powerful theorem relating the étale $H^1_{\text{ét}}$ and the étale fundamental group for LCC group sheaves. Suppose $\bar{x} : \mathrm{Spec}\,\bar{k} \to S$ is a geometric point of a connected scheme S. Note that the group of left G-torsors is equivalent (by the LCC classification and the equivalence of sheaves over $\mathrm{Spec}\,k$ and Galois modules) to the group of left $G_{\bar{x}}$-torsors over $\mathrm{Spec}\,\bar{k}$.

Theorem 2.49. *If S is connected and $\bar{x}$ a geometric point of S, and if $G \in \mathrm{AbShv}(S)$ is an LCC group object, then there is a bifunctorial isomorphism of groups*

$$H^1_{\text{ét}}(X, G) \cong H^1\big(\pi_1^{\text{ét}}(X, \bar{x}), G_{\bar{x}}\big),$$

where the cohomology on the right is group cohomology.

2.9 Generalities on Topology Comparisons

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two Grothendieck topologies on categories $\mathcal{C}$ and $\mathcal{C}'$ respectively. For a site $(\mathcal{C}, \mathcal{T})$ we denote $\mathrm{Shv}(\mathcal{T})$ (resp $\mathrm{PreShv}(\mathcal{T})$) to be the category of sheaves (resp. pre-sheaves) of sets on $(\mathcal{C}, \mathcal{T})$. Similarly for $\mathrm{AbShv}(\mathcal{T})$ and $\mathrm{AbPreShv}(\mathcal{T})$.

Definition. A morphism of topologies $F : \mathcal{T} \to \mathcal{T}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ of the underlying categories such that for any $U \in \mathcal{C}$ and for any covering $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I} \in \mathrm{Cov}_{\mathcal{T}}(U)$, we have

- $\{F(f_i) : F(U_i) \to F(U)\}_{i \in I} \in \mathrm{Cov}_{\mathcal{T}'}(F(U))$; and
- for any morphism $g : V \to U$ in $\mathcal{C}$ the canonical morphism

  $$F(U_i \times_U V) \to F(U_i) \times_{F(U)} F(V)$$

  is an isomorphism for all $i \in I$.

Notice that a morphism of topologies $F : \mathcal{T} \to \mathcal{T}'$ induces a functor

$$F^* : \mathrm{PreShv}(\mathcal{T}') \to \mathrm{PreShv}(\mathcal{T}), \quad \mathcal{F} \mapsto \mathcal{F} \circ F$$

which restricts naturally to a functor $F^* : \mathrm{Shv}(\mathcal{T}') \to \mathrm{Shv}(\mathcal{T})$. This functor has a left-adjoint

$$F_* : \mathrm{PreShv}(\mathcal{T}) \to \mathrm{PreShv}(\mathcal{T}')$$

(defined in a similar way to f_*), and upon sheafification it induces a functor $F_* : \mathrm{Shv}(\mathcal{T}) \to \mathrm{Shv}(\mathcal{T}')$. This functor F_* on sheaves is again left-adjoint to F^*. In particular, we record the following important consequence.

Proposition 2.50. *The functor $F^* : \mathrm{Shv}(\mathcal{T}') \to \mathrm{Shv}(\mathcal{T})$ is left-exact.*

Example 2.51. If $f : S' \to S$ is a morphism of schemes, then we have a corresponding base-change functor $\hat{f} : \mathrm{Sch}_S^{\text{ét}} \to \mathrm{Sch}_{S'}^{\text{ét}}$ given by $\hat{f}(X) = X \times_S S'$ for all $X \in \mathrm{Sch}_S^{\text{ét}}$. In this case, the functor $\hat{f}^*$ is the usual pull-back f^*, and the functor $\hat{f}_*$ is the usual push-forward f_*. In fact, most of the results from the previous section can be generalized to an arbitrary Grothendieck topology on any category with all finite products and a final object. This is the perspective taken in [Tam94].

It is important to be able to compute the pull-back functor F^* as well as its right derived functors (recall that $\mathrm{Shv}(\mathcal{T}')$ has enough injectives). We have the following result.

Theorem 2.52. *Suppose $F : \mathcal{T} \to \mathcal{T}'$ is a morphism of topologies with induced map $F^* : \mathrm{AbShv}(\mathcal{T}') \to \mathrm{AbShv}(\mathcal{T})$ For each $\mathcal{F}' \in \mathrm{AbShv}(\mathcal{T}')$ there is an isomorphism*

$$R^q F^*(\mathcal{F}') \cong \left(F^* \underline{H}^q_{\mathcal{T}'}(\mathcal{F}')\right)^{sh},$$

i.e. the sheaf $R^q F^*(\mathcal{F}') \in \mathrm{AbShv}(\mathcal{T})$ is the sheaf associated to the pre-sheaf $U \mapsto H^q_{\mathcal{T}'}(FU, \mathcal{F}')$ on $\mathcal{T}$.

Proof. Notice that F^* is the composition of functors

$$\mathrm{AbShv}(\mathcal{T}') \hookrightarrow \mathrm{AbPreShv}(\mathcal{T}') \xrightarrow{F^*} \mathrm{AbPreShv}(\mathcal{T}) \xrightarrow{(-)^{sh}} \mathrm{AbShv}(\mathcal{T}).$$

Here, the first functor $\mathrm{AbShv}(\mathcal{T}') \hookrightarrow \mathrm{AbPreShv}(\mathcal{T}')$ is left-exact whereas the composition $\mathrm{AbPreShv}(\mathcal{T}') \to \mathrm{AbShv}(\mathcal{T})$ is exact. Since the right-derived functors of $\mathrm{AbShv}(\mathcal{T}') \hookrightarrow \mathrm{AbPreShv}(\mathcal{T}')$ is the pre-sheaf $U' \mapsto H^q_{\mathcal{T}'}(U', -)$, the result follows. ☐

A basic computation shows the following.

Proposition 2.53. *Suppose $F : \mathcal{T} \to \mathcal{T}'$ is such that the functor $F : \mathcal{C} \to \mathcal{C}'$ sends fibre products to fibre products. Then, we have an equality of Čech complexes*

$$C^\bullet(\{U_i \to U\}, F^*\mathcal{F}') \cong C^\bullet(\{FU_i \to FU\}, \mathcal{F}')$$

for all $\mathcal{F}' \in \mathrm{AbShv}(\mathcal{T}')$. Thus Čech cohomology on $\mathcal{T}$ and $\mathcal{T}'$ coincides.

Corollary 2.53.1. *F^* sends flabby sheaves in $\mathrm{AbShv}(\mathcal{T}')$ to flabby sheaves in $\mathrm{AbShv}(\mathcal{T})$.*

If a morphism of topologies is sufficiently nice, we get comparison theorems on cohomology as follows.

Theorem 2.54. *Suppose $(\mathcal{C}, \mathcal{T})$ and $(\mathcal{C}', \mathcal{T}')$ are two sites, such that $\mathcal{C}'$ is a full-subcategory of $\mathcal{C}$. Suppose the inclusion functor $\iota : \mathcal{C}' \to \mathcal{C}$ induces a morphism on topologies, such that for each $U \in \mathcal{C}'$ and each covering $\{U_i \to \iota U\} \in \mathrm{Cov}_{\mathcal{T}}(\iota U)$ there exists a cover $\{U'_j \to U\} \in \mathrm{Cov}_{\mathcal{T}'}(U)$ such that $\{\iota U'_j \to \iota U\}$ is a refinement of $\{U_i \to \iota U\}$.*

Then, the following holds.

- *The morphism $\rho_{\mathcal{F}'} : \mathcal{F}' \to \iota^* \iota_* \mathcal{F}'$ corresponding to the image of $Id_{\iota_* \mathcal{F}'}$ under the adjunction*

$$\mathrm{Hom}_{\mathrm{Shv}(\mathcal{T})}(\iota_*\mathcal{F}', \iota_*\mathcal{F}') \cong \mathrm{Hom}_{\mathrm{Shv}(\mathcal{T}')}(\mathcal{F}', \iota^*\iota_*\mathcal{F}')$$

  is an isomorphism for all $\mathcal{F}' \in \mathrm{Shv}(\mathcal{T}')$.

- *The functor ι^* is an exact functor.*

- *For all $U' \in \mathcal{C}'$, all abelian sheaves $\mathcal{F} \in \mathrm{AbShv}(\mathcal{T})$ and $\mathcal{F}' \in \mathrm{AbShv}(\mathcal{T}')$, we have functorial isomorphisms*

$$H^\bullet_{\mathcal{T}'}(U', \iota^*\mathcal{F}) \cong H^\bullet_{\mathcal{T}}(\iota U', \mathcal{F}) \quad \text{and} \quad H^\bullet_{\mathcal{T}'}(U', F') \cong H^\bullet_{\mathcal{T}}(\iota U', \iota_*\mathcal{F}').$$

See [Tam94, Theorem 3.9.2 & Corollary 3.9.3] for a proof.

Strengthening the conditions in the previous theorem a little, we get the following proposition (c.f. [Tam94, Theorem 3.9.1]).

Theorem 2.55. *Suppose $(\mathcal{C}, \mathcal{T})$ and $(\mathcal{C}', \mathcal{T}')$ are two sites, such that $\mathcal{C}'$ is a full-subcategory of $\mathcal{C}$. and the inclusion functor $\iota : \mathcal{C}' \to \mathcal{C}$ induces a morphism on topologies. Suppose also that any covering $\{U_i \to U\}$ in $\mathcal{T}$ with U_i and U objects in $\mathcal{C}'$ is in fact a covering in the topology $\mathcal{T}'$, and moreover that each object U in $\mathcal{C}$ has a covering $\{U_i \to U\}$ such that $U_i \in \mathcal{C}'$.*

Then, the functors $\iota^* : \mathrm{Shv}(\mathcal{T}) \to \mathrm{Shv}(\mathcal{T}')$ and $\iota_* : \mathrm{Shv}(\mathcal{T}') \to \mathrm{Shv}(\mathcal{T})$ are quasi-inverse equivalences (i.e. $\iota_* \circ \iota^*$ and $\iota^* \circ \iota_*$ are naturally equivalent to the identity functors on their respective categories).

2.10 Comparing Zariski and Étale Cohomology

Suppose S is a scheme. Since any open embedding is étale, we have an obvious fully-faithful functor ι that embeds the Zariski topology on a scheme S into the étale site $\mathrm{Sch}_S^{\text{ét}}$. We know that the corresponding functor $\iota^* : \mathrm{Shv}_{\text{ét}}(S) \to \mathrm{Shv}_{Zar}(S)$ is left-exact. For any Zariski sheaf $\mathcal{F}$, we have an obvious map

$$H^0_{Zar}(S, \mathcal{F}) \cong \mathcal{F}(S) \to \iota^* \mathcal{F}(S) \cong H^0_{\text{ét}}(S, \iota_* \mathcal{F})$$

which must extend to a unique map of δ-functors

$$H^\bullet_{Zar}(S, \mathcal{F}) \to H^\bullet_{\text{ét}}(S, \iota_* \mathcal{F}).$$

This map is rarely an isomorphism.

Suppose now that the Zariski sheaf $\mathcal{F}$ is an $\mathcal{O}_S$-module (here, $\mathcal{O}_S$ denotes the usual Zariski structure sheaf of a scheme). The natural map $\mathcal{O}_S \to \iota^* \mathcal{O}_{S,\text{ét}}$ of Zariski sheaves of rings induces by adjointness a map $\iota_* \mathcal{O}_S \to \mathcal{O}_{S,\text{ét}}$ of étale sheaves of rings. This map is in general not an isomorphism (see the following lemma). However, we get an étale $\mathcal{O}_{S,\text{ét}}$-module

$$\mathcal{F}_{\text{ét}} := \mathcal{O}_{S,\text{ét}} \otimes_{\iota_* \mathcal{O}_S} \iota_* \mathcal{F}.$$

We have thus defined a functor $\mathcal{F} \mapsto \mathcal{F}_{\text{ét}}$ from the category $\mathrm{Mod}_{Zar}(\mathcal{O}_S)$ of Zariski $\mathcal{O}_S$ modules to the category $\mathrm{Mod}_{\text{ét}}(\mathcal{O}_{S,\text{ét}})$ of étale $\mathcal{O}_{S,\text{ét}}$ modules.

Example 2.56. We have

$$(\mathcal{O}_S)_{\text{ét}} \cong \mathcal{O}_{S,\text{ét}} \ \big(\cong \mathbb{G}_{a,S}\big)$$

where the last congruence ignores the ring structure on $\mathcal{O}_{S,\text{ét}}$.

Example 2.57. Recall that $\mathrm{Pic}(S)$ is the group of (Zariski) line bundles on the scheme S. The functor $(-)_{\text{ét}}$ induces a morphism $\mathrm{Pic}(S) \to \mathrm{Pic}_{\text{ét}}(S)$.

The following lemma is a straightforward computation using an explicit construction of the Henselization of a ring.

Lemma 2.58. *Consider a geometric point $\bar{s} : \mathrm{Spec}\, \bar{k} \hookrightarrow S$ with image the point $s \in S$ (with $k = \kappa(s)$). The stalk of the étale sheaf $\mathcal{O}_{S,\text{ét}}$ at $\bar{s}$ is written as $\mathcal{O}_{S,\text{ét},\bar{s}}$, which we consider as simply a ring (by taking global sections). Then,*

$$(\mathcal{O}_{S,s})^{sh} \cong \mathcal{O}_{S,\text{ét},\bar{s}}.$$

Since strict henselization is a flat, we have the following corollary.

Corollary 2.58.1. *The functor* $\mathrm{Mod}_{Zar}(\mathcal{O}_S) \to \mathrm{Mod}_{\text{ét}}(\mathcal{O}_{S,\text{ét}}), \mathcal{F} \mapsto \mathcal{F}_{\text{ét}}$ *is exact.*

Proposition 2.59. *Suppose $h : X \to S$ is étale, and $\mathcal{F}$ a quasi-coherent Zariski $\mathcal{O}_S$-module. Then $h^* \mathcal{F}_{\text{ét}} \cong (h^* \mathcal{F})_{\text{ét}}$.*

In other words, pushing forward to the étale site and then pulling back by h to the étale site $\mathrm{Sch}_X^{\text{ét}}$ *is the same as first pulling back $\mathcal{F}$ to a Zariski sheaf on X and then pushing forward to the étale site on X.*

See [Con, Example 1.2.6.1] for the proof.

Remark 2.60. While the proof is not important, the following description of $\mathcal{F}_{\text{ét}}$ for $\mathcal{F}$ quasi-coherent arising from the proof is useful: for any étale S-scheme U, we have

$$\mathcal{F}_{\text{ét}}(U \xrightarrow{h} S) = \Gamma(U, h^* \mathcal{F}).$$

As a consequence of this proposition, as well as by fpqc descent of quasi-coherent sheaves (recall étale covers are fpqc covers as well), we have the following.

Corollary 2.60.1. *The functor $\mathcal{F} \mapsto \mathcal{F}_{\text{ét}}$ yields an isomorphism* $\mathrm{Pic}(S) \cong \mathrm{Pic}_{\text{ét}}(S)$ *of groups. In particular, it follows that*

$$H^1_{Zar}(S, \mathbb{G}_m) \cong H^1_{\text{ét}}(S, \mathbb{G}_m).$$

Since $\mathrm{Mod}_{Zar}(\mathcal{O}_S) \to \mathrm{Mod}_{\text{ét}}(\mathcal{O}_{S,\text{ét}}), \mathfrak{F} \mapsto \mathfrak{F}_{\text{ét}}$ is an exact functor, we can compose the Zariski-étale comparison morphism to define another δ-functorial comparison morphism

$$H^\bullet_{Zar}(S, \mathcal{F}) \to H^\bullet_{\text{ét}}(S, \mathcal{F}_{\text{ét}})$$

for all $\mathcal{F} \in \mathrm{Mod}_{Zar}(\mathcal{O}_S)$.

Theorem 2.61. *The above comparison morphism*

$$H^{\bullet}_{Zar}(S,\mathcal{F}) \to H^{\bullet}_{\text{ét}}(S,\mathcal{F}_{\text{ét}})$$

is an isomorphism whenever $\mathcal{F}$ is a quasi-coherent Zariski $\mathcal{O}_S$-module.

Proof. We follow the proof given in [Tam94]. Fix a quasi-coherent Zariski $\mathcal{O}_S$-module $\mathcal{F}$. The composition of functors

$$\mathrm{AbShv}_{\text{ét}}(S) \xrightarrow{\iota^*} \mathrm{AbShv}_{Zar}(S) \xrightarrow{\Gamma_{Zar}} \mathrm{Ab}$$

is the global sections functor on étale sheaves. If $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(S)$ is an injective sheaf, then it is in particular flabby, and so by Corollary 2.53.1 it is Γ_{Zar}-acyclic. The Grothendieck spectral sequence applied to the object $\mathcal{F}_{\text{ét}} \in \mathrm{Mod}_{\text{ét}}(\mathcal{O}_{S,\text{ét}}) \subset \mathrm{AbShv}_{\text{ét}}(S)$ is then

$$H^q_{Zar}(S, R^p\iota^*(\mathcal{F}_{\text{ét}})) \Rightarrow H^{p+q}_{\text{ét}}(S,\mathcal{F}_{\text{ét}}).$$

Notice that, if we can show that $R^p\iota^*(\mathcal{F}_{\text{ét}}) = 0$, then the spectral sequence above immediately forces all the edge morphisms

$$H^q_{Zar}(S,\mathcal{F}) \to H^q_{\text{ét}}(S,\mathcal{F}_{\text{ét}})$$

are isomorphisms, where we use the obvious fact that $\iota^*(\mathcal{F}_{\text{ét}}) \cong \mathcal{F}$ for any sheaf $\mathcal{F}$. However, by Theorem 2.52, we know that $R^p\iota^*(\mathcal{F}_{\text{ét}})$ is the sheafification of the pre-sheaf

$$U \mapsto H^p_{\text{ét}}(U,\mathcal{F}_{\text{ét}}) \cong H^p_{\text{ét}}(U,\mathcal{F}_{\text{ét}}|_U)$$

for any open embedding $U \hookrightarrow S$. By Proposition 2.59, we have $\mathcal{F}_{\text{ét}}|_U \cong (\mathcal{F}|_U)_{\text{ét}}$. Since affine opens form a basis for the Zariski topology, it suffices to show that $H^p_{\text{ét}}(U,(\mathcal{F}|_U)_{\text{ét}}) = 0$ for any affine open U of S.

So suppose $U = \operatorname{Spec} A$ is affine, and $\mathcal{F}|_U = \tilde{M}$. Let T be the site whose underlying category is the full-subcategory of all affine schemes in $\mathrm{Sch}^{\text{ét}}_U$, and whose topology is the induced topology. Notice that the inclusion morphism $T \hookrightarrow \mathrm{Sch}^{\text{ét}}_U$ satisfies the conditions of Theorem 2.54 (since open embeddings are étale). Thus, we have

$$H^p_{\text{ét}}(U,(\mathcal{F}|_U)_{\text{ét}}) = H^p_T(U,(\mathcal{F}|_U)_{\text{ét}})$$

(we abuse notation by writing $(\mathcal{F}|_U)_{\text{ét}}$ for the induced sheaf in $\mathrm{AbShv}(T)$ as well). Thus it suffices to work with affine schemes only. We claim that $(\mathcal{F}|_U)_{\text{ét}}$ is flabby as a sheaf in $\mathrm{AbShv}(T)$; it suffices to show this for Čech cohomology, i.e. we want to show that

$$\check{H}^p_T(\mathcal{U},(\mathcal{F}|_U)_{\text{ét}}) = 0$$

for all $p \geq 1$ and all coverings $\mathcal{U}$ of U in T. Since affine schemes are quasi-compact, we may suppose $\mathcal{U}$ is a finite covering. Let $\mathcal{U} = \{U_1,\dots,U_n\}$ where each U_i is affine, $U_i \to U$ is étale, and $\bigsqcup_i U_i \to U$ is surjective. However, as there are only finitely many affines, we see that $\bigsqcup_i U_i$ is also affine, and moreover

$$\check{H}^p_T(\mathcal{U},(\mathcal{F}|_U)_{\text{ét}}) = \check{H}^p_T\left(\bigsqcup_i U_i \to U, (\mathcal{F}|_U)_{\text{ét}}\right).$$

Thus, we have reduced to the case that $\mathcal{U} = \{V \to U\}$ where V is affine and étale over U, and that $V \to U$ is surjective. Let $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, so that B is an étale algebra over A. Writing $\mathcal{F}|_U = \tilde{M}$ for an A-module M, the Čech complex is the complex

$$M \otimes_A B \to M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A B \otimes_A B \to \cdots$$

(here, we use the fact that $\tilde{M}(\operatorname{Spec} B) \cong M \tilde{\otimes}_A B$). However, as $V \to U$ is surjective and étale (in particular flat), B/A is faithfully flat. It follows that the above complex is exact. Hence the Čech cohomology groups vanish, yielding the theorem. ☐

Corollary 2.61.1. *If S is affine and $\mathcal{F}$ a quasi-coherent Zariski $\mathcal{O}_S$-module, then $H^p_{\text{ét}}(S,\mathcal{F}_{\text{ét}}) = 0$ for all $p \geq 1$. In particular, we have that $H^p_{\text{ét}}(S,\mathbb{G}_{a,S}) = H^p_{\text{ét}}(S,\mathcal{O}_{S,\text{ét}}) = 0$ for all $p \geq 1$ and all affines S.*

2.11 Kummer and Artin-Schreier Sequences

Fix a scheme S. Recall the representable abelian sheaf $\mathbb{G}_m \in \mathrm{AbShv}_{\text{ét}}(S)$ given by $\mathbb{G}_m(U) = \mathcal{O}_U(U)^\times$. For any $n \in \mathbb{N}$, the map $x \mapsto x^n$ on $\mathbb{G}_m(U)$ induces a morphism $\mathbb{G}_m \to \mathbb{G}_m$; the kernel of this map is the representable sheaf μ_n. This sheaf is representable by the finite flat commutative group scheme $\operatorname{Spec} k[t]/\langle t^n - 1\rangle$.

Definition. We say that $n \in \mathbb{N}$ is *invertible* on a scheme S, or is a *unit* on S, if n is non-zero in the residue field $\kappa(x)$ for all $x \in S$.

Proposition 2.62 (Kummer Exact Sequence)**.** *We have the exact sequence*

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m$$

of étale abelian sheaves. If n is a unit on S, then

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \to 0$$

is an exact sequence, called the Kummer sequence.

Proof. The first exact sequence is obvious since μ_n is by definition the kernel of $\mathbb{G}_m \to \mathbb{G}_m$. Now suppose n is a unit on S. To see that $\mathbb{G}_m \to \mathbb{G}_m, x \mapsto x^n$ is surjective, we need to show that for any étale S-scheme X and any element $u \in \mathbb{G}_m(X)$, there exists an étale covering $\{U_i\}$ of X and sections $a_i \in \mathbb{G}_m(U_i)$ such that $a_i^n = u|_{U_i}$.

Take an affine open covering $\{\operatorname{Spec} A_i\}$ of X, and let $u_i := u|_{\operatorname{Spec} A_i} \in A_i$. Since n is a unit on S, it must be non-zero in A_i, and so the algebra

$$\tilde{A}_i = \frac{A[t]}{\langle t^n - u_i\rangle}$$

is étale over A_i (the derivative of $t^n - u$ is nt^{n-1} which is non-zero). The inclusions $A_i \hookrightarrow \tilde{A}_i$ induce an étale cover $\{\operatorname{Spec} \tilde{A}_i\}$ of X. Now, $\tilde{a}_i = [t] \in \tilde{A}_i$ satisfies $\tilde{a}_i^n = u_i|_{\operatorname{Spec} \tilde{A}_i} = u|_{\operatorname{Spec} A_i}$, as required. ☐

Notation. For an abelian group A, we write $A[n]$ to be the n-torsion points of A.

Taking the long exact sequence in étale cohomology associated to the Kummer exact sequence, we get the following corollary.

Corollary 2.62.1. *If n is a unit on S, then we have the exact sequence*

$$0 \to \frac{\mathcal{O}_S(S)^\times}{(\mathcal{O}_S(S)^\times)^n} \to H^1_{\text{ét}}(S, \mu_n) \to \mathrm{Pic}_{\text{ét}}(S)[n] \cong \mathrm{Pic}(S)[n] \to 0.$$

This allows us to compute $H^1_{\text{ét}}(S, \mu_n)$ in some special cases.

Corollary 2.62.2. *If $S = \operatorname{Spec} A$ for A a local ring, and if n is invertible in A, then*

$$H^1_{\text{ét}}(S, \mu_n) \cong A^\times/(A^\times)^n.$$

Corollary 2.62.3. *If S is a reduced proper scheme over a separably closed field k, and if $n \neq 0$ in k, then*

$$H^1_{\text{ét}}(S, \mu_n) \cong Pic(S)[n].$$

Remark 2.63. Just as in analytic theory the exponential exact sequence is very useful, the Kummer sequence is analogously very useful in applications to cohomology.

Suppose now p is prime and the characteristic of S is p, i.e. $p \cdot \mathcal{O}_{S,s} = 0$ for all $s \in S$. In characteristic p the map $t \mapsto t^p - t$ is additive, and it defines a map $\wp : \mathbb{G}_a \to \mathbb{G}_a$ in $\mathrm{AbShv}_{\text{ét}}(S)$.

Proposition 2.64 (Artin-Schreier Sequence)**.** *If S is a scheme with characteristic p, then we have the exact sequence*

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}} \to \mathbb{G}_a \xrightarrow{\wp} \mathbb{G}_a \to 0.$$

Proof. It is clear that $\underline{\mathbb{Z}/p\mathbb{Z}} \hookrightarrow \mathbb{G}_a$, since S has characteristic p. We need to show that $\ker \wp = \underline{\mathbb{Z}/p\mathbb{Z}}$ and $\mathbb{G}_a \to \mathbb{G}_a$ is surjective. That $\ker \wp = \underline{\mathbb{Z}/p\mathbb{Z}}$ follows from the fact that $s^p - s = \prod_{j \in \mathbb{Z}/p\mathbb{Z}}(s - j)$, and so $s \in \ker \wp(X)$ if and only if $s|_{X'} \in \mathbb{Z}/p\mathbb{Z}$ as X' runs through all connected components of X.

Now suppose $X \in \mathrm{Sch}_S^{\text{ét}}$ and $s \in \mathbb{G}_a(X)$. Take an affine open covering $\{\operatorname{Spec} A_i\}$ of X, with $s_i := s|_{\operatorname{Spec} A_i} \in A_i$. Define $\tilde{A}_i = A[x]/\langle x^p - x - s_i\rangle$; since $(x^p - x - s_i)' = -1$ it follows that $\tilde{A}_i$ is étale over A, and so $\{\operatorname{Spec} \tilde{A}_i\}$ is an étale cover of X. Taking $\tilde{s}_i = [t] \in \tilde{A}_i$, we have $\wp(\tilde{s}_i) = s_i|_{\operatorname{Spec} \tilde{A}_i} = s|_{\operatorname{Spec} \tilde{A}_i}$. Hence $\wp$ is surjective. ☐

Notation. If A is a group and $\phi : A \to A$ an endomorphism of the group, we write A^{ϕ} for the subgroup of A fixed pointwise by ϕ, or equivalently, the invariant subgroup of A given by the action of the cyclic group $\langle \phi \rangle \leq \mathrm{End}(A)$.

As before, the long exact sequence in étale cohomology associated to the Artin-Schreier sequence yields the following corollaries.

Corollary 2.64.1. *If S has characteristic p, we have the exact sequence*

$$0 \to \frac{\mathcal{O}_S(S)}{\wp(\mathcal{O}_S(S))} \to H^1_{\text{ét}}(S, \underline{\mathbb{Z}/p\mathbb{Z}}) \to H^1_{\text{ét}}(S, \mathcal{O}_S)^{\mathrm{Frob}} \to 0.$$

Corollary 2.64.2. *If $X = \operatorname{Spec} A$, with $\operatorname{char} A = p$, then*

$$H^q_{\text{ét}}(X, \underline{\mathbb{Z}/p\mathbb{Z}}) = \begin{cases} A/\wp(A) & q = 1 \\ 0 & q > 1. \end{cases}$$

Proof. As $\mathcal{O}_{\operatorname{Spec} A}$ is quasi-coherent over $\operatorname{Spec} A$, its higher cohomology groups vanish. The result follows. □

Corollary 2.64.3. *If k is a separably closed field of characteristic p and X is a reduced proper k-scheme, then*

$$H^1_{\text{ét}}(S, \underline{\mathbb{Z}/p\mathbb{Z}}) \cong H^1_{\text{ét}}(S, \mathcal{O}_S)^{\mathrm{Frob}}.$$

Remark 2.65. In all of these corollaries, we use the fact that $\mathcal{O}_S$ is a Zariski quasi-coherent sheaf over S, and so the Zariski sheaf cohomology and the étale sheaf cohomology coincide.

Definition. Let S be a scheme. An abelian étale sheaf $\mathcal{F}$ on $\mathrm{Sch}_S^{\text{ét}}$ is a *torsion sheaf* if every section is locally killed by a non-zero integer, or equivalently, that all the stalks are torsion abelian groups. Write $\mathrm{AbShv}_{\text{ét}}^{tors}(S)$ for the full subcategory of torsion abelian étale sheaves on $\mathrm{Sch}_S^{\text{ét}}$.

If all sections are locally killed by powers of a prime p, then $\mathcal{F}$ is a *p-power torsion sheaf.*

We now state a hard theorem that results from the Artin-Schreier and Kummer sequences. For a reference to the proof see [Con, Theorem 1.2.7.3].

Theorem 2.66. *Let X be a separated finite type scheme of dimension ≤ 1 over a separably closed field k. For any torsion sheaf $\mathcal{F}$ on X, the groups $H^i_{\text{ét}}(X, \mathcal{F})$ vanish for $i \geq 3$. If $\mathcal{F}$ is constructible, then $H^i_{\text{ét}}(X, \mathcal{F})$ is finite for $i \leq 2$.*

Moreover, we have $H^2_{\text{ét}}(X, \mathcal{F}) = 0$ if either

- *X is affine and the torsion orders of $\mathcal{F}$ are not divisible by* $\operatorname{char}(k)$*; or*
- *X is proper,* $\operatorname{char}(k) = p > 0$*, and $\mathcal{F}$ is a p-power torsion sheaf.*

3 Cohomology of Torsion Sheaves

3.1 Compatibility of Limits and Cohomology

In this subsection, we write down extremely important results on limits/colimits of schemes and the behaviour of étale cohomology with such limits. These results are extremely powerful tools to reduce proving a theorem to only proving a very special case. The proof of most of the big results that follow after this subsection will almost always use one of the theorems listed here, though we will omit all proofs in subsequent sections.

Suppose I is a filtered category, and consider a contravariant functor $i \mapsto X_i$ from $I^op \to \mathrm{Sch}$. We have the following theorem.

Theorem 3.1. *Suppose for each $i \to j$ in I the morphism $X_j \to X_i$ is affine. Then, the functor*

$$\mathrm{Sch}^{op} \to \mathrm{Set}, \quad Y \mapsto \lim_{i \in I} X_i(Y)$$

is representable by a scheme X (i.e. the limit of the diagram $I^{op} \to \mathrm{Sch}$ exists).

Proof. We follow the proof in [Tam94]. Fix some $i_0 \in I$. For each $i \to i_0$, since $X_{i_0} \to X_i$ is affine, there is an X_{i_0}-isomorphism of schemes

$$X_i \cong \underline{\mathrm{Spec}}\, \mathcal{A}_i$$

where $\mathcal{A}_i$ is a quasi-coherent sheaf of $\mathcal{O}_{X_{i_0}}$-algebras. Then, we have a filtered system of quasi-coherent sheaves $\mathcal{A}_i$ of $\mathcal{O}_{X_{i_0}}$-algebras indexed by the over-category I/i_0 of arrows $i \to i_0$. Since colimits of quasi-coherent sheaves always exist, we can find a quasi-coherent $\mathcal{O}_{X_{i_0}}$-algebra $\mathcal{A}$ such that

$$\mathcal{A} = i \overset{\mathrm{colim}}{\to} i_0 \mathcal{A}_i.$$

Define $X = \underline{\mathrm{Spec}}\, \mathcal{A}$. The morphisms $\mathcal{A}_i \to \mathcal{A}$ induce morphisms $u_i : X \to X_i$ for all $i \in I$ such that $i \to i_0$. Since I is filtered, it follows that these morphisms extend to give morphisms $u_i : X \to X_i$ for all $i \in I$. Then X is the limit of the diagram (X_i). □

We get some preliminary properties.

Proposition 3.2. *Let X and $(X_i)_{i \in I}$ be as above. Let $u_i : X \to X_i$ be the canonical morphisms.*

1. *Each morphism u_i is an affine morphism.*
2. *The underlying topological space of X is canonically homeomorphic to the limit of the underlying topological spaces X_i.*
3. *The structure sheaf $\mathcal{O}_X$ is canonically isomorphic to $\mathrm{colim}_{i \in I} u_i^{-1} \mathcal{O}_{X_i}$.*
4. *If all the $(X_i)_{i \in I}$ are S-schemes and the morphisms $X_j \to X_i$ are S-morphisms, then X is also the limit of the diagram (X_i) in the category Sch_S.*
5. *If $T \to S$ is given, then*
$$\lim_{i \in I} (X_i \times_S T) \cong X \times_S T.$$

Now suppose for each $i \in I$ we have an abelian sheaf $\mathcal{F}_i \in \mathrm{AbShv}_{\text{ét}}(X_i)$ in such a way that for each arrow $i \to j$ in I the sheaf F_j is the inverse image of $\mathcal{F}_i$ under the corresponding morphism $f_{ij} : X_j \to X_i$. Since $f_{ij} \circ u_j = u_i$, it follows that $u_i^* \mathcal{F}_i = u_j^* \mathcal{F}_j$. Thus $u_i^* \mathcal{F}_i$ is independent of $i \in I$, and so we can define a sheaf $\mathcal{F} = u_i^* \mathcal{F}_i \in \mathrm{AbShv}_{\text{ét}}(X)$. Now, each u_i induces a canonical morphism of δ-functors

$$H^\bullet_{\text{ét}}(X_i, \mathcal{F}_i) \to H^\bullet_{\text{ét}}(X, \mathcal{F})$$

which is compatible with the canonical morphisms $H^\bullet_{\text{ét}}(X_i, \mathcal{F}_i) \to H^\bullet_{\text{ét}}(X_j, \mathcal{F}_j)$ induced by the arrow $i \to j$ in I. Therefore, we obtain a canonical homomorphism

$$\operatorname*{colim}_{i \in I} H^q(X_i, \mathcal{F}_i) \to H^q(X, \mathcal{F})$$

for all $q \geq 0$. We have the following two hard theorems.

Theorem 3.3. *If X is a quasi-compact and quasi-separated scheme, and $\{\mathcal{F}_i\}$ a filtered directed system of abelian sheaves on $X_{\text{ét}}$ with colimit $\mathcal{F}$, then the map*

$$\mathrm{colim}_i H^n_{\text{ét}}(X, \mathcal{F}_i) \to H^n_{\text{ét}}(X, \mathcal{F}) = H^n_{\text{ét}}(X, \mathrm{colim}_i \mathcal{F}_i).$$

Theorem 3.4. *Suppose I is filtered and $i \mapsto X_i$ a contravariant functor $I^{op} \to \mathrm{Sch}_S$. Assume as before that $X_j \to X_i$ is affine for all $i \to j$ so that $X = \lim_i X_i$ exists. Assume also that each X_i is quasi-compact and quasi-separated. Suppose also that for each $i \in I$ we have $\mathcal{F}_i \in \mathrm{AbShv}_{\text{ét}}(X_i)$ is given such that for each $i \to j$ with corresponding morphism $f_{ij} : X_j \to X_i$, we have $\mathcal{F}_j = f_{ij}^{-1}\mathcal{F}_i$ (not just $\mathcal{F}_j = f_{ij}^{*}\mathcal{F}_i$). Let $\mathcal{F} = u_i^{-1}\mathcal{F}_i$ where $u_i : X \to X_i$ are the canonical morphisms; as before $\mathcal{F}$ is independent of i. Then, the canonical homomorphism*

$$\operatorname*{colim}_{i \in I} H^q(X_i, \mathcal{F}_i) \to H^q(X, \mathcal{F})$$

is an isomorphism for all $q \geq 0$.

For a proof of both theorems see [Con]. We summarize the key points here. The crucial step is to notice that if U is quasi-compact and quasi-separated over S, then any étale cover of U over S admits a finite refinement $\{U_i\}$ with each étale map $U_i \to U$ quasi-compact and quasi-separated. One can then use this fact and the filtered assumption on $\mathcal{F}_i$ to compute that the canonical homomorphism between the *Čech cohomology* in degree 0 is an isomorphism. Next, one inducts on the degree and does a similar computation as above to prove an isomorphism in Čech cohomology for all degrees, by using the fact that in a finite étale cover $\{U_i\}$ of U where each $U_i \to U$ is quasi-compact and quasi-separated, all finite products are quasi-compact and quasi-separated. Finally, an argument involving the Čech to étale cohomology spectral sequence shows that the commuting of $\check{H}^{\bullet}_{\text{ét}}$ with colimits implies the commuting of $H^{\bullet}_{\text{ét}}$ with colimits. For the proof of the degree 0 isomorphism in Čech cohomology in the second theorem above, one also needs to use a clever argument involving topoi of subcategories of $\mathrm{Sch}_S^{\text{ét}}$. For details see [Con, Theorem 1.3.2.3].

We now list some applications of the above theorem. The most important of such applications is the computation of fibres.

Corollary 3.4.1. *Suppose $f : X \to Y$ is a quasi-compact and quasi-separated morphism of schemes, and $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(X)$. Let $\bar{y}$ be a geometric point of Y, and set $\bar{Y} := \operatorname{Spec} \mathcal{O}_{Y,\text{ét},\bar{y}}$ (here, the stalk of the étale sheaf of rings $\mathcal{O}_{Y,\text{ét}}$ can be identified as a ring). Define $\bar{X} := \bar{Y} \times_Y X$, and let $\bar{\mathcal{F}}$ the inverse image of $\mathcal{F}$ under $\bar{X} \to X$. Then the canonical homomorphism*

$$\big(R^q f_*(\mathcal{F})\big)_{\bar{y}} \to H^q_{\text{ét}}(\bar{X}, \bar{\mathcal{F}})$$

is an isomorphism for all $q \geq 0$.

Proof. Since $R^q f_*$ is the sheafification of the pre-sheaf $U \mapsto H^q_{\text{ét}}(U \times_Y X, \mathcal{F})$, we see that

$$\big(R^q f_*(\mathcal{F})\big)_{\bar{y}} = \operatorname*{colim}_{Y' \to Y \text{affine étale nbd of } \bar{y}} H^q_{\text{ét}}(X \times_Y Y', \mathcal{F}).$$

However, we have that

$$\lim_{Y' \to Y \text{affine étale nbd of } \bar{y}} X \times_Y Y' \cong X \times_Y \left(\lim_{Y' \to Y \text{affine étale nbd of } \bar{y}} Y' \right) \cong X \times_Y \bar{Y}.$$

The result then follows from the theorem, noting that $X \times_Y Y'$ for $Y' \to Y$ affine étale is quasi-compact and quasi-separated over Y' since $f : X \to Y$ is quasi-compact and quasi-separated. ☐

A simple computation of the cohomology groups $H^q_{\text{ét}}(\bar{X}, \bar{\mathcal{F}})$ yields the following result.

Corollary 3.4.2. *Let $f : X \to Y$ a finite morphism of schemes and $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(X)$. Then*

1. *For each geometric point $\bar{y}$ of Y, we have*

$$(f_*\mathcal{F})_{\bar{y}} \cong \prod_{\bar{x} \in f^{-1}(\bar{y})} \mathcal{F}_{\bar{x}}.$$

2. *$f_*\mathcal{F}$ commutes with arbitrary base change, i.e. if $g : Y' \to Y$ is arbitrary and we have a pull-back square*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \big\downarrow{\scriptstyle g'} & \lrcorner & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y, \end{array}$$

then we have a natural isomorphism $g^*(f_*\mathcal{F}) \cong f'_*(g'^*\mathcal{F})$.

3. $R^q f_*(\mathcal{F}) = 0$ *for all* $q \geq 1$. *Thus* f_* *is exact when it is finite, and we have canonical isomorphisms*

$$H^q_{\text{ét}}(Y, f_*\mathcal{F}) \cong H^q_{\text{ét}}(X, \mathcal{F}).$$

This corollary also relies on the following lemma, which can be established by a simple computation.

Lemma 3.5. *Suppose A is strictly Henselian. Let $s : \operatorname{Spec} A/\mathfrak{m} \to \operatorname{Spec} A$ be the closed point of $\operatorname{Spec} A$, viewed as a geometric point. Then, there is a functorial isomorphism*

$$\mathcal{F}_s \cong H^0(\operatorname{Spec} A, \mathcal{F}) \quad \textit{and} \quad H^q(\operatorname{Spec} A, \mathcal{F}) = 0 \ \ \forall\, q \geq 1,$$

for all $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(\operatorname{Spec} A)$.

As another application of the above two theorems, we describe the Galois action on an abelian sheaf over a k-scheme X. Let $\bar{k}$ be the separable algebraic closure of k.

Proposition 3.6. *Let $f : X \to k$ be some map, and let $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(X)$. The $\mathrm{Gal}(\bar{k}/k)$-action coming from the structure of $R^q f_*(\mathcal{F})$ as an étale k-sheaf induces a $\mathrm{Gal}(\bar{k}/k)$-action on the geometric fibre $R^q f_*(\mathcal{F})_{\bar{k}}$. This action coincides with the action of $\mathrm{Gal}(\bar{k}/k)$ on the cohomology $H^q_{\text{ét}}(X \times_k \bar{k}, \mathcal{F}_{\bar{k}})$ (induced by cohomological pull-back by $(1 \times \sigma) : X \times_k \bar{k} \to X \times_k \bar{k}$ for $\sigma \in \mathrm{Gal}(\bar{k}/k)$), under the isomorphism*

$$\big(R^q f_*(\mathcal{F})\big)_{\bar{k}} \cong H^q_{\text{ét}}(X \times_k \bar{k}, \mathcal{F}_{\bar{k}}).$$

We finish this section with a description of Noetherian descent.

Definition. For P a set of primes, a *P-sheaf* is a torsion sheaf whose torsion-orders are not divisible by primes in P.

Theorem 3.7 (Noetherian Descent)**.** *Let S be a quasi-compact and quasi-separated scheme and P a set of primes.*

1. *There exists an inverse system of Noetherian schemes $\{S_i\}$ with affine transition maps such that $S \cong \lim_i S_i$. Moreover, the S_i may be taken to be finite type over $\mathbb{Z}$.*
2. *For such a system $\{S_i\}$ as above, any P-sheaf $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(S)$ is a colimit of P-sheaves $\{\mathcal{F}_\lambda\} \subset \mathrm{AbShv}_{\text{ét}}(S)$ such that each $\mathcal{F}_\lambda$ is a pull-back under the canonical morphism $S \to S_i$ of a constructible P-sheaf on some $\mathrm{AbShv}_{\text{ét}}(S_i)$.*
3. *For any quasi-separated finite-type S-scheme X there exists a closed immersion $X \hookrightarrow \bar{X}$ into a finitely presented S-scheme $\bar{X}$. If X is separated over S, then $\bar{X}$ can be chosen to be separated over S as well.*
4. *For X a quasi-separated finite-type S-scheme, and for $\{S_i\}$ a Noetherian cover as in (1), there exists i_0 and a finite type S_{i_0}-scheme $\bar{X}_{i_0}$ such that $\bar{X} \cong \bar{X}_{i_0} \times_{S_{i_0}} S$. Defining $\bar{X}_i := S_i \times_{S_{i_0}} \bar{X}_{i_0}$ for all i such that $i_0 \to i$, the map $\bar{X}_i \to S_i$ is separated (resp. proper) for some large enough i if and only if $\bar{X} \to S$ is separated (resp. proper).*

As usual the proof is omitted.

3.2 Smooth and Proper Base Change

Consider a Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ \big\downarrow{\scriptstyle g'} & \lrcorner & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S, \end{array}$$

and a sheaf $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(X)$. Then, there is a natural *base-change map*

$$g^*\big(R^q f_*(\mathcal{F})\big) \to R^q f'_*({g'}^*\mathcal{F}).$$

There are two perspectives to take on the construction of this base-change map.

- The sheafification of the pull-back maps
$$H^q_{\text{ét}}(f^{-1}(U),\mathcal{F}) \to H^q_{\text{ét}}\left({f'}^{-1}\big(g^{-1}(U)\big), {g'}^*\mathcal{F}\right) = H^q_{\text{ét}}\left({g'}^{-1}\big(f^{-1}(U)\big), {g'}^*\mathcal{F}\right)$$
for étale $U \to S$ yields a map
$$R^q f_*\mathcal{F} \to (g_* \circ R^q f'_* \circ {g'}^*)(\mathcal{F}).$$
By the (g^*, g_*)-adjunction, we then get a map
$$(g^* \circ R^q f_*)(\mathcal{F}) \to (R^q f'_* \circ {g'}^*)(\mathcal{F}).$$
- We have a natural map $\mathcal{F} \mapsto g'_*({g'}^*\mathcal{F})$ adjoint to the identity on ${g'}^*\mathcal{F}$. Composing with f_*, we get a natural map $f_*\mathcal{F} \to f_* g'_*({g'}^*\mathcal{F}) = g_* f'_*({g'}^*\mathcal{F})$. By the (g^*, g_*)-adjunction, we then get a map
$$(g^* \circ f_*)(\mathcal{F}) \to (f'_* \circ {g'}^*)(\mathcal{F}).$$
This then induces a map of (universal) δ-functors
$$g^* R^\bullet f_* \to R^\bullet f'_* \circ {g'}^*,$$
which is the base-change map.

In many nice cases, this base-change map is an isomorphism. The significance of such base-change isomorphisms is that they give isomorphisms on cohomology, since recall that the higher direct images are sheafifications of cohomology.

Theorem 3.8 (Proper Base Change)**.** *Let S be a scheme, and $f : X \to S$ a proper map. Let $\mathcal{F} \in \text{AbShv}_{\text{ét}}(X)$ a torsion sheaf. For any Cartesian square*
$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ \downarrow{\scriptstyle g'} & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S, \end{array}$$
the natural base-change map
$$g^*\big(R^q f_*(\mathcal{F})\big) \to R^q f'_*({g'}^*\mathcal{F})$$
is an isomorphism for all $q \geq 0$.

We omit the proof of this (c.f. [Con, Theorem 1.3.4.1] or [Sta22, Tag 095S]).

Corollary 3.8.1. *Let $f : X \to S$ be proper, $\mathcal{F} \in \text{AbShv}_{\text{ét}}(X)$ a torsion sheaf, and let $\bar{s} : \operatorname{Spec} \bar{k} \to S$ a geometric point of S. Denote $X \times_{S,\bar{s}} \operatorname{Spec} \bar{k} =: X_{\bar{s}}$. Then, for all $q \geq 0$, the fibre of $R^q f_*(\mathcal{F})$ at $\bar{s}$ is $H^q_{\text{ét}}(X_{\bar{s}}, \mathcal{F}|_{X_{\bar{s}}})$.*

Corollary 3.8.2. *Let $f : X \to S$ be a proper morphism all of whose fibres have dimension $\leq n$. Then, $R^q f_*(\mathcal{F}) = 0$ for all $q > 2n$ and for all torsion sheaves $\mathcal{F} \in \text{AbShv}_{\text{ét}}(X)$.*

The following corollary is very important, for instance for the inclusions $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ or $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$.

Corollary 3.8.3. *Suppose k and k' are separably closed fields with $k \subset k'$. Let X be a proper k-scheme and $X' := X \times_k k'$. Write $p : X' \to X$ for the projection morphism. Then,*
$$H^q(X, \mathcal{F}) \cong H^q\big(X', p^{-1}(\mathcal{F})\big)$$
for all $q \geq 0$ and all torsion sheaves $\mathcal{F} \in \text{AbShv}_{\text{ét}}(X)$.

We now move onto the smooth base change theorem; for the proof see [Con, Theorem 1.3.5.2] (though there is a typo in his statement of the theorem) or [Sta22, Tag 0EYQ].

Theorem 3.9 (Smooth Base Change)**.** *Suppose S a scheme and $f : X \to S$ a quasi-compact and quasi-separated morphism. Let $\mathcal{F} \in \text{AbShv}_{\text{ét}}(X)$ be a torsion sheaf whose torsion-orders are invertible on S. Consider a Cartesian square*
$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ \downarrow{\scriptstyle g'} & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S, \end{array}$$
with $S' = \lim_i S_i$ where $\{S_i\}$ is an inverse system of smooth S-schemes such that the transition maps $S_i \to S_j$ are affine. Then, the natural base-change map
$$g^*\big(R^q f_*(\mathcal{F})\big) \to R^q f'_*({g'}^*\mathcal{F})$$
is an isomorphism for all $q \geq 0$.

Remark 3.10. Notice that g need not actually be a smooth map, but more generally can be the limit of smooth maps.

Corollary 3.10.1. *If K/k is an extension of separably closed fields and X is a k-scheme, then we have natural isomorphisms*

$$H^q_{\text{ét}}(X, \mathcal{F}) \cong H^q_{\text{ét}}(X_K, \mathcal{F}|_{X_K})$$

for all $q \geq 0$ and all torsion sheaves $\mathcal{F}$ whose torsion orders are relatively prime to the characteristic.

Example 3.11. It is necessary that the torsion orders of $\mathcal{F}$ are invertible on S. Suppose k is an algebraically closed field of characteristic $p > 0$. Let K/k be a non-trivial field extension; since k is perfect K is the limit of smooth k-algebras. We claim that the base-change morphism

$$H^1_{\text{ét}}(\mathbb{A}^1_k, \underline{\mathbb{Z}/p\mathbb{Z}}) \to H^1_{\text{ét}}(\mathbb{A}^1_K, \underline{\mathbb{Z}/p\mathbb{Z}})$$

is not an isomorphism. Indeed, Corollary 2.64.2 implies that for any field F of characteristic p, we have

$$H^1_{\text{ét}}(\mathbb{A}^1_F, \mathbb{Z}/p\mathbb{Z}) \cong \operatorname{coker}\big(F[t] \to F[t], f \mapsto f^p - f\big).$$

Thus, one checks that the base-change morphism is the obvious map

$$\frac{k[t]}{\{f^p - f : f \in k[t]\}} \hookrightarrow \frac{K[t]}{\{f^p - f : f \in K[t]\}},$$

and this is not an isomorphism since it isn't surjective (at^{p-1} is not in the image of the base-change morphism whenever $a \in K \setminus k$).

3.3 More on the Shriek-Pushforward, and Cohomology with Compact Supports

For $f : X \to S$ étale, recall the functor $f_! : \text{AbShv}_{\text{ét}}(X) \to \text{AbShv}_{\text{ét}}(S)$ that is left-adjoint to to the pull-back. We write down a few more properties (see [FK88, Lemma 8.2] for proofs).

Lemma 3.12. *Suppose $f : X \to S$ étale.*

1. *$f_!$ sends constructible sheaves to constructible sheaves.*
2. *Suppose $y : \operatorname{Spec} \bar{k} \to Y$ be a geometric point of Y. Then there is a canonical isomorphism*

$$f_!(\mathcal{F})_y \cong \prod_{x : \operatorname{Spec} \bar{k} \to X, f(x) = y} \mathcal{F}_x.$$

3. *$f_!$ is exact and compatible with arbitrary colimits.*
4. *If $g : Y \to Z$ is étale, then $(g \circ f)_! = g_! \circ f_!$ as functors (on the nose).*
5. *If we are given arbitrary $g : Z \to Y$, and suppose we have the induced projection maps $\hat{f} : X \times_Y Z \to Z$ and $\hat{g} : X \times_Y Z \to X$. Then the functors $g^* \circ f_!$ is naturally isomorphic to $\hat{f}_! \circ \hat{g}^*$.*

We now justify the use of the term "extension by zero" [FK88, Lemma 8.3].

Lemma 3.13. *Let $j : X \to Y$ be an open embedding, $\mathcal{F} \in \text{AbShv}_{\text{ét}}(X)$, and $\mathcal{G} \in \text{AbShv}_{\text{ét}}(Y)$. Then $\mathcal{G}$ is canonically isomorphic to $j_!\mathcal{F}$ if and only if $\mathcal{F} \cong j^*\mathcal{G}$ and the restriction of $\mathcal{G}$ to the complement $Y \setminus jX$ vanishes.*

We now sweep a lot of details under the rug in the following definition; to see that the following definition is actually well-defined, one needs to use derived functor arguments and the proper base change theorem (see [FK88, Section 8] for a complete proof).

Definition. Suppose $f : X \to S$ be a morphism. It is *compactifiable over S* if there is a commutative diagram

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & \bar{X} \\ & \searrow^{f} & \downarrow{\scriptstyle \bar{f}} \\ & & S \end{array}$$

where j is an open embedding and $\bar{f}$ is a proper morphism. Such a triple $(\bar{X}, j, \bar{f})$ is said to be a *compactification over S*.

Example 3.14. All quasi-projective morphisms are compactifiable. If X is compactifiable over S, then X is separated and of finite type over S.

The following two theorems (black-boxed) tell us that all nice enough morphisms are compactifiable.

Theorem 3.15 (Nagata's Compactification Theorem)**.** *If S is quasi-compact and quasi-separated, then X is compactifiable over S if and only if X is separated and of finite type over S.*

Theorem 3.16 (Zariski's Main Theorem)**.** *Suppose S is quasi-compact and separated, and suppose $f : X \to S$ is separated, quasi-finite, and finitely presented. Then X is compactifiable over S, and moreover there is a compactification $(\bar{X}, j, \bar{f})$ of f such that $\bar{f}$ is finite.*

Definition. Suppose $f : X \to S$ is a separated and finite type map of schemes, with S quasi-compact and quasi-separated. All of our functors here are restricted to the category of torsion sheaves.

Pick any compactification $(\bar{X}, j, \bar{f})$ of f over S. The (derived) *shriek pushforward* is the functor

$$\mathrm{R}f_! := (\mathrm{R}\bar{f}_*) \circ j_! : D^+\mathrm{AbShv}_{\text{ét}}^{tor}(X) \to D^+\mathrm{AbShv}_{\text{ét}}^{tor}(S).$$

The *higher direct images with proper support* $R^\bullet f_!$ are the terms in the δ-functor $R^\bullet \bar{f}_* \circ j_!$ on the category $\mathrm{AbShv}_{\text{ét}}^{tors}(X)$ of torsion sheaves; equivalently, they are the n-th cohomologies of $\mathrm{R}f_!$. This definition is independent of j and $\bar{X}$.

If $S = \operatorname{Spec} k$ for k separably closed, we also write $H^\bullet_{c,\text{ét}}$ for $R^\bullet \bar{f}_* \circ j_!$, i.e.

$$H^\bullet_{c,\text{ét}} := H^\bullet_{\text{ét}}(\bar{X}, j_!\mathcal{F})$$

for all torsion abelian sheaves $\mathcal{F}$. We write $\mathrm{R}\Gamma_c(X, -)$ for the composition $\Gamma(S, -) \circ \mathrm{R}f_!$.

If S is not quasi-compact nor quasi-separated, then by glueing on open affines we can also define $R^\bullet f_!$ and $H^\bullet_{c,\text{ét}}$.

Remark 3.17. Despite the abuse of notation, $R^\bullet f_!$ is *NOT* the classical derived functor (in the sense of universal δ-functors) associated to the exact functor $f_!$, and $\mathrm{R}f_!$ is *NOT* the total derived functor of '$f_!$'.

Lemma-Definition. For any compactifiable morphism $f : X \to S$, the functor

$$f_! : \mathrm{AbShv}_{\text{ét}}^{tors}(X) \to \mathrm{AbShv}_{\text{ét}}^{tors}(S)$$

given by

$$f_!(\mathcal{F}) := (R^0 f_!)(\mathcal{F})$$

is called the *shriek-pushforward* by f.

If f is étale, this coincides with the previous definition for the shriek-pushforward. In other words, if f is étale, then $\mathrm{R}f_!$ defined previously coincides with the induced functor $f_! : \mathrm{AbShv}_{\text{ét}}^{tors}(X) \to \mathrm{AbShv}_{\text{ét}}^{tors}(S)$.

Definition. Suppose $s \in \mathcal{F}(U)$ where $U \to S$ is étale and $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}(S)$. Let V be the largest Zariski open in U such that $s|_V = 0$. The *support of* s is $\operatorname{supp}(s) := U \setminus V$.

Clearly, $\operatorname{supp}(s)$ is a scheme, with a closed embedding into U.

Lemma 3.18. *For any compactifiable morphism $f : X \to S$, the sheaf $f_!(\mathcal{F})$ for $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}^{tors}(X)$ is a subsheaf of $f_*(\mathcal{F})$ via*

$$f_!(\mathcal{F})(U) = \{s \in \mathcal{F}(U \times_S X) : \operatorname{supp}(s) \to U \textit{ is proper}\}.$$

Lemma 3.19. *If f is proper, then $f_! = f_*$.*

Proposition 3.20. *If $f : X \to S$ is étale, then $R^n f_! = 0$ for all $n \geq 1$.*

The shriek-pushforward has the following nicer properties when compared to the direct pushforward (see [Con, Remark 1.3.6.2] and [FK88, Theorems 8.7-8.10]).

Theorem 3.21. *Let $f : X \to S$ be compactifiable.*

1. *(*Base Change*) Let $g : T \to S$, and consider the projections $g' : T \times_S X \to X$ and $f' : T \times_S X \to T$. Then, f' is compactifiable, and there is a natural base change isomorphism of δ-functors*

   $$g^* \circ R^\bullet f_! \cong R^\bullet f'_! \circ g'^*$$

   from $\mathrm{AbShv}_{\text{ét}}^{tors}(X) \to \mathrm{AbShv}_{\text{ét}}(T)$.

   In terms of derived categories, we have a natural isomorphism of (derived)-functors $g^* \circ \mathrm{R}f_! = \mathrm{R}f'_! \circ g'^*$.

2. (Proper Pullback) *If $h : Y \to X$ is proper, then there is a canonical pullback map $R^\bullet f_! \to R^\bullet h_! \circ h^*$. In terms of derived functors, there is a natural transformation $\mathrm{R}f_! \to \mathrm{R}h_! \circ h^*$.*

3. (Composition) *Suppose $f : X \to Y$ and $g : Y \to S$, and let $h = g \circ f : X \to S$. Let g and h be compactifiable. Then, f is compactifiable and there is a spectral sequence*

$$R^q g_! \circ R^p f_! \Rightarrow R^{p+q} h_!.$$

In terms of derived functors, we have $\mathrm{R}h = \mathrm{R}g \circ \mathrm{R}f$.

4. (Excision) *Let $f : X \to S$ be a compactifiable map, $j : U \to X$ an open embedding, and $i : A \to X$ a closed embedding. Suppose, as underlying sets, that $i(A) \sqcup j(U) = X$. For each $\mathcal{F} \in \mathrm{AbShv}^{tors}_{\text{ét}}(X)$, there is a functorial long exact sequence*

$$0 \to (f\circ j)_!(j^*\mathcal{F}) \to f_!\mathcal{F} \to (f\circ i)_!(i^*\mathcal{F}) \to R^1(f\circ j)_!(j^*\mathcal{F}) \to R^1 f_!\mathcal{F} \to R^1(f\circ i)_!(i^*\mathcal{F}) \to R^2(f\circ j)_!(j^*\mathcal{F}) \to \cdots.$$

5. *If $\mathcal{F} \in \mathrm{AbShv}^{tors}_{\text{ét}}(X)$ is constructible, then $R^n f_!(\mathcal{F})$ is constructible for all $n \geq 0$. More generally, $R^n f_!(\mathcal{F}^\bullet)$ is constructible for any complex of sheaves $\mathcal{F} \in D^+\mathrm{AbShv}^{tors}_{\text{ét}}(X)$ whose cohomology sheaves are constructible.*

6. *If X is a compactifiable k-scheme where k is separably closed, then $H^n_{c,\text{ét}}(X, \mathcal{F})$ is a finite group and it vanishes for $n > 2\dim(X)$.*

7. *Suppose f is smooth and proper, and $\mathcal{F}$ an LCC torsion sheaf on X with torsion orders invertible on S. Then, all sheaves $R^n f_!\mathcal{F} = R^n f_*\mathcal{F}$ are LCC sheaves as well for all $n \geq 0$. Its formation moreover commutes with arbitrary base change.*

8. *Let $\mathcal{F} \in \mathrm{AbShv}^{tors}_{\text{ét}}(X)$, and let $\dim(X/S)$ be the maximum of the dimensions of all geometric fibres of f. Then*

$$R^n f_!(\mathcal{F}) = 0$$

for all $n > 2\dim(X/S)$.

3.4 Comparing Singular and Étale Cohomology

Definition. Suppose X is a $\mathbb{C}$-scheme locally of finite type. Endow the set $X(\mathbb{C})$ with the topology given by the basis

$$\{x \in U(\mathbb{C}) : |f_i(x)| < \epsilon \text{ for all } i = 1, ..., r\}$$

where $U \subset X$ is Zariski-open, $f_1, ..., f_r \in \Gamma(U, \mathcal{O}_X)$, and $\epsilon > 0$ arbitrary. Let X^{an} be this topological space regarded as a complex analytic space.

The following propositions appear in section 11 of chapter 1 of [FK88].

Proposition 3.22. *If $f : X \to X'$ is an arbitrary morphism of schemes locally of finite over $\mathbb{C}$, then the induced map $f^{an} : X^{an} \to X'^{an}$ (given pointwise by $f : X(\mathbb{C}) \to X'(\mathbb{C})$) is in fact holomorphic as a map of complex analytic spaces.*

If f is moreover étale, then f^{an} is a local isomorphism of complex analytic spaces.

The map f is an isomorphism of $\mathbb{C}$-schemes if and only if $f^{an} : X^{an} \to X'^{an}$ is an isomorphism of complex analytic spaces.

Proposition 3.23. *The natural morphism of locally ringed spaces $X^{an} \to X$ induced by the functor $(-)^{an}$ induces an isomorphism*

$$\mathrm{Hom}(X, Z) \cong \mathrm{Hom}(X^{an}, Z)$$

for all locally ringed spaces Z, where the Homs *are in the category of locally ringed spaces.*

For a complex analytic space Z, we can define the site $\mathrm{Et}(Z)$ whose underlying category is the category of all complex analytic spaces $q : Y \to Z$ over Z for which q is a local isomorphism.

Proposition 3.24. *The functor $\iota = \iota_X : \mathrm{Sch}^{\text{ét}}_S \to \mathrm{Et}(S^{an})$ given by $(X \to S) \mapsto (X^{an} \to S^{an})$ is a fully-faithful embedding.*

Now that we have the morphism of sites ι defined above, we get maps induced on the category of sheaves in both directions, which we denote by

$$(-)_{an} : \mathrm{Shv}_{\text{ét}}(X) \to \mathrm{Shv}(\mathrm{Et}(X^{an}))$$

and

$$(-)_{al} : \mathrm{Shv}(\mathrm{Et}(X^{an})) \to \mathrm{Shv}_{\text{ét}}(X)$$

for 'analytification' and 'algebraization'. It is a fact that $(-)_{an}$ is an exact functor left-adjoint to the left-exact functor $(-)_{al}$. In fact, $(-)_{an}$ commutes with all colimits and all finite direct limits.

Proposition 3.25. *Let $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_{\text{ét}}(X)$. Then the map*

$$\mathrm{Hom}_{\mathrm{Shv}_{\text{ét}}(X)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{Shv}(\mathrm{Et}(X))}(\mathcal{F}_{an}, \mathcal{G}_{an})$$

induced by $(-)_{an}$ is injective. If $\mathcal{G}$ is constructible, then it is bijective.

Theorem 3.26. *Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F} \in \mathrm{AbShv}_{\text{ét}}^{tor}(X)$.*

1. *We have canonical isomorphisms $\left(R^n f_!(\mathcal{F})\right)_{an} \cong R^n(f_{an})_!(\mathcal{F})$ and $H^n_c(X, \mathcal{F}) \cong H^n_c(X_{an}, \mathcal{F}_{an})$. In the derived setting, for any $\mathcal{F} \in D^+\mathrm{AbShv}_{\text{ét}}^{tors}(X)$ we have natural isomorphisms*

$$\left(\mathrm{R}f_!(\mathcal{F})\right)_{an} \cong \mathrm{R}(f_{an})_!(\mathcal{F})$$

   of complexes in $D\mathrm{AbShv}(X_{an})$.

2. *If $\mathcal{F}$ is constructible, then we have the canonical isomorphism $\left(R^n f_!(\mathcal{F})\right)_{an} \cong R^n(f_{an})_!(\mathcal{F})$.*

4 ℓ-adic Sheaves and Cohomology

We have thus seen that torsion sheaves behave very nicely under higher direct images and under cohomology. However, we would like to have a good theory of cohomology for sheaves with coefficients in characteristic zero, since we would like to have non-torsion cohomology groups. We thus need to give a 'better' definition for étale cohomology valued in $\mathbb{Q}_\ell$. We fix a prime ℓ throughout.

Remark 4.1. We only work with the ℓ-adic integers, though one can develop a fully analogous theory for an arbitrary complete local Noetherian rings. This is what Brian Conrad does in Section 1.4 of his notes [Con].

4.1 Commutative Algebra Setup

Recall that a projective system $F = (F_n, u_n : F_n \to F_{n-1})_{n\in\mathbb{Z}}$ of torsion $\mathbb{Z}_\ell$-modules is a collection of torsion $\mathbb{Z}_\ell$-modules F_n equipped with morphisms $u_n : F_n \to F_{n-1}$. We say that a projective system is bounded below if there is some $n_0 \in \mathbb{Z}$ such that $F_n = 0$ for all $n \le n_0$. For each $r \in \mathbb{Z}$ we can also define the shifted projective system $F[r] = (F'_n, u'_n)$ such that $F'_n = F_{n+r}$ and $u'_n = u_{n+r}$. A morphism of projective systems is a collection of $\mathbb{Z}_\ell$ morphisms $F_n \to G_n$ such that

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & F_{n+1} & \longrightarrow & F_n & \longrightarrow & F_{n-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & G_{n+1} & \longrightarrow & G_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots \end{array}$$

commutes.

Remark 4.2. The category of projective systems $\mathrm{PSTB}(\mathbb{Z}_\ell)$ of torsion $\mathbb{Z}_\ell$-modules bounded below is an abelian category. The shift functors $F \mapsto F[r]$ are an additive exact functor for all $r \in \mathbb{Z}$.

Definition. Suppose $F \in \mathrm{PSTB}(\mathbb{Z}_\ell)$ is a projective system of torsion $\mathbb{Z}_\ell$-module bounded below.

1. F satisfies the *Mittag-Leffler condition* (or ML condition) if for every $n \in \mathbb{Z}$ there exists $t \ge n$ such that
$$\mathrm{Im}(F_m \to F_n) = \mathrm{Im}(F_t \to F_n)$$
for all $m \ge t$.
2. F satisfies the *Mittag-Leffler-Artin-Rees condition* (or MLAR condition) if there exists $t \ge 0$ such that
$$\mathrm{Im}(F[r] \to F) = \mathrm{Im}(F[t] \to F)$$
for all $r \ge t$.
3. F is a *null-system* if there is $t \ge 0$ such that $F[t] \to F$ is the zero mapping.

The following lemmas are straightforward.

Lemma 4.3. *A projective system $(F_n) \in \mathrm{PSTB}(\mathbb{Z}_\ell)$ in which all the $\mathbb{Z}_\ell$-modules F_n have finite length automatically satisfies the ML condition.*

Lemma 4.4. *For a short exact sequence of projective systems*

$$0 \to (F_n) \to (G_n) \to (H_n) \to 0$$

in $\mathrm{PSTB}(\mathbb{Z}_\ell)$*, if (F_n) satisfies the ML condition, then the sequence*

$$0 \to \lim_n F_n \to \lim_n G_n \to \lim_n H_n \to 0$$

is exact.

Definition. The *A-R category of projective systems* $\mathrm{AR}(\mathbb{Z}_\ell)$ is the category whose objects are projective systems of torsion $\mathbb{Z}_\ell$-modules bounded below, and whose hom sets are

$$\mathrm{Hom}_{AR}(F, G) = \mathrm{colim}_{r\ge 0}\mathrm{Hom}(F[r], G).$$

If $f \in \mathrm{Hom}_{AR}(F, G)$ and $g \in \mathrm{Hom}_{AR}(G, H)$ are represented by $f : F[r] \to G$ and $g : G[s] \to H$, then $g \circ f$ is represented by $g \circ (f[s]) : F[r + s] \to H$.

Here, notice that the maps $\mathrm{Hom}(F[r], G) \to \mathrm{Hom}(F[r+1], G)$ is induced by the natural maps $F[r+1] \to F[r]$ given by $u_{n+r+1} : F_{n+r+1} \to F_{n+r}$.

Remark 4.5. The A-R stands for Artin-Rees.

Remark 4.6. The A-R category is an abelian category, where the kernel and cokernel in the A-R category are the kernel and cokernel in the original category $\mathrm{PSTB}(\mathbb{Z}_\ell)$.

Remark 4.7. A morphism in the A-R category is an isomorphism if and only if its kernel and cokernel are null-systems.

If F is a projective system, then we obviously have $\lim_n F_n \cong \lim_n F_{n+r}$, and so the limit

$$F \mapsto \lim_n F_n$$

is naturally a functor from the A-R category of projective systems to the category of $\mathbb{Z}_\ell$-modules. The previous lemma then shows that the limit functor is exact on the subcategory of projective systems consisting of finite modules.

Definition. A projective system $F = (F_n, u_n)$ is an *ℓ-adic system* if the following conditions hold:

- $F_n = 0$ for $n < 0$;
- $\ell^{n+1} F_n = 0$ for all n (so that F_n is naturally a $\mathbb{Z}/\ell^{n+1}\mathbb{Z}$-module);
- for all $n \geq 1$, the morphism u_n induces an isomorphism

  $$F_n \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} = F_n/\ell^n F_n \xrightarrow[u_n \bmod \ell^n]{\simeq} F_{n-1}$$

  of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules;
- F_n is a module of finite length for all $n \geq 0$.

A projective system is *A-R ℓ-adic* if it is A-R isomorphic to an ℓ-adic system.

The following lemma is straightforward.

Lemma 4.8. *If F is an ℓ-adic system and G a projective system with $\ell^{n+1} G_n = 0$ for all n, then*

$$\mathrm{Hom}(F, G) = \mathrm{Hom}_{AR}(F, G).$$

We now state an important result from commutative algebra.

Proposition 4.9 (Artin-Rees Lemma). *If I is an ideal in a Noetherian ring R, and if M is a finitely generated R-module and N a submodule of M, then there exists $k \geq 1$ so that for all $n \geq k$, we have*

$$I^n M \cap N = I^{n-k}(I^k M \cap N)$$

We now define a functor going from finitely generated $\mathbb{Z}_\ell$-modules to the category of ℓ-adic systems. If M is a finitely generated $\mathbb{Z}_\ell$-module, then set $M_n := M/\ell^{n+1}M$. We have natural 'mod ℓ' maps $M_n \to M_{n-1}$, and so we get a projective system (M_n). It is easy to see that (M_n) is an ℓ-adic system. The following corollary is an immediate consequence of the Artin-Rees Lemma.

Corollary 4.9.1. *Suppose $M \to N$ is a morphism of $\mathbb{Z}_\ell$-modules. Let $K = \ker(M \to N)$ and $C = \mathrm{coker}\,(M \to N)$. Then, the ℓ-adic systems (K_n) and (C_n) constructed as above are A-R isomorphic to $\ker((M_n) \to (N_n))$ and $\mathrm{coker}\,((M_n) \to (N_n))$, the latter kernels and cokernels being taken in the category of A-R ℓ-adic projective systems.*

As a result, we get the following.

Proposition 4.10. *If F is an ℓ-adic system and $M := \lim_n F_n$, then M is a finite $\mathbb{Z}_\ell$-module and*

$$F_n \cong M/\ell^{n+1}M.$$

Thus the limit functor establishes an equivalence between the full subcategory of the A-R category consisting of all A-R ℓ-adic systems, and the category of all finitely generated $\mathbb{Z}_\ell$-modules.

The category of A-R ℓ-adic projective systems is an exact subcategory of the A-R category, i.e. kernels and cokernels of an A-R morphism of A-R ℓ-adic systems are again ℓ-adic systems.

Moreover, if $0 \to F \to G \to H$ is a short exact sequence in the A-R category such that F and H are A-R ℓ-adic and there exists t such that $\ell^{n+t} G_n = 0$ for all n, then G is also A-R ℓ-adic.

Proposition 4.11. *Consider a projective system of complexes of $\mathbb{Z}_\ell$-modules $(K_n^\bullet)$ such that $K_n^\bullet = 0$ for $n < 0$, satisfying the following conditions:*

- $\ell^{n+1}K_n^i = 0$ *for all i, so that $K_n^\bullet$ is naturally a complex of $\mathbb{Z}/\ell^{n+1}\mathbb{Z}$ modules;*
- *each K_n^i is flat over $\mathbb{Z}/\ell^{n+1}\mathbb{Z}$;*
- *the complexes are uniformly bounded, i.e. there exists $r > 0$ such that $K_n^i = 0$ for all n and all $|i| > r$;*
- *the cohomology modules $H^i(K_n^\bullet)$ are finite;*
- *the map $K_n^\bullet \to K_{n-1}^\bullet$ induces a quasi-isomorphism*

$$K_n^\bullet \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} (\mathbb{Z}/\ell^n\mathbb{Z}) \to K_{n-1}^\bullet$$

  for all $n \geq 1$.

Then, there is a bounded complex $K^\bullet$ of finitely generated free $\mathbb{Z}_\ell$-modules together with quasi-isomorphisms

$$K^\bullet/\ell^{n+1}K^\bullet \to K_n^\bullet$$

for which the diagram

$$\begin{array}{ccc} K^\bullet/\ell^{n+1}K^\bullet & \longrightarrow & K_n^\bullet \\ \downarrow & & \downarrow \\ K^\bullet/\ell^n K^\bullet & \longrightarrow & K_{n-1}^\bullet \end{array}$$

is homotopy-commutative. Moreover, for each i the projective systems $(H^i(K_n^\bullet))_{n\in\mathbb{Z}}$ of cohomology modules are A-R ℓ-adic.

4.2 ℓ-adic sheaves

Throughout, suppose X is a scheme on which ℓ is invertible. Analogous to the above definitions for $\mathbb{Z}_\ell$-modules, we can say what it means for a projective system to satisfy the ML condition, the MLAR condition, and what it means for a system to be null. We can also define the A-R category of projective systems $\mathcal{F} = (\mathcal{F}_n)$ of ℓ-torsion abelian étale sheaves, i.e. each sheaf $\mathcal{F}_n \in \mathrm{AbShv}_{\text{ét}}^{tors}(X)$ are ℓ-torsion, so that

$$\mathcal{F}_n = \bigcup_m \ker(\mathcal{F}_n \xrightarrow{\ell^m} \mathcal{F}_n)$$

and we also assume that this projective system is bounded below. For two systems $\mathcal{F} = (\mathcal{F}_n)$ and $\mathcal{G} = (\mathcal{G}_n)$ of ℓ-torsion sheaves, we define

$$\mathrm{Hom}_{AR}(\mathcal{F}, \mathcal{G}) = \mathrm{colim}_{r\geq 0}\mathrm{Hom}(\mathcal{F}[r], \mathcal{G}).$$

We extend the definition of ℓ-adic systems to sheaves:

Definition. A projective system of $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ of abelian étale sheaves on X is an *ℓ-adic sheaf* if

- each sheaf $\mathcal{F}_n$ is constructible;
- we have $\mathcal{F}_n = 0$ for $n < 0$ and $\ell^{n+1}\mathcal{F}_n = 0$ for $n \geq 0$;
- the morphisms $\mathcal{F}_n \to \mathcal{F}_{n-1}$ induce isomorphisms

$$\mathcal{F}_n \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} = \mathcal{F}_n/\ell^n \xrightarrow{\simeq} \mathcal{F}_{n-1}$$

  for all $n \geq 1$.

A system $\mathcal{F}$ is an *A-R ℓ-adic sheaf* is it is A-R isomorphic to an ℓ-adic sheaf, i.e. there exists an ℓ-adic sheaf $\mathcal{G}$ and a morphism of projective systems $\mathcal{G}[r] \to \mathcal{F}$ such that the kernel and cokernel are null systems.

An ℓ-adic $\mathcal{F} = (\mathcal{F}_n)$ is *locally constant* or *lisse* if all of the sheaves $\mathcal{F}_n$ are locally constant.

Lemma 4.12. *If $\mathcal{F} = (\mathcal{F}_n)$ is an ℓ-adic sheaf and $\mathcal{G}$ a projective system such that $\ell^{n+1}\mathcal{G}_n = 0$, then we have*

$$\mathrm{Hom}_{AR}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G}).$$

Example 4.13. If M is a finitely generated $\mathbb{Z}_\ell$-module, then the projective system $M_X = \left(\underline{M/\ell^{n+1}M}_X\right)$ is an ℓ-adic sheaf. It is clear that M_X is locally constant.

Example 4.14. Let $\mu_{\ell^n,X} := \ker(\mathbb{G}_{m,X} \xrightarrow{\ell^n} \mathbb{G}_{m,X})$ be the étale sheaf of ℓ^n-th roots of unity. Then,

$$\mathbb{Z}_\ell(m) := \left(\mu_{\ell^{n+1},X}^{\otimes m}; \mu_{\ell^{n+1},X}^{\otimes m} \xrightarrow{\zeta_1\otimes\cdots\otimes\zeta_m \mapsto \zeta_1^\ell\otimes\cdots\otimes\zeta_m^\ell} \mu_{\ell^n,X}^{\otimes m}\right)$$

is a lisse ℓ-adic sheaf.

We can also define, for $m < 0$, the étale sheaf

$$\mu_{\ell^n,X}^{\otimes m} := \mathrm{Hom}(\mu_{\ell^n,X}^{\otimes|m|}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}_X)$$

on X, and thus we can define the lisse ℓ-adic sheaf

$$\mathbb{Z}_\ell(m) := \left(\mu_{\ell^{n+1},X}^{\otimes m}\right).$$

Example 4.15. Let $\mathcal{F}$ be a constructible ℓ-torsion étale sheaf with $\ell^m\mathcal{F} = 0$. The (stationary) system

$$\mathcal{F}_n := \begin{cases} \mathcal{F}/\ell^{n+1}\mathcal{F} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

is an ℓ-adic system. The functor $\mathcal{F} \mapsto (\mathcal{F}_n)_n$ identifies the category of constructible ℓ-torsion sheaves with a full-subcategory of all ℓ-adic sheaves.

The following result is useful to carry out Noetherian induction.

Lemma 4.16. *Suppose $\mathcal{F}$ is a projective system of sheaves with $\ell^{n+1}\mathcal{F}_n$ for $n \geq 0$ and $\mathcal{F}_n = 0$ for $n < 0$. Then $\mathcal{F}$ is an A-R ℓ-adic sheaf if and only if*

- *$\mathcal{F}$ satisfies the MLAR condition, i.e. there is an $r > 0$ such that*

$$\mathrm{im}\,(\mathcal{F}[t] \to \mathcal{F}) = \mathrm{im}\,(\mathcal{F}[r] \to \mathcal{F}) =: \bar{\mathcal{F}}$$

  for all $t \geq r$; and

- *there is an integer $s \geq 0$ with*

$$\bar{\mathcal{F}}_{n+t}/\ell^{n+1}\bar{\mathcal{F}}_{n+t} \cong \bar{\mathcal{F}}_{n+s}/\ell^{n+1}\bar{\mathcal{F}}_{n+s}$$

  for all $n \geq 0$ and all $t \geq s$.

Moreover, if the two conditions hold, then in fact $(\bar{\mathcal{F}}_{n+s}/\ell^{n+1}\bar{\mathcal{F}}_{n+s})$ is an ℓ-adic sheaf isomorphic to $\mathcal{F}$.

As a consequence, the following results hold.

1. *$\mathcal{F}$ is an A-R ℓ-adic sheaf if and only if it is an ℓ-adic sheaf locally with respect to the étale topology.*
2. *If A is a closed subscheme and U an open subscheme of X such that $A \sqcup U = X$, then $\mathcal{F}$ is an A-R ℓ-adic sheaf iff $(\mathcal{F}_n|_U)$ and $(\mathcal{F}_n|_A)$ are A-R ℓ-adic.*
3. *Suppose X is connected and the $\mathcal{F}_n$ are constructible and locally constant. Suppose $\bar{s}$ is a geometric point of X. Then, $\mathcal{F}$ is an A-R ℓ-adic sheaf if and only if the projective system $\left((\mathcal{F}_n)_{\bar{s}}\right)$ is an A-R ℓ-adic system of $\mathbb{Z}_\ell$-modules.*

Definition. If $\mathcal{F}$ is an A-R ℓ-adic sheaf, and $\bar{s}$ a geometric point of X, then we define its *stalk* to be

$$\mathcal{F}_{\bar{s}} = \lim_n (\mathcal{F}_n)_{\bar{s}},$$

which is a $\mathbb{Z}_\ell$-module.

Lemma 4.17. *An A-R ℓ-adic sheaf $\mathcal{F}$ vanishes in the A-R category (i.e. is a null system) if and only if all of its stalks vanishes.*

The following is Example 1.4.4.5 of [Con].

Proposition 4.18. *If X is connected and $\bar{x}$ a geometric point of X, then the stalk functor at $\bar{x}$ establishes an equivalence of categories between lisse ℓ-adic sheaves and continuous linear representations of $\pi_1(X, \bar{x})$ on finite $\mathbb{Z}_\ell$-modules.*

The following is Proposition 12.10 of [FK88].

Proposition 4.19. *Let $\mathcal{F}$ be an ℓ-adic sheaf on X. Then there is a Zariski-dense open subscheme U for which the restricted system $\mathcal{F}|_U := (\mathcal{F}_n|_U)$ is locally constant.*

The following is Proposition 12.11 of [FK88].

Proposition 4.20. *The category of A-R ℓ-adic sheaves is an exact subcategory of the A-R category, i.e. kernels and cokernels of an A-R morphism of A-R ℓ-adic systems are again ℓ-adic systems.*

Moreover, if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is a short exact sequence in the A-R category such that $\mathcal{F}$ and $\mathcal{H}$ are A-R ℓ-adic and there exists t such that $\ell^{n+t}\mathcal{G}_n = 0$ for all n, then $\mathcal{G}$ is also A-R ℓ-adic.

The following is Proposition 12.13 of [FK88].

Proposition 4.21. *For every ℓ-adic sheaf $\mathcal{G}$ there is an ordinary étale ℓ-torsion sheaf $\mathcal{H} \in \mathrm{AbShv}_{\text{ét}}^{tors}(X)$, a torsion free ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_n)$ (i.e. $\mathcal{F} \xrightarrow{\ell} \mathcal{F}$ is A-R injective) such that each $\mathcal{F}_n$ is $(\mathbb{Z}/\ell^{n+1}\mathbb{Z})$-flat, and an A-R exact sequence*

$$0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{F} \to 0.$$

Analogous to the previous subsection, we have the following result (Lemma 12.14 of [FK88]) which allows us to construct many examples of A-R ℓ-adic sheaves.

Proposition 4.22. *Let $(\mathcal{K}_n^\bullet)$ be a projective system of complexes of sheaves, such that the following hypotheses hold:*

- $\ell^{n+1}\mathcal{K}_n^\bullet = 0$ *for all* $n \geq 0$, *and* $\mathcal{K}_n^\bullet = 0$ *for* $n < 0$;
- $\mathcal{K}_n^i$ *are* $(\mathbb{Z}/\ell^{n+1}\mathbb{Z})$*-flat;*
- *the cohomology sheaves* $H^i(\mathcal{K}_n^\bullet)$ *are constructible;*
- *there is an* $r > 0$ *such that* $\mathcal{K}_n^i = 0$ *for all* n *and all* $|i| > r$;
- *the map* $\mathcal{K}_n^\bullet \to \mathcal{K}_{n-1}^\bullet$ *induces a quasi-isomorphism*

$$\mathcal{K}_n^\bullet \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} = \mathcal{K}_n^\bullet/\mathcal{K}_n^\bullet \to \mathcal{K}_{n-1}^\bullet.$$

Then for each $i \in \mathbb{Z}$ the projective systems $\left(H^i(\mathcal{K}_n^\bullet)\right)_n$ of cohomology sheaves are A-R ℓ-adic sheaves.

4.3 Cohomology of ℓ-adic sheaves

As in the previous section, we fix a prime ℓ. Throughout, we only consider those schemes on which ℓ is invertible.

Suppose $F : \mathrm{AbShv}_{\text{ét}}^{tors}(X) \to \mathrm{AbShv}_{\text{ét}}^{tors}(Y)$. Then, we can extend F to becomes a functor from the category $AR_\ell(X)$ of A-R ℓ-adic sheaves on X to the category $AR_\ell(Y)$ of A-R ℓ-adic sheaves on Y, by defining it component-wise

$$F(\mathcal{F}_n) := (F\mathcal{F}_n).$$

It is easy to see that properties of $F : \mathrm{AbShv}_{\text{ét}}^{tors}(X) \to \mathrm{AbShv}_{\text{ét}}^{tors}(Y)$ transfer to properties of $F : AR_\ell(X) \to AR_\ell(Y)$. Thus for instance, for any morphism $f : X \to Y$ of Noetherian schemes, we have

- the *pull-back functor* $f^* : AR_\ell(Y) \to AR_\ell(X)$, which is exact and sends ℓ-adic sheaves to ℓ-adic sheaves;
- the *direct image functors* $R^i f_* : AR_\ell(X) \to AR_\ell(Y)$; and
- for compactifiable f the *direct image functor with compact support* $R^i f_! : AR_\ell(X) \to AR_\ell(Y)$.

The last functor is a non-trivial result, the proof of which requires the use of the Godement resolution.

Theorem 4.23. *Suppose $f : X \to Y$ is a compactifiable mapping, and $\mathcal{F} = (\mathcal{F}_n)$ an A-R ℓ-adic sheaf on X. Then the system $R^i f_!(\mathcal{F})$ is an A-R ℓ-adic sheaf for each $i \geq 0$.*

In this way, the system of functors $(R^i f_*)$ and $(R^i f_!)$ each form a system of δ-functors on the A-R categories.

Lemma 4.24. *If $f : X \to Y$ and $g : Y \to Z$ then for each $\mathcal{F} \in AR_\ell(X)$ we have a spectral sequence*

$$R^q g_* \circ R^p f_*(\mathcal{F}) \Rightarrow R^{p+q}(g \circ f(\mathcal{F}))_*,$$

functorial in $\mathcal{F}$. If f, g, and $g \circ f$ are all compactifiable, then we also have

$$R^q g_! \circ R^p f_!(\mathcal{F}) \Rightarrow R^{p+q}(g \circ f(\mathcal{F}))_!$$

functorially in $\mathcal{F} \in AR_\ell(X)$.

Similarly, the base-change theorems carry over to the category of A-R ℓ-adic sheaves.

Definition. For a scheme X (on which ℓ is invertible) over a separably closed field $\bar{k}$, the ordinary *cohomology* of an A-R ℓ-adic sheaf

cohomology with compact supports of an A-R ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_n)$ is given by

$$H^i_{c,\text{ét}}(X, \mathcal{F}) := \lim_n H^i_{c,\text{ét}}(X, \mathcal{F}_n).$$

These are finitely generated $\mathbb{Z}_\ell$-modules.

One can also compare singular cohomology with the above defined cohomology of (AR) ℓ-adic sheaves, though we omit this. See [FK88].

4.4 Sheaves of $\mathbb{Q}_\ell$-Vector Spaces

In this section, we describe the 'quotient' category of the category of all A-R ℓ-adic sheaves by the subcategory of ℓ-torsion sheaves.

Definition. The category $\mathrm{Vec}_\ell^{\text{ét}}(X)$ of *sheaves of* $\mathbb{Q}_\ell$*-vector spaces* is the category in which

- the objects are the same as in the A-R category of projective systems $\mathcal{F} = (\mathcal{F}_n)$ of ℓ-torsion sheaves, though we denote such a sheaf $\mathcal{F}$ considered as a sheaf of $\mathbb{Q}_\ell$-vector spaces by $\mathcal{F} \otimes \mathbb{Q}_\ell$;
- the hom sets are
$$\mathrm{Hom}(\mathcal{F} \otimes \mathbb{Q}_\ell, \mathcal{G} \otimes \mathbb{Q}_\ell) := \mathrm{Hom}_{AR}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Composition in this category is induced by the composition in the A-R category of projective systems.

This category is an abelian category, and the natural functor from the A-R category into the category of sheaves of $\mathbb{Q}_\ell$-vector spaces is exact.

Definition. A sheaf of $\mathbb{Q}_\ell$-vector spaces is *constructible* if it is isomorphic to a sheaf $\mathcal{F} \otimes \mathbb{Q}_\ell$ for an A-R ℓ-adic sheaf $\mathcal{F}$. A constructible sheaf of $\mathbb{Q}_\ell$-vector spaces is *lisse* if the above $\mathcal{F}$ is a locally constant (i.e. lisse) constructible ℓ-adic sheaf.

Example 4.25. For ℓ invertible on X, the ℓ-adic sheaves $\mathbb{Q}_\ell(r) := \mathbb{Z}_\ell(r) \otimes \mathbb{Q}_\ell$ are lisse sheaves of $\mathbb{Q}_\ell$-vector spaces for all $r \in \mathbb{Z}$.

Lemma 4.26. *Every constructible sheaf is isomorphic to a sheaf $\mathcal{F} \otimes \mathbb{Q}_\ell$ with $\mathcal{F}$ a torsion free ℓ-adic sheaf.*

Theorem 4.27. *If X is a connected Noetherian scheme with a geometric point $\bar{x}$, then the stalk functor at $\bar{x}$ gives an equivalence of categories between lisse $\mathbb{Q}_\ell$-sheaves on X and the category of continuous $\mathbb{Q}_\ell$-linear representations of $\pi_1(X, \bar{x})$ on finite dimensional $\mathbb{Q}_\ell$-vector spaces.*

See [Con, Theorem 1.4.5.4] for the proof.

Corollary 4.27.1. *Suppose X is a normal and Noetherian scheme. Then, the property of being a lisse $\mathbb{Q}_\ell$-sheaf is local for the étale topology on X.*

We can extend previous sheaf-theoretic concepts to a sheaf $\mathcal{F} \otimes \mathbb{Q}_\ell \mathrm{Vec}_\ell^{\text{ét}}(X)$ by simply tensoring with $\mathbb{Q}_\ell$:

- For $\bar{s} : \operatorname{Spec} \bar{k} \to X$ a geometric point, the *stalk* is $(\mathcal{F} \otimes \mathbb{Q}_\ell)_{\bar{s}} := \mathcal{F}_{\bar{s}} \otimes \mathbb{Q}_\ell$.
- $\Gamma(X, \mathcal{F} \otimes \mathbb{Q}_\ell) := \Gamma(X, \mathcal{F}) \otimes \mathbb{Q}_\ell$
- $H^i_{\text{ét}}(X, \mathcal{F} \otimes \mathbb{Q}_\ell) := H^i(X, \mathcal{F}) \otimes \mathbb{Q}_\ell$
- $H^i_{c,\text{ét}}(X, \mathcal{F} \otimes \mathbb{Q}_\ell) := H^i_{c,\text{ét}}(X, \mathcal{F}) \otimes \mathbb{Q}_\ell$
- $R^i f_*(X, \mathcal{F} \otimes \mathbb{Q}_\ell) := R^i f_*(X, \mathcal{F}) \otimes \mathbb{Q}_\ell$
- $R^i f_!(X, \mathcal{F} \otimes \mathbb{Q}_\ell) := R^i f_!(X, \mathcal{F}) \otimes \mathbb{Q}_\ell$

A Abstract Nonsense

A.1 Universal δ-Functors

Definition. A (covariant) *cohomological δ-functor* between $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $T^n : \mathcal{A} \to \mathcal{B}$ (with the convention $T^n = 0$ for all $n < 0$) together with morphisms $\delta^n : T^n(C) \to T^{n+1}(A)$ (called the connecting morphism) defined for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, such that

1. for each short exact sequence $0 \to A \to B \to C \to 0$, there is a long exact sequence
$$\cdots \to T^{n-1}C \xrightarrow{\delta^{n-1}} T^n A \to T^n B \to T^n C \xrightarrow{\delta^n} T^{n+1}A \to \cdots ;$$

2. for any morphism
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$
of short exact sequences $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ in $\mathcal{A}$, we have the commutative diagram
$$\begin{array}{ccc} T^n C & \xrightarrow{\delta^n} & T^{n+1}A \\ \downarrow & & \downarrow \\ T^n C' & \xrightarrow{\delta^n} & T^{n+1}A'. \end{array}$$

Example A.1. Cohomology gives a cohomological δ-functor H^* from $\mathrm{Ch}^{\geq 0}(\mathcal{A})$ to $\mathcal{A}$.

Definition. A morphism $S^* \to T^*$ of δ-functors is a system of natural transformations $f^n : S^n \to T^n$ commuting with the δs. In other words, a morphism $S^* \to T^*$ of δ-functors is the data of a commutative ladder
$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\delta} & S^n A & \longrightarrow & S^n B & \longrightarrow & S^n C & \xrightarrow{\delta} & T^{n+1}A & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \xrightarrow{\delta} & T^n A & \longrightarrow & T^n B & \longrightarrow & T^n C & \xrightarrow{\delta} & T^{n+1}A & \longrightarrow & \cdots \end{array}$$
attached to every short exact sequence $0 \to A \to B \to C \to 0$.

Definition. A cohomological δ-functor T is *universal* if, given any δ-functor S and any natural transformation $f^0 : T^0 \to S^0$, there exists a unique morphism $T \to S$ of δ-functors extending f^0.

Example A.2. If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor, then $T^0 = F$ and $T^n = 0$ for all $n \geq 1$ defines a universal δ-functor.

The following lemma is obvious from the universality of the above property.

Lemma A.3. *Given any additive functor $F : \mathcal{A} \to \mathcal{B}$, there exists at most one universal δ-functor T^* such that $T^0 = F$.*

Definition. An additive functor $F : \mathcal{A} \to \mathcal{B}$ is *effaceable* if for every $A \in \mathcal{A}$ there is a monomorphism $A \hookrightarrow J$ with $F(A \hookrightarrow J) = 0$.

Proposition A.4. *Let T be a covariant δ-functor from $\mathcal{A} \to \mathcal{B}$. Suppose for each $i > 0$ the functor T^i is effaceable. Then, T is a universal δ-functor.*

Proof. Suppose given any δ-functor S and any natural transformation $f^0 : T^0 \to S^0$. Suppose inductively that each $f^i : T^i \to S^i$ for $0 \leq i < n$ is defined and that they commute with all the appropriate δ^is. Given $A \in \mathcal{A}$, let $J \in \mathcal{A}$ be such that $A \hookrightarrow J$ and $T^n(A \hookrightarrow J) = 0$; such a J exists since T^n is effaceable. Consider the exact sequence $0 \to A \to J \to C \to 0$ where $C = \mathrm{coker}\,(A \hookrightarrow J)$. We get a commutative diagram
$$\begin{array}{ccccccc} T^{n-1}J & \longrightarrow & T^{n-1}C & \xrightarrow{\delta_T^{n-1}} \!\!\!\!\rightarrow & T^n A & \xrightarrow{0} & T^n J \\ \downarrow{\scriptstyle f^{n-1}J} & & \downarrow{\scriptstyle f^{n-1}C} & & \vdots & & \\ S^{n-1}J & \longrightarrow & S^{n-1}C & \xrightarrow{\delta_S^{n-1}} & S^n A & \longrightarrow & S^n J \end{array}$$

where both rows are exact. In particular, we have $T^nA = \operatorname{coker}(T^{n-1}J \to T^{n-1}C)$. Consider the map $\delta_S^{n-1} \circ f^{n-1}C : T^{n-1}C \to S^nA$. Pre-composing this with $T^{n-1}J \to T^{n-1}C$ we get zero since the left square commutes and since $S^{n-1}J \to S^{n-1}C \xrightarrow{\delta_S^{n-1}} S^nA$ is the zero map. The universal property of cokernels then implies the existence of a unique map $T^nA \to S^nA$ such that the entire diagram above commutes. Thus, we have a well-defined map $f_J^nA : T^nA \to S^nA$ which *a priori* depends on the choice of J. To see that it is independent of J, by taking products and noting that T is additive we may suppose that $J \hookrightarrow I$ with $T^n(A \hookrightarrow I) = 0$. Then, we have two exact sequences $0 \to A \to J \to C \to 0$ and $0 \to A \to I \to C' \to 0$, and we have a morphism of exact sequences

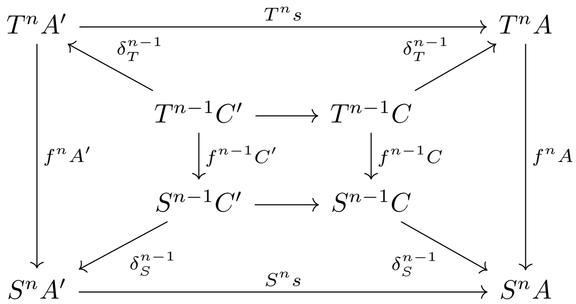
$$\begin{CD} 0 @>>> A @>>> J @>>> C @>>> 0\\ @. @V{\mathrm{id}}VV @VVV @VVV @.\\ 0 @>>> A @>>> I @>>> C' @>>> 0. \end{CD}$$

Running through the above argument, it is easy to see that $f_J^nA = f_I^nA$. Hence $f^nA : T^nA \to S^nA$ is independent of the choice of A.

Now, consider an arbitrary morphism $s : A \to A'$ with J as given above. Consider the fibred coproduct I of A' and J over A. Pick a monomorphism $I \hookrightarrow J'$ such that $T^n(I \hookrightarrow J) = 0$. Then, we have $A' \hookrightarrow J'$ satisfies $T^n(A' \hookrightarrow J') = 0$. We can thus extend the morphism s to a morphism of exact sequences

$$\begin{CD} 0 @>>> A @>>> J @>>> C @>>> 0\\ @. @V{s}VV @VVV @VVV @.\\ 0 @>>> A' @>>> J' @>>> C' @>>> 0 \end{CD}$$

We get a diagram



in which each small quadrilateral commutes. One then checks that $S^ns \circ f^nA' \circ \delta_T^{n-1} = f^nA \circ T^ns \circ \delta_T^{n-1}$. Since $T^nJ = 0 = T^nJ'$, it follows that δ_T^{n-1} is epic, and so we have $S^ns \circ f^nA' = f^nA \circ T^ns$. Therefore, $f^n : S^n \to T^n$ is a well-defined natural transformation.

It remains to check that f^n commutes with the δ^{n-1}s. Consider any short exact sequence $0 \to A \to B \to C \to 0$. Choose a monomorphism $B \hookrightarrow J$ for which $T^n(B \hookrightarrow J) = 0$; then $A \hookrightarrow J$ satisfies $T^n(A \hookrightarrow J) = 0$. We then get a map of short exact sequences

$$\begin{CD} 0 @>>> A @>>> B @>>> C @>>> 0 \qquad \text{(SES1)}\\ @. @| @VVV @VVV @.\\ 0 @>>> A @>>> J @>>> C' @>>> 0 \qquad \text{(SES2)} \end{CD}$$

which then yields

$$\begin{CD} T^{n-1}C @>{\delta_T^{\mathrm{SES1}}}>> T^nA\\ @V{T^{n-1}g}VV @|\\ T^{n-1}C' @>>{\delta_T^{\mathrm{SES2}}}> T^nA \end{CD} \qquad \begin{CD} S^{n-1}C @>{\delta_S^{\mathrm{SES1}}}>> S^nA\\ @V{S^{n-1}g}VV @|\\ S^{n-1}C' @>>{\delta_S^{\mathrm{SES2}}}> S^nA. \end{CD}$$

We thus have the commutative diagram

$$\begin{CD} T^{n-1}C @>{T^{n-1}g}>> T^{n-1}C' @>{\delta_T^{\mathrm{SES2}}}>> T^nA\\ @V{f^{n-1}C}VV @V{f^{n-1}C'}VV @V{f^nA}VV\\ S^{n-1}C @>{S^{n-1}g}>> S^{n-1}C' @>{\delta_S^{\mathrm{SES2}}}>> S^nA \end{CD}$$

which implies that the square

$$\begin{array}{ccc} T^{n-1}C & \xrightarrow{\delta_T^{\mathrm{SES1}}} & T^nA \\ \downarrow{\scriptstyle f^{n-1}C} & & \downarrow{\scriptstyle f^nA} \\ S^{n-1}C & \xrightarrow{\delta_S^{\mathrm{SES1}}} & S^nA \end{array}$$

commutes. This is precisely the statement that f^n commutes with δ^{n-1}, as required. □

Suppose $\mathcal{A}$ has enough injectives. If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor with right derived functors R^*F, then notice that each R^nF is effaceable. This follows since $R^nF(I) = 0$ for all injective I and for all $n \geq 1$, and existence of such I is guaranteed by assumption on $\mathcal{A}$. This yields the following corollary.

Corollary A.4.1. *If $\mathcal{A}$ has enough injectives, then for any left exact functor $F : \mathcal{A} \to \mathcal{B}$ the right derived functors R^*F form a universal δ-functor.*

Lemma A.5. *If $\mathcal{A}$ has enough injectives, a functor $F : \mathcal{A} \to \mathcal{B}$ is effaceable if and only if $F(I) = 0$ for all injective objects $I \in \mathcal{A}$.*

Proof. The converse is trivial, and so it suffices to show that $F(I) = 0$ for an arbitrary injective object I if F is effaceable. Since F is effaceable, there exists an object $A \in \mathcal{A}$ and a monomorphism $u : I \hookrightarrow A$ such that $F(u) = 0$. We have an exact sequence $0 \to I \to A \to \operatorname{coker} u \to 0$, and since I is injective this exact sequence splits. There thus exists a morphism $v : A \to I$ such that $vu = \mathrm{id}_I$. Then, we have

$$\mathrm{id}_{F(I)} = F(\mathrm{id}_I) = F(vu) = F(v) \circ F(u) = 0,$$

which is only possible if $F(I) = 0$. □

Proposition A.6. *Suppose $\mathcal{A}$ has enough injectives and T^* is a δ-functor from $\mathcal{A}$ to $\mathcal{B}$. Then T^* is universal if and only if T^n is effaceable for all $n \geq 1$.*

A.2 Generalities on Spectral Sequences

Throughout we fix an abelian category $\mathcal{A}$.

Definition. Suppose $A \in \mathcal{A}$. A (decreasing) *filtration* of A is a family $(F^pA)_{p\in\mathbb{Z}}$ of sub-objects F^pA of A such that $F^{p+1}A \hookrightarrow F^pA$ for all p.

The *pth graded piece* is $gr_pA = \operatorname{coker}(F^{p+1}A \hookrightarrow F^pA)$.

Given filtered objects A and B in $\mathcal{A}$, a morphism $u : A \to B$ is said to be *compatible with the filtration* if for every $p \in \mathbb{Z}$ the map $F^pA \hookrightarrow A \xrightarrow{u} B$ factors through $F^pB \hookrightarrow B$.

Definition. A (rightward oriented) *spectral sequence* in $\mathcal{A}$ is a system

$$E = (E_\bullet^{\bullet,\bullet}, E^\bullet)$$

consisting of the data:

- objects $E_r^{pq} \in \mathcal{A}$ for all $(p,q) \in \mathbb{Z}^2$ and $r \geq 0$;
- morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p-r+1,q+r}$ such that $d_r^{p-r+1,q+r} \circ d_r^{pq} = 0$;
- isomorphisms $\alpha_r^{pq} : \ker(d_r^{pq})/\mathrm{im}\,(d_r^{p+r-1,q-r}) \xrightarrow{\cong} E_{r+1}^{pq}$, where we assume that for each $(p,q) \in \mathbb{Z}^2$ the morphisms d_r^{pq} and $d_r^{p+r-1,q-r}$ vanish for all r large enough so that $E_r^{pq} = E_{r+1}^{pq} = E_{r+2}^{pq} = \cdots =: E_\infty^{pq}$ for some sufficiently large r;
- decreasingly filtered objects $E^n \in \mathcal{A}$ for $n \in \mathbb{Z}$, where we assume that $F^pE^n = E^n$ for sufficiently small p and $F^pE^n = 0$ for sufficiently large p;
- isomorphisms $\beta^{pq} : E_\infty^{pq} \xrightarrow{\cong} \mathrm{gr}_pE^{p+q}$.

For each $r \geq 0$, we call the collection of complexes $E_r^{\bullet,\bullet}$ the *r'th page of the spectral sequence* E. The filtered objects E^n are the *limit terms* of the spectral sequence.

A spectral sequence is called a *cohomological spectral sequence* if $E_2^{pq} = 0$ whenever $p < 0$ or $q < 0$.

Remark A.7. An *upward oriented spectral sequence* can be similarly defined by simply swapping the pair of indices in the superscript E_r^{pq}. Thus, rightward oriented spectral sequences (E_r^{pq}, E^n) can be made into an upward oriented spectral sequence (E_r^{qp}, E^n), and vice versa. In order to make explicit the orientation of a spectral sequence, we write $E_{\bullet,\to}^{\bullet;\bullet}$ for a rightward oriented spectral sequence.

Remark A.8. We use the definition given in [Tam94], though we define both rightward oriented and upward oriented spectral sequences. One should note that the spectral sequences considered in [Tam94] are all 'upward-oriented' spectral sequences in the sense of [Vak].

Remark A.9. Very often, we only know (canonically) the second page of a spectral sequence (see Grothendieck's spectral sequence below, for instance). In such a situation, we write $E_2^{pq} \Rightarrow E^{p+q}$ to mean that there is a spectral sequence $E = (E_\bullet^{\bullet,\bullet}, E^\bullet)$ whose second page is given by E_2^{pq} and whose limit terms are E^{p+q}.

Definition. A morphism $u : E \to E'$ of spectral sequences in $\mathcal{C}$ is a system of morphisms $u_r^{pq} : E_r^{pq} \to E'^{pq}_r$ and $u^n : E^n \to E'^n$ where the u^n are compatible with the filtrations of $E^\bullet$ and $E'^\bullet$, and the u_r^{pq} and u^n naturally commute with d_r^{pq}, α_r^{pq}, and the β^{pq}.

In this way, the spectral sequences in $\mathcal{A}$ form an additive category. It is also easy to see that cohomological spectral sequences form a full subcategory of the category of spectral sequences. An additive functor from an abelian category to a category of spectral sequences in a *spectral functor*.

Lemma A.10. *For any cohomological spectral sequence E in $\mathcal{A}$, we have $F^0E^n = E^n$ and $F^{n+1}E^n = 0$ for all $n \in \mathbb{Z}$.*

Proof. Since $E_2^{pq} = 0$ for all $p < 0$ or $q < 0$, it follows that $E_r^{pq} = 0$ for all $p < 0$ or $q < 0$. Thus $E_\infty^{pq} = 0$ for all $p < 0$ or $q < 0$. It follows that $gr_pE^{p+q} \cong 0$ whenever $p < 0$ or $q < 0$. In particular, $gr_{-i}E^n = 0$ for all $i > 0$ and all $n \in \mathbb{Z}$. This implies that $F^0E^n \cong F^{-1}E^n \cong F^{-2}E^n \cong \cdots \cong E^n$. Similarly, $gr_{n+i}E^n = 0$ for all $i > 0$ and so $F^{n+1}E^n \cong F^{n+2}E^n \cong \cdots \cong 0$. □

Proposition A.11. *For any cohomological spectral sequence E in $\mathcal{A}$ there are morphisms $E_2^{0,n} \to E^n$ and $E^n \to E_2^{n,0}$ that are functorial in E, called the* edge morphisms.

Moreover, the edge morphism $E_2^{0,n} \to E^n$ factors as $E_2^{0,n} \twoheadrightarrow E_\infty^{0,n} \hookrightarrow E^n$, while the edge morphism $E^n \to E_2^{n,0}$ factors as $E^n \twoheadrightarrow E_\infty^{n,0} \hookrightarrow E_2^{n,0}$.

Proof. We construct the edge morphisms as follows. Since $F^{n+1}E^n = 0$, we have monomorphisms

$$E_\infty^{0,n} \cong \operatorname{coker}(F^{n+1}E^n \hookrightarrow F^nE^n) \cong F^nE^n \hookrightarrow E^n$$

for all $n \in \mathbb{Z}$. For $r \geq 2$, we have

$$E_{r+1}^{0,n} \cong \frac{\ker(E_r^{0,n} \to E_r^{-r+1,n+r})}{\operatorname{im}(E_r^{r-1,n-r} \to E_r^{0,n})} \cong \frac{E_r^{0,n}}{\operatorname{im}(E_r^{r-1,n-r} \to E_r^{0,n})}$$

since $E_r^{-r+1,n+r} = 0$. In particular, we have an epimorphism $E_r^{0,n} \twoheadrightarrow E_{r+1}^{0,n}$ for all $r \geq 2$, and so $E_2^{0,n} \twoheadrightarrow E_\infty^{0,n} \hookrightarrow E^n$, giving us the first edge morphism. It is obvious that this morphism is functorial in E.

Since $F^0E^n = E^n$, we have an epimorphism

$$E^n = F^0E^n \twoheadrightarrow F^0E^n/F^1E^n \cong E_\infty^{n,0}.$$

Also, for $r \geq 2$, we have

$$E_{r+1}^{n,0} \cong \frac{\ker(E_r^{n,0} \to E_r^{n-r+1,r})}{\operatorname{im}(E_r^{-r,n+r-1} \to E_r^{0,n})} \cong \ker(E_r^{n,0} \to E_r^{n-r+1,r}) \hookrightarrow E_r^{n,0},$$

and thus a monomorphism $E_\infty^{n,0} \hookrightarrow E_2^{n,0}$. The result now follows. □

Proposition A.12. *Assume that for a cohomological spectral sequence $E_2^{pq} \Rightarrow E^{p+q}$ the terms E_2^{pq} vanish for $0 < p < n$ ($q \in \mathbb{Z}$ arbitrary). Then, the edge morphism induces an isomorphism $E_2^{0,m} \cong E^m$ for all $m < n$, and the sequence*

$$0 \to E_2^{0,n} \to E^n \to E_2^{n,0} \to E_2^{0,n+1} \to E_2^{n+1}$$

is exact.

Proof. Note first that

$$gr_pE^m \cong E_\infty^{p,m-p} \cong 0$$

for all $1 \leq p \leq n-1$, and all $m < n$. Thus $F^pE^m \cong F^{p+1}E^m$ for all $1 \leq p \leq n-1$. If $m < n$, we then get $F^1E^m \cong F^{m+1}E^m \cong 0$, and so $E^m \cong F^0E^m \cong gr_0E^m \cong E_\infty^{0,m}$ as required. It remains to show the above sequence is exact.

For $r \geq 2$, we have

$$E_{r+1}^{0,n} \cong \frac{\ker(E_r^{0,n} \to E_r^{-r+1,n+r})}{\operatorname{im}(E_r^{r-1,n-r} \to E_r^{0,n})} \cong \frac{\ker(E_r^{0,n} \to 0)}{\operatorname{im}(0 \to E_r^{0,n})} \cong E_r^{0,n}.$$

Here, we need to use the fact that $E_r^{r-1,n-r} = 0$ for all $r \geq 2$ where it follows for $2 \leq r \leq n$ by assumption, whereas it follows for $r > n$ as E is cohomological. Thus $E_2^{0,n} \cong E_\infty^{0,n}$, and the definition of the edge map then implies exactness at $E_2^{0,n}$.

By Proposition A.11, we know that $\operatorname{im}(E_2^{0,n} \to E^n) \cong E_\infty^{0,n} \cong gr_0 E^n$, and

$$\ker(E^n \to E_2^{n,0}) = \ker(E^n \to E_\infty^{n,0}) \cong \ker(F^0 E^n \to gr_n E^n).$$

Now, as $E_2^{pq} = 0$ for $0 < p < n$, it follows that $E_\infty^{pq} = 0$ for all $p = 1, 2, ..., n-1$. Hence $gr_p E^{p+q} = 0$ for all $1 \leq p \leq n-1$ and all $q \geq 0$, and so $F^p E^{p+q} \cong F^{p+1} E^{p+q}$ for all $1 \leq p \leq n-1$ and all $q \geq 0$. Taking $q = n - p$, it then follows that $F^p E^n \cong F^{p+1} E^n$ for all $1 \leq p \leq n-1$, and so $F^1 E^n \cong F^n E^n \cong gr_n E^n$. Thus

$$\ker(E^n \to E_2^{n,0}) \cong \ker(F^0 E^n \to F^1 E^n) \cong gr_0 E^n \cong \operatorname{im}(E_2^{0,n} \to E^n).$$

Exactness follows at E^n.

Next, we have $\operatorname{im}(E^n \to E_2^{n,0}) \cong E_\infty^{n,0}$. We describe the map $E_2^{n,0} \to E_2^{0,n+1}$: notice that

$$E_{r+1}^{0,n+1} \cong \frac{\ker(E_r^{0,n+1} \to E_r^{-r+1,n+r+1})}{\operatorname{im}(E_r^{r-1,n+1-r} \to E_r^{0,n+1})} \cong \frac{E_r^{0,n+1}}{\operatorname{im}(E_r^{r-1,n+1-r} \to E_r^{0,n+1})} \cong \begin{cases} E_r^{0,n+1} & \text{if } r \neq n+1, \\ E_{n+1}^{0,n+1}/\operatorname{im}(d_{n+1}^{n,0}), & \text{if } r = n+1, \end{cases}$$

for all $r \geq 2$. Thus $E_2^{0,n+1} \cong E_{n+1}^{0,n+1}$ and $E_\infty^{0,n+1} \cong E_{n+1}^{0,n+1}/\operatorname{im} d_{n+1}^{n,0}$. Similarly, we have for $r \geq 2$,

$$E_{r+1}^{n,0} \cong \frac{\ker(E_r^{n,0} \to E_r^{n-r+1,r})}{\operatorname{im}(E_r^{n+r-1,-r} \to E_r^{n,0})} \cong \ker(E_r^{n,0} \to E_r^{n-r+1,r}) \begin{cases} E_r^{n,0} & \text{if } r \neq n+1, \\ \ker d_{n+1}^{n,0} & \text{if } r = n+1, \end{cases}$$

and so $E_2^{n,0} \cong E_{n+1}^{0,n+1}$ and $E_\infty^{n,0} \cong \ker d_{n+1}^{n,0}$. The map $E_2^{n,0} \to E_2^{0,n+1}$ is simply the composition

$$E_2^{n,0} \cong E_{n+1}^{n,0} \xrightarrow{d_{n+1}^{0,n+1}} E_{n+1}^{0,n+1} \cong E_2^{0,n+1},$$

with kernel isomorphic to $\ker d_{n+1}^{0,n+1} \cong E_\infty^{n,0} \cong \operatorname{im}(E^n \to E_2^{n,0})$. Hence the sequence is exact at $E_2^{n,0}$.

Finally, note that $\operatorname{im}(E_2^{n,0} \to E_2^{0,n+1}) \cong \operatorname{im} d_{n+1}^{n,0}$ while the kernel of the edge map is

$$\ker(E_2^{0,n+1} \to E_2^{n+1}) \cong \ker(E_2^{0,n+1} \to E_\infty^{0,n+1}) \cong \ker(E_{n+1}^{0,n+1} \to E_\infty^{0,n+1}) \cong \operatorname{im} d_{n+1}^{n,0}.$$

Therefore the entire sequence is exact. □

Taking $n = 1$ we have the following corollary.

Corollary A.12.1. *For a cohomological spectral sequence $E_2^{pq} \Rightarrow E^{p+q}$, we have the exact sequence*

$$0 \to E_2^{0,1} \to E^1 \to E_2^{1,0} \xrightarrow{d_2^{1,0}} E_2^{0,2} \to E^2.$$

This exact sequence is called the five term exact sequence *of the spectral sequence $E_2^{pq} \Rightarrow E^{p+q}$.*

Corollary A.12.2. *In a cohomological spectral sequence $E_2^{pq} \Rightarrow E^{p+q}$, suppose $E_2^{pq} \cong 0$ for all $p > 0$. Then the edge morphism induces an isomorphism $E_2^{0,n} \cong E^n$ for all n. In such a case, the spectral sequence is said to be* trivial*.*

A.3 Grothendieck Spectral Sequence

Throughout, we let $\mathcal{C}$ be an abelian category. We state and prove the Grothendieck Spectral Sequence using the method given in [Vak].

Lemma A.13. *Suppose there is an exact sequence $0 \to A \to B \to C \to 0$. Let I and J be injectives such that $A \hookrightarrow I$ and $C \hookrightarrow J$. Then, there exists an injective K such that $B \hookrightarrow K$ and we have the following commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \dashrightarrow & K & \dashrightarrow & J & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

in which both rows are exact.

Proof. Recall that in an abelian category the (finite) product and coproduct coincide. Let $K = I \times J$. Since $A \hookrightarrow B$ and I is an injective, we have a map $B \to I$. We also have a map $B \to C \to J$. The universal property of products implies that we have a map $B \to K$ such that the following diagram except for the map $I \hookrightarrow K$ commutes.

$$\begin{array}{ccccc} I & \overset{\twoheadleftarrow}{\hookrightarrow} & K & \twoheadrightarrow & J \\ \uparrow & \nwarrow & \uparrow & & \uparrow \\ A & \hookrightarrow & B & \twoheadrightarrow & C \end{array}$$

Suppose $\iota : I \hookrightarrow K$ and $\pi : K \twoheadrightarrow I$. Then $\pi \circ \iota = \mathrm{Id}$. Denoting the map $B \to I$ by f and $B \to K$ by g, notice that $\pi \circ g = f$. It follows that $\pi \circ g = \pi \circ (\iota \circ f)$, and since π is epic, we have $g = \iota \circ f$. Hence, the following diagram commutes.

$$\begin{array}{ccccc} I & \hookrightarrow & K & \twoheadrightarrow & J \\ \uparrow & & \uparrow & & \uparrow \\ A & \hookrightarrow & B & \twoheadrightarrow & C \end{array}$$

The five lemma then implies that the central arrow is injective, as required. $\square$

Applying this lemma inductively to an injective resolution, we get the following corollary.

Corollary A.13.1. *Suppose we have an exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathcal{C}$. *Let* $I^\bullet$ *and* $J^\bullet$ *be injective resolutions for* A *and* C *respectively. Then, there exists an injective resolution* $K^\bullet$ *for* B *such that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^0 & \dashrightarrow & K^0 & \dashrightarrow & J^0 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

commutes and such that the complexes fit into the exact sequence

$$0 \to I^\bullet \to K^\bullet \to J^\bullet \to 0.$$

Given a complex $K^\bullet$, we use the usual notation

$$Z^p(K^\bullet) = \ker(K^p \to K^{p+1}), \quad B^p(K^\bullet) = \mathrm{im}(K^{p-1} \to K^p), \quad \text{and} \quad H^p(K^\bullet) = \frac{Z^p(K^\bullet)}{B^p(K^\bullet)}.$$

Lemma-Definition. Suppose $C^\bullet$ is a complex in an abelian category $\mathcal{C}$, indexed by non-negative integers. Then, there exists a double complex $I^{\bullet,\bullet}$ satisfying the following properties

- $I^{p,q}$ is an injective object of $\mathcal{C}$, and moreover $I^{p,\bullet}$ is an injective resolution of C^p;
- $Z^{p,*} := Z^p(I^{\bullet,*})$ is an injective resolution for $Z^p(C^\bullet)$;
- $B^{p,*} := B^p(I^{\bullet,*})$ is an injective resolution for $B^p(C^\bullet)$; and
- $H^{p,*} := H^p(I^{\bullet,*})$ is an injective resolution for $H^p(C^\bullet)$.

Such a double complex $I^{\bullet,\bullet}$ is the *Cartan-Eilenberg resolution of* $C^\bullet$.

Proof. Note that for each $p \geq 0$ we have exact sequences

$$0 \to B^p(C^\bullet) \to Z^p(C^\bullet) \to H^p(C^\bullet) \to 0.$$

Let $B^{p,*}$ and $H^{p,*}$ be injective resolutions of $B^p(C^\bullet)$ and $H^p(C^\bullet)$ respectively; then, we can construct an injective resolution $Z^{p,*}$ of $Z^p(C^\bullet)$ such that

$$0 \to B^{p,*} \to Z^{p,*} \to H^{p,*} \to 0$$

is an exact sequence of complexes and everything commutes with the exact sequence $0 \to B^p(C^\bullet) \to Z^p(C^\bullet) \to H^p(C^\bullet) \to 0$. Now, we have the exact sequence

$$0 \to Z^p(C^\bullet) \to C^p \to B^{p+1}(C^\bullet) \to 0$$

with injective resolutions $Z^{p,*}$ and $B^{p+1,*}$ for $Z^p(C^\bullet)$ and $B^{p+1}(C^\bullet)$. We can thus find an injective resolution $I^{p,\bullet}$ for C^p such that

$$0 \to Z^{p,*} \to I^{p,*} \to B^{p+1,*} \to 0.$$

We now claim that $I^{\bullet,\bullet}$ is the desired double complex. Indeed, consider the double complex

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & I^{p-1,2} & \longrightarrow & I^{p,2} & \longrightarrow & I^{p+1,2} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & I^{p-1,1} & \longrightarrow & I^{p,1} & \longrightarrow & I^{p+1,1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & I^{p-1,0} & \longrightarrow & I^{p,0} & \longrightarrow & I^{p+1,0} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^{p-1} & \longrightarrow & C^{p} & \longrightarrow & C^{p+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the columns are exact. Here, the maps $I^{p,q} \to I^{p+1,q}$ arise by the composition

$$I^{p,q} \twoheadrightarrow B^{p+1,q} \hookrightarrow Z^{p+1,q} \hookrightarrow I^{p+1,q}.$$

Let us take the cohomology of the complex $I^{\bullet,q}$ for $q \geq 1$. Notice that

$$\ker(I^{p,q} \to I^{p+1,q}) = \ker(I^{p,q} \to B^{p+1,q}) = Z^{p,q}$$

and

$$\operatorname{im}(I^{p,q} \to I^{p+1,q}) = B^{p+1,q}.$$

Hence, we see that $Z^p(I^{\bullet,*}) = Z^{p,*}$, $B^p(I^{\bullet,*}) = B^{p,*}$, and $H^p(I^{\bullet,*}) = H^{p,*}$. The result follows. □

Definition. Suppose $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is an additive left-exact functor of abelian categories, with right-derived functors R^iF. An object $C \in \mathcal{C}$ is said to be *F-acyclic* if $R^iF(C) = 0$ for all $i \geq 1$.

Theorem A.14 (Grothendieck Spectral Sequence)**.** *Suppose $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are left-exact additive (covariant) functors of abelian categories. Notice that $G \circ F : \mathcal{A} \to \mathcal{C}$ is also left-exact. Suppose $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and that F sends injective elements of $\mathcal{A}$ to G-acyclic elements of $\mathcal{B}$.*

Then, for each $X \in \mathcal{A}$, there is a first-quadrant spectral sequence $E_{\bullet,\to}^{\bullet,\bullet}$ with rightward orientation whose second page is $E_{2,\to}^{p,q} := R^qG\big(R^pF(X)\big)$ (the second page is uniquely defined, though the first two pages require choices) such that

$$R^qG\big(R^pF(X)\big) \Rightarrow R^{p+q}(G \circ F)(X).$$

Proof. Choose an injective resolution $J^\bullet$ of X. By left-exactness of F, we have

$$0 \to FX \to FJ^0 \to FJ^1 \to \cdots .$$

Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of the complex $FJ^\bullet$. Consider the spectral sequence whose 0th page is $E_0^{\bullet,\bullet} = GI^{\bullet,\bullet}$. Taking upward orientation first, and noticing that $I^{p,\bullet}$ is an injective resolution of FJ^p, we see that

$$E_{1,\uparrow}^{p,q} \cong (R^qG)(FJ^p).$$

However, by assumption FJ^p is acyclic for all p, and so $(R^qG)(FJ^p) = 0$ for all p and all $q \geq 1$. For $q = 0$, we simply have $G(FJ^p) = (G \circ F)(J^p)$. Taking cohomology again, we see that the spectral sequence converges on the second page to $R^p(G \circ F)(X)$.

We now evaluate the second page of this spectral sequence with rightward orientation. Clearly $E_{1,\to}^{p,*} = H^p(GI^{\bullet,*})$. Now, applying the left exact functor G to

$$0 \to Z^p(I^{\bullet,q}) \to I^{p,q} \to I^{p+1,q},$$

we get that

$$G\big(Z^p(I^\bullet, q)\big) \cong \ker(GI^{p,q} \to GI^{p+1,q}) =: Z^p(GI^{\bullet,q}).$$

Thus G "commutes" with Z^p. Applying the left exact G to the split exact sequences

$$0 \to B^p(I^{\bullet,q}) \to Z^p(I^{\bullet,q}) \to H^p(I^{\bullet,q}) \to 0$$

$$0 \to Z^p(I^{\bullet,q}) \to I^{p,q} \to B^{p+1}(I^{\bullet,q}) \to 0$$

we have the exact sequences

$$0 \to GB^p(I^{\bullet,q}) \to GZ^p(I^{\bullet,q}) \to GH^p(I^{\bullet,q}) \to 0$$

$$0 \to GZ^p(I^{\bullet,q}) \to GI^{p,q} \to GB^{p+1}(I^{\bullet,q}) \to 0.$$

The latter exact sequence implies that G "commutes" with B^p, and the former then implies that G "commutes" with H^p. Hence, $E_{1,\to}^{p,*} = GH^p(I^{\bullet,*})$. Since $I^{\bullet,\bullet}$ is a Cartan-Eilenberg resolution, $H^p(I^{\bullet,*})$ is an injective resolution of $H^p(FJ^\bullet) =: (R^pF)(X)$, and so taking cohomology of $GH^p(I^{\bullet,*}) = E_{1,\to}^{p,*}$ we get

$$E_{2,\to}^{p,q} = (R^qG)\big((R^pF)(X)\big).$$

Since we know that $E_\bullet^{\bullet,\bullet}$ converges to $R^\bullet(G \circ F)(X)$, the result follows. □

In fact, more is true though we won't prove it in this complete generality.

Theorem A.15. *With the same conditions on F and G as above, there is a cohomological spectral functor*

$$X \mapsto E(X) = (E_\bullet^{\bullet,\bullet}(X), E^\bullet(X))$$

from $\mathcal{A}$ to the category of (cohomological) spectral sequences in $\mathcal{C}$ given by

$$E_2^{p,q}(X) = R^qG(R^pF(X)), \quad \text{and} \quad E^n = R^n(G \circ F)(X).$$

A.4 The Category of Chain Complexes, and Derived Categories

We follow [Wei94], though we only work with cochain complexes.

A.4.1 Chain Complexes

Definition. Given an abelian category $\mathcal{A}$, define the *category* $\mathrm{Ch}^\bullet(\mathcal{A})$ *of cochain complexes* whose objects are cochain complexes

$$C^\bullet : \cdots \xrightarrow{d^{-3}} C^{-2} \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots$$

where each $C^i \in \mathcal{A}$ and every composition $d^{n+1} \circ d^n$ is the zero map. The maps d^n are called the *coboundary maps* of $C^\bullet$.

A cochain complex is *exact* if $\operatorname{coker} d^n = \ker d^{n+1}$.

Morphisms of cochain complexes $f^\bullet : C^\bullet \to D^\bullet$ such that $d_D^n \circ f^n = f^{n+1} \circ d_C^n$ for all $n \in \mathbb{Z}$. Addition of morphisms is done degree wise.

The full-subcategory $\mathrm{Ch}_{\geq 0}^\bullet(\mathcal{A})$ of those cochain complexes $C^\bullet$ such that $C^n = 0$ for all $n < 0$.

Lemma A.16. *Suppose $f^\bullet : C^\bullet \to D^\bullet$ is a morphism of cochain complexes in $\mathrm{Ch}^\bullet(\mathcal{A})$.*

1. $\ker f^\bullet$ *is the cochain complex given by* $(\ker f^\bullet)^n = \ker f^n$ *with the coboundary maps induced by those on* $C^\bullet$.

2. $\operatorname{coker} f^\bullet$ *is the cochain complex given by* $(\operatorname{coker} f^\bullet)^n = \operatorname{coker} f^n$ *with the coboundary maps induced by those on* $D^\bullet$.

In particular, every morphism in $\mathrm{Ch}^\bullet(\mathcal{A})$ has a kernel and a cokernel. Also, the kernel and cokernel of a morphism in the full-subcategory $\mathrm{Ch}_{\geq 0}^\bullet(\mathcal{A})$ also lies in $\mathrm{Ch}_{\geq 0}^\bullet(\mathcal{A})$.

Remark A.17. This lemma is very powerful, since it allows us to extend results about kernels and cokernels in $\mathcal{A}$ to results about kernels and cokernels in $\mathrm{Ch}^\bullet(\mathcal{A})$. For instance, it shows that the sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is an exact sequence of chain modules if and only if for each n the sequence $0 \to A^n \to B^n \to C^n \to 0$ is exact.

Proof. We prove the first statement only, since the second statement is formally dual. The final statement that $\mathrm{Ch}_{\geq 0}^\bullet(\mathcal{A})$ is closed under taking kernels and cokernels follows because $f^n : 0 \to 0$ for all $n < 0$.

Denote $\iota^n : \ker f^n \to C^n$ the kernel map. The morphism $d_C^n \circ \iota^n : \ker f^n \to C^{n+1}$ satisfies

$$f^{n+1} \circ (d_C^n \circ \iota^n) = f^{n+1} \circ d_C^n \circ \iota^n = d_D^n \circ f^n \circ \iota^n = 0,$$

and so factors through $\ker f^{n+1}$. Thus, there exist maps $d^n : \ker f^n \to \ker f^{n+1}$ such that $d_C^n \circ \iota^n = \iota^{n+1} \circ d^n$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \dashrightarrow & \ker f^{n-1} & \dashrightarrow & \ker f^n & \dashrightarrow & \ker f^{n+1} & \dashrightarrow & \cdots \\
 & & \hookdownarrow{\scriptstyle \iota^{n-1}} & & \hookdownarrow{\scriptstyle \iota^{n}} & & \hookdownarrow{\scriptstyle \iota^{n+1}} & & \\
\cdots & \longrightarrow & C^{n-1} & \longrightarrow & C^n & \longrightarrow & C^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^{n}} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & D^{n-1} & \longrightarrow & D^n & \longrightarrow & D^{n+1} & \longrightarrow & \cdots
\end{array}$$

Since

$$\iota^{n+1} \circ d^n \circ d^{n-1} = d_C^n \circ \iota^n \circ d^{n-1} = d_C^n \circ d_C^{n-1} \circ \iota^{n-1} = 0$$

where ι^{n-1} is monic, it follows that $d^n \circ d^{n-1}$ is 0. Hence $K^n := \ker f^n$ with coboundary maps $d_K^n = d^n$ gives a cochain complex, and moreover $\iota^\bullet : K^\bullet \to C^\bullet$ is a chain morphism.

It remains to show that $K^\bullet$ satisfies the universal property of kernels. So suppose that $g^\bullet : B^\bullet \to C^\bullet$ is a cochain morphism such that $f^\bullet \circ g^\bullet$ is the zero morphism from $B^\bullet \to D^\bullet$. Then, at each n, the map $f^n \circ g^n : B^n \to D^n$ is the zero map. The universal property of kernels in $\mathcal{A}$ then implies the existence of $h^n : B^n \to K^n$ such that $g^n = \iota^n \circ h^n$. We claim that $h^\bullet$ is a chain map. Indeed, we have

$$\iota^{n+1} \circ (d_K^n \circ h^n) = d_C^n \circ \iota^n \circ h^n = d_C^n \circ g^n = g^{n+1} \circ d_B^n = \iota^{n+1} \circ h^{n+1} \circ d_B^n.$$

Since ι^{n+1} is monic, it then follows that $d_K^n \circ h^n = h^{n+1} \circ d_B^n$. Therefore $h^\bullet$ is a chain map, and the claim follows. □

Corollary A.17.1. *A morphism $f^\bullet : C^\bullet \to D^\bullet$ is monic in $\mathrm{Ch}^\bullet(\mathcal{A})$ if and only if $f^n : C^n \to D^n$ is monic in $\mathcal{A}$ for all n. Similarly, $f^\bullet : C^\bullet \to D^\bullet$ is epic in $\mathrm{Ch}^\bullet(\mathcal{A})$ if and only if $f^n : C^n \to D^n$ is epic in $\mathcal{A}$ for all n.*

Proof. The first statement follows from the general category-theoretic fact that, in a category $\mathcal{C}$ with a zero object, a morphism $A \to B$ with kernel $K \to A$ is monic if and only if $K = 0$. The second statement is formally dual. □

Proposition A.18. *$\mathrm{Ch}^\bullet(\mathcal{A})$ is an abelian category whenever $\mathcal{A}$ is an abelian category. Moreover, $\mathrm{Ch}^\bullet_{\geq 0}(\mathcal{A})$ is an abelian subcategory.*

Proof. The lemma already shows that all kernels and cokernels exist. The previous corollary implies that for $f^\bullet : C^\bullet \to D^\bullet$ monic, each f^n is monic and so $f^n = \ker \operatorname{coker} f^n$. Since kernels and cokernels are constructed degree-wise, it follows that $f^\bullet = \ker \operatorname{coker} f^\bullet$. Similarly, all epic maps in $\mathrm{Ch}^\bullet(\mathcal{A})$ are the cokernels of their kernels. That $\mathrm{Ch}^\bullet_{\geq 0}(\mathcal{A})$ is an abelian category follows because it is a full-subcategory of the abelian category $\mathrm{Ch}^\bullet(\mathcal{A})$ that is closed under taking kernels and cokernels. □

Definition. Given $C^\bullet \in \mathrm{Ch}^\bullet(\mathcal{A})$ and given $m \in \mathbb{Z}$, define the complex $C^\bullet[m] \in \mathrm{Ch}^\bullet(\mathcal{A})$ by

$$\big(C^\bullet[m]\big)^n := C^{n+m}, \qquad \text{and} \qquad d^n_{C^\bullet[m]} = (-1)^m d^{n+m}.$$

It is easy to see that the assignment $C^\bullet \mapsto C^\bullet[m]$ defines an additive exact functor $(-)[m] : \mathrm{Ch}^\bullet(\mathcal{A}) \to \mathrm{Ch}^\bullet(\mathcal{A})$.

Definition. Given a cochain $C^\bullet \in \mathrm{Ch}^\bullet(\mathcal{A})$ with coboundary maps $d^n : C^n \to C^{n+1}$, we have the *n-cocycles* $Z^n(C^\bullet) := \ker d^n$ and the *n-coboundaries* $B^n(C^\bullet) := \operatorname{im} d^{n-1}$.

Lemma A.19. *There is a canonically induced monomorphism $B^n(C^\bullet) \hookrightarrow Z^n(C^\bullet)$.*

Proof. Fix n. The map $d^{n-1} : C^{n-1} \to C^n$ factors as $C^{n-1} \twoheadrightarrow B^n(C^\bullet) \hookrightarrow C^n$. Since $d^n \circ d^{n-1} = 0$, the composition $C^{n-1} \twoheadrightarrow B^n \to C^{n+1}$ is zero, and so the map $B^n \hookrightarrow C^n \xrightarrow{d^n} C^{n+1}$ is zero. Hence, this map factors as $B^n \to Z^n$. It remains to show that this is monic. However, we know that $B^n \to Z^n \hookrightarrow C^n$ is itself the monic map $B^n \hookrightarrow C^n$, which implies that $B^n \to Z^n$ is monic. □

Definition. The cokernel of the map $B^n(C^\bullet) \hookrightarrow Z^n(C^\bullet)$ is called the *nth cohomology of $C^\bullet$*, denoted by $H^n(C^\bullet)$.

Notice that, by definition, we have for each $n \in \mathbb{Z}$ the exact sequence

$$0 \to B^n(C^\bullet) \to Z^n(C^\bullet) \to H^n(C^\bullet) \to 0.$$

Also, since im = coker ker in an abelian category, it follows that $B^{n+1}(C^\bullet) = \operatorname{im} d^n = \operatorname{coker}(\ker d^n) = \operatorname{coker}(Z^n \hookrightarrow C^n)$. We thus have the exact sequence

$$0 \to Z^n(C^\bullet) \to C^n \to B^{n+1}(C^\bullet) \to 0$$

as well.

Lemma A.20. *A morphism $f^\bullet : C^\bullet \to D^\bullet$ induces unique natural morphisms $Z^n(C^\bullet) \to Z^n(D^\bullet)$, $B^n(C^\bullet) \to B^n(D^\bullet)$, and $H^n(C^\bullet) \to H^n(D^\bullet)$ for all $n \in \mathbb{Z}$ such that for each $n \in \mathbb{Z}$ the diagrams*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^n(C^\bullet) & \longrightarrow & Z^n(C^\bullet) & \longrightarrow & H^n(C^\bullet) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^n(D^\bullet) & \longrightarrow & Z^n(D^\bullet) & \longrightarrow & H^n(D^\bullet) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z^n(C^\bullet) & \longrightarrow & C^n & \xrightarrow{d_C^n} & B^{n+1}(C^\bullet) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^n(D^\bullet) & \longrightarrow & D^n & \xrightarrow{d_D^n} & B^{n+1}(D^\bullet) & \longrightarrow & 0
\end{array}$$

with exact rows commutes.

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
C^{n-1} & \xrightarrow{d_C^{n-1}} & Z^n(C^\bullet) & \hookrightarrow & C^n & \xrightarrow{d_C^n} & B^{n+1}(C^\bullet) & \hookrightarrow & C^{n+1} \\
\downarrow{\scriptstyle f^{n-1}} & & \vdots & & \downarrow{\scriptstyle f^n} & & \vdots & & \downarrow{\scriptstyle f^{n+1}} \\
D^{n-1} & \xrightarrow{d_D^{n-1}} & Z^n(D^\bullet) & \hookrightarrow & D^n & \xrightarrow{d_D^n} & B^{n+1}(D^\bullet) & \hookrightarrow & D^{n+1}
\end{array}$$

The composition $Z^n(C^\bullet) \hookrightarrow C^n \xrightarrow{f^n} D^n \xrightarrow{d_D^n} D^{n+1}$ is zero since $Z^n(C^\bullet) = \ker d_C^n$ and since the right solid rectangle commutes. Hence, the composition $Z^n(C^\bullet) \hookrightarrow C^n \xrightarrow{f^n} D^n$ uniquely factors through $\ker d_D^n = Z^n(D^\bullet)$. This uniquely induces the leftmost dashed arrow, which moreover commutes with everything in sight. Similarly, the composition $d_D^n \circ f^n : C^n \to B^{n+1}(D^\bullet)$ is zero when composed with $Z^n(C^\bullet) \to C^n$, and so uniquely factors through $\operatorname{coker}(Z^n(C^\bullet) \hookrightarrow C^n) = B^{n+1}(C^\bullet)$. This yields the rightmost dashed arrow, which moreover commutes with everything in sight. This gives the second diagram in the statement of the proposition. Since $H^n = \operatorname{coker}(B^n \to Z^n)$, it follows that the maps $Z^n(C^\bullet) \to Z^n(D^\bullet)$ and $B^n(C^\bullet) \to B^n(D^\bullet)$ induce the map $H^n(C^\bullet) \to H^n(D^\bullet)$. The lemma follows. $\square$

Proposition A.21. *For each $n \in \mathbb{Z}$, the three assignments $C^\bullet \mapsto Z^n(C^\bullet)$, $C^\bullet \mapsto B^n(C^\bullet)$, and $C^\bullet \mapsto H^n(C^\bullet)$ give covariant additive functors Z^n, B^n, H^n from $\mathrm{Ch}^\bullet(\mathcal{A})$ to $\mathcal{A}$.*

Proof. Uniqueness in the previous lemma implies that H^n sends the identity to the identity. Moreover, if we have morphisms $C^\bullet \xrightarrow{f^\bullet} D^\bullet \xrightarrow{g^\bullet} E^\bullet$, then we may graft the commutative diagrams from the previous lemma induced by $f^\bullet$ and $g^\bullet$ to get the two commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^n(C^\bullet) & \longrightarrow & Z^n(C^\bullet) & \longrightarrow & H^n(C^\bullet) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle B^n(f^\bullet)} & & \downarrow{\scriptstyle Z^n(f^\bullet)} & & \downarrow{\scriptstyle H^n(f^\bullet)} & & \\
0 & \longrightarrow & B^n(D^\bullet) & \longrightarrow & Z^n(D^\bullet) & \longrightarrow & H^n(D^\bullet) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle B^n(g^\bullet)} & & \downarrow{\scriptstyle Z^n(g^\bullet)} & & \downarrow{\scriptstyle H^n(g^\bullet)} & & \\
0 & \longrightarrow & B^n(E^\bullet) & \longrightarrow & Z^n(E^\bullet) & \longrightarrow & H^n(E^\bullet) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z^n(C^\bullet) & \longrightarrow & C^n & \longrightarrow & B^{n+1}(C^\bullet) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Z^n(f^\bullet)} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle B^{n+1}(f^\bullet)} & & \\
0 & \longrightarrow & Z^n(D^\bullet) & \longrightarrow & D^n & \longrightarrow & B^{n+1}(D^\bullet) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle Z^n(g^\bullet)} & & \downarrow{\scriptstyle g^n} & & \downarrow{\scriptstyle B^{n+1}(g^\bullet)} & & \\
0 & \longrightarrow & Z^n(E^\bullet) & \longrightarrow & E^n & \longrightarrow & Bn+1n(E^\bullet) & \longrightarrow & 0
\end{array}$$

for all $n \in \mathbb{Z}$, in which all rows are exact. The uniqueness clause of the previous lemma then implies that $Z^n(g^\bullet) \circ Z^n(f^\bullet) = Z^n(g^\bullet \circ f^\bullet)$, $B^n(g^\bullet) \circ B^n(f^\bullet) = B^n(g^\bullet \circ f^\bullet)$, and $H^n(g^\bullet) \circ H^n(f^\bullet) = H^n(g^\bullet \circ f^\bullet)$. The result follows. $\square$

Theorem A.22 (Long Exact Sequence)**.** *Given a short exact sequence $0 \to C^\bullet \xrightarrow{f^\bullet} D^\bullet \xrightarrow{g^\bullet} E^\bullet \to 0$ of chain complexes in $\mathrm{Ch}(\mathcal{A})$, there exist natural maps $\partial : H^n(E^\bullet) \to H^{n+1}(C^\bullet)$ called* connecting homomorphisms*, such we have a long exact sequence*

$$\cdots \xrightarrow{H^{n-1}(g^\bullet)} H^{n-1}(E^\bullet) \xrightarrow{\partial} H^n(C^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(D^\bullet) \xrightarrow{H^n(g^\bullet)} H^n(E^\bullet) \xrightarrow{\partial} H^{n+1}(C^\bullet) \xrightarrow{H^{n+1}(f^\bullet)} \cdots.$$

The connecting homomorphisms are natural in the sense that for any commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1^\bullet & \longrightarrow & D_1^\bullet & \longrightarrow & E_1^\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_2^\bullet & \longrightarrow & D_2^\bullet & \longrightarrow & E_2^\bullet & \longrightarrow & 0
\end{array}$$

there is a commutative ladder diagram

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial} & H^n(C_1^\bullet) & \longrightarrow & H^n(D_1^\bullet) & \longrightarrow & H^n(E_1^\bullet) & \xrightarrow{\partial} & H^{n+1}(C_1^\bullet) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\partial} & H^n(C_2^\bullet) & \longrightarrow & H^n(D_2^\bullet) & \longrightarrow & H^n(E_2^\bullet) & \xrightarrow{\partial} & H^{n+1}(C_2^\bullet) & \longrightarrow & \cdots
\end{array}$$

Proof. The Snake Lemma applied to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^n & \longrightarrow & D^n & \longrightarrow & E^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_C^n} & & \downarrow{\scriptstyle d_D^n} & & \downarrow{\scriptstyle d_E^n} & & \\
0 & \longrightarrow & C^{n+1} & \longrightarrow & D^{n+1} & \longrightarrow & E^{n+1} & \longrightarrow & 0
\end{array}$$

with exact rows yields the exact sequence

$$0 \to Z^n(C^\bullet) \to Z^n(D^\bullet) \to Z^n(E^\bullet) \to \operatorname{coker} d_C^n \to \operatorname{coker} d_D^n \to \operatorname{coker} d_E^n \to 0.$$

In particular, in the commutative diagram

$$\begin{array}{ccccccccc}
 & & \operatorname{coker} d_C^{n-1} & \longrightarrow & \operatorname{coker} d_D^{n-1} & \longrightarrow & \operatorname{coker} d_E^{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_C^n} & & \downarrow{\scriptstyle d_D^n} & & \downarrow{\scriptstyle d_E^n} & & \\
0 & \longrightarrow & Z^{n+1}(C^\bullet) & \longrightarrow & Z^{n+1}(D^\bullet) & \longrightarrow & Z^{n+1}(E^\bullet) & &
\end{array}$$

both rows are exact. Let us study the kernel and cokernel of the map $\operatorname{coker} d^{n-1} \to Z^{n+1}C^\bullet$, where we write d^{n-1} for d_C^{n-1}. Recall that this map is the composition $\operatorname{coker} d^{n-1} \twoheadrightarrow \operatorname{im} d^n = B^{n+1}C^\bullet \hookrightarrow Z^{n+1}C^\bullet$. In particular, this implies that

$$\operatorname{coker}\left(\operatorname{coker} d^{n-1} \to Z^{n+1}C^\bullet\right) = \operatorname{coker}\left(B^{n+1}C^\bullet \hookrightarrow Z^{n+1}C^\bullet\right) =: H^{n+1}C^\bullet.$$

Similarly,

$$\ker\left(\operatorname{coker} d^{n-1} \to Z^{n+1}C^\bullet\right) = \ker\left(\operatorname{coker} d^{n-1} \twoheadrightarrow B^{n+1}\right).$$

However, the Snake Lemma applied to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^n & \longrightarrow & C^{n-1} & \longrightarrow & \operatorname{coker} d^{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^n & \longrightarrow & C^n & \longrightarrow & B^{n+1} & \longrightarrow & 0
\end{array}$$

where both rows are exact shows that

$$\ker\left(\operatorname{coker} d^{n-1} \twoheadrightarrow B^{n+1}\right) = \operatorname{coker}\left(B^n \to Z^n\right) = H^n C^\bullet.$$

Thus, the Snake Lemma applied to the commutative diagram with exact rows

$$\begin{array}{ccccccc} & & \operatorname{coker} d_C^{n-1} & \longrightarrow & \operatorname{coker} d_D^{n-1} & \longrightarrow & \operatorname{coker} d_E^{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_C^n} & & \downarrow{\scriptstyle d_D^n} & & \downarrow{\scriptstyle d_E^n} & & \\ 0 & \longrightarrow & Z^{n+1}(C^\bullet) & \longrightarrow & Z^{n+1}(D^\bullet) & \longrightarrow & Z^{n+1}(E^\bullet) & & \end{array}$$

yields the six term exact sequence

$$H^n C^\bullet \to H^n D^\bullet \to H^n E^\bullet \to H^{n+1} C^\bullet \to H^{n+1} D^\bullet \to H^{n+1} E^\bullet.$$

Patching these six term exact sequences together yields the long exact sequence.

We now need to show functoriality. By functoriality of H^n, it suffices to show that the diagram

$$\begin{array}{ccc} H^n E_1^\bullet & \xrightarrow{\partial} & H^{n+1} C_1^\bullet \\ \downarrow & & \downarrow \\ H^n E_2^\bullet & \xrightarrow{\partial} & H^{n+1} C_2^\bullet \end{array}$$

commutes. This follows from the corresponding functoriality properties of the connecting homomorphism in the Snake Lemma. □

A.4.2 (Co)Chain Homotopies and Quasi-Isomorphisms

Definition. A morphism $f^\bullet : C^\bullet \to D^\bullet$ is said to be *null-homotopic* if there exist maps $s^n : C^n \to D^{n-1}$ such that

$$f^n = d_D^{n-1} \circ s^n + s^{n+1} \circ d_C^n$$

as maps from $C^n \to D^n$. The sequence of maps $\{s^n\}$ is called the *cochain contraction* of f.

Two morphisms $f^\bullet, g^\bullet : C^\bullet \to D^\bullet$ are said to be *(cochain) homotopic* if $f^\bullet - g^\bullet$ is null-homotopic. The cochain contraction $\{s^n\}$ corresponding to $f^\bullet - g^\bullet$ is called a *cochain homotopy* from $f^\bullet$ to $g^\bullet$. It is obvious that the relationship of being cochain homotopic induces an equivalence relation on $\operatorname{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)$.

A morphism $f^\bullet : C^\bullet \to D^\bullet$ is a *cochain homotopy equivalence* if there exists a morphism $g^\bullet : D^\bullet \to C^\bullet$ such that $g^\bullet \circ f^\bullet$ and $f^\bullet \circ g^\bullet$ are cochain homotopic to the respective identity morphisms on $C^\bullet$ and $D^\bullet$ respectively.

Lemma A.23. *A null-homotopic cochain morphism $f^\bullet : C^\bullet \to D^\bullet$ induces the zero map on cohomology, i.e. $H^n(f^\bullet) = 0$ for all $n \in \mathbb{Z}$.*

Proof. Write $f^n = d_D^{n-1} \circ s^n + s^{n+1} \circ d_C^n$ for a cochain contraction $\{s^n\}$. It suffices to show that $Z^n(f^\bullet) : Z^n C^\bullet \to Z^n D^\bullet$ factors through $B^n D^\bullet \to Z^n D^\bullet$. Notice that $Z^n(f^\bullet)$ is the map induced by the composition $Z^n C^\bullet \hookrightarrow C^n \xrightarrow{f^n} D^n$. Now, as $Z^n C^\bullet = \ker d_C^n$, it follows that the composition $Z^n C^\bullet \hookrightarrow C^n \xrightarrow{s^{n+1} d_C^n} D^n$ is identically zero. Thus the composition $Z^n C^\bullet \hookrightarrow C^n \xrightarrow{f^n} D^n$ is equal to the composition $Z^n C^\bullet \hookrightarrow C^n \xrightarrow{d_D^{n-1} s^n} D^n$. However, this factors through $B^n D^\bullet \hookrightarrow D^n$ since d_D^{n-1} factors through $B^n D^\bullet \hookrightarrow D^n$. Hence $Z^n(f^\bullet)$ factors through $B^n D^\bullet \hookrightarrow D^n$ as required. □

Corollary A.23.1. *If two cochain morphisms $f^\bullet, g^\bullet : C^\bullet \to D^\bullet$ are homotopic, then $H^n f^\bullet = H^n g^\bullet$ as morphisms in $\mathcal{A}$ for all $n \in \mathbb{Z}$.*

Definition. A cochain morphism $f^\bullet : C^\bullet \to D^\bullet$ is said to be a *quasi-isomorphism* if $H^n f^\bullet : H^n C^\bullet \to H^n D^\bullet$ is an isomorphism in $\mathcal{A}$ for all $n \in \mathbb{Z}$. Two cochains are quasi-isomorphic if there exists a quasi-isomorphism from one to the other.

Lemma A.24. *The following are equivalent for a cochain $C^\bullet \in \mathrm{Ch}^\bullet(\mathcal{A})$:*

1. *$C^\bullet$ is exact, i.e. $B^n C^\bullet \cong Z^n C^\bullet$ for all $n \in \mathbb{Z}$;*
2. *$H^n C^\bullet = 0$ for all $n \in \mathbb{Z}$;*
3. *The canonical cochain map $0^\bullet \hookrightarrow C^\bullet$ is a quasi-isomorphism.*

Proof. The equivalence (1) ⇔ (2) is trivial since $\operatorname{coker}(B^nC^\bullet \hookrightarrow Z^nC^\bullet) = H^nC^\bullet$, and since in an abelian category a morphism is epic if and only if it has trivial cokernel, and a morphism is an isomorphism if and only if it is both monic and epic. The equivalence (2) ⇔ (3) is also trivial, since $H^n(0^\bullet \hookrightarrow C^\bullet) : 0 \hookrightarrow H^nC^\bullet$ is an isomorphism if and only if $H^nC^\bullet = 0$. □

Lemma A.25. *Consider the equivalence relation on* $\operatorname{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)$ *given by* $f^\bullet \sim g^\bullet$ *if and only if* $f^\bullet$ *and* $g^\bullet$ *are homotopic. Then, the set of equivalence classes* $\operatorname{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)/\sim$ *naturally inherits the abelian group structure on* $\operatorname{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)$.

Proof. Notice that $f^\bullet \sim g^\bullet$ if and only if $f^\bullet - g^\bullet$ is null-homotopic. The result then follows if we show that the subset of $\operatorname{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)$ consisting of null-homotopic maps is a subgroup of the abelian group $\operatorname{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)$. This is obvious however, since the zero map $0^\bullet : C^\bullet \to D^\bullet$ is obviously null-homotopic, and if $f^\bullet$ and $g^\bullet$ are null-homotopic with cochain contractions $\{s^n\}$ and $\{t^n\}$, then

$$f^n + g^n = d_D^{n-1}s^n + s^{n+1}d_C^n + d_D^{n-1}t^n + t^{n+1}d_C^n = d_D^{n-1}(s^n + t^n) + (s^{n+1} + t^{n+1})d_C^n$$

and so $f^\bullet + g^\bullet$ is null-homotopic with cochain contraction $\{s^n + t^n\}$. □

Lemma A.26. *Suppose we have cochain morphisms* $u^\bullet : B^\bullet \to C^\bullet$, $f^\bullet, g^\bullet : C^\bullet \to D^\bullet$, *and* $v^\bullet : D^\bullet \to E^\bullet$. *If* $f^\bullet$ *and* $g^\bullet$ *are homotopic, then so are* $v^\bullet f^\bullet u^\bullet$ *and* $v^\bullet g^\bullet u^\bullet$.

Proof. If $\{s^n\}$ is a cochain homotopy from $f^\bullet$ to $g^\bullet$, then it is easy to see that $\{v^{n-1} \circ s^n \circ u^n\}$ is a cochain homotopy from $v^\bullet f^\bullet u^\bullet$ to $v^\bullet g^\bullet u^\bullet$. □

The previous two lemmas yield the following.

Lemma-Definition. There is an additive category $\mathbb{K}^\bullet(\mathcal{A})$ whose objects are the same as objects of $\mathrm{Ch}^\bullet(\mathcal{A})$ (i.e. cochain complexes) and whose morphisms are the set of homotopy equivalence classes of $\operatorname{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)$. Moreover, the obvious functor $\mathrm{Ch}^\bullet(\mathcal{A}) \to \mathbb{K}^\bullet(\mathcal{A})$ is additive.

This category is the *homotopy category* of chain complexes of $\mathcal{A}$. The functor $\mathrm{Ch}^\bullet(\mathcal{A}) \to \mathbb{K}^\bullet(\mathcal{A})$ is often called the *projection functor*.

For any full subcategory $\mathcal{C}$ of $\mathrm{Ch}^\bullet(\mathcal{A})$, we have a full subcategory $\mathcal{K} \subset \mathbb{K}^\bullet(\mathcal{A})$ whose objects are the cochain complexes in $\mathcal{C}$. Then, the restriction of the projection functor $\mathrm{Ch}^\bullet(\mathcal{A}) \to \mathbb{K}^\bullet(\mathcal{A})$ to $\mathcal{C}$ yields another projection functor $\mathcal{C} \to \mathcal{K}$, in the sense that $\operatorname{Hom}_{\mathcal{K}}$ is still the quotient of $\operatorname{Hom}_{\mathcal{C}}$ by homotopy equivalence. In particular, we have the full subcategory $\mathbb{K}^\bullet_{\geq 0}(\mathcal{A})$ of $\mathbb{K}(\mathcal{A})$ corresponding to $\mathrm{Ch}^\bullet_{\geq 0}(\mathcal{A})$.

Remark A.27. The homotopy category is usually *not* an abelian category.

Remark A.28. From now on, we will drop the superscript bullets in our notation, for simplicity. The reason is that in the literature, the derived category is always constructed from the category of cochain complexes, and so in this appendix we will always fix our numbering to be cohomological.

We will also now drop the superscript bullet from $C^\bullet \in \mathrm{Ch}(\mathcal{A})$ and from $f^\bullet : C^\bullet \to D^\bullet$, instead writing $C \in \mathrm{Ch}(\mathcal{A})$ and $f : C \to D$. Again, this shouldn't cause confusion.

Since cochain homotopic maps induce the same morphisms on cohomology, we have the following lemma.

Lemma A.29. *The cohomology* $H^n(C)$ *of a cochain complex* C *induces a well-defined family of functors* $H^n : \mathbb{K}(\mathcal{A}) \to \mathcal{A}$. *Moreover, the composition of functors* $\mathrm{Ch}(\mathcal{A}) \to \mathbb{K}(\mathcal{A}) \xrightarrow{H^n} \mathcal{A}$ *corresponds to the family of functors* $H^n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ *given in the previous subsection.*

We omit the proof of the following proposition (see [Wei94, Proposition 10.1.2])

Proposition A.30 (Universal Property of the Homotopy Category)**.** *Suppose* $F : \mathrm{Ch}(\mathcal{A}) \to \mathcal{D}$ *is any functor that sends homotopy equivalences to isomorphisms. Then,* F *factors uniquely as* $\mathrm{Ch}(\mathcal{A}) \to \mathbb{K}(\mathcal{A}) \to \mathcal{D}$.

A.5 The Derived Category

In order to define the derived category, we need to define localizations of categories.

Definition. Let S be a collection of morphisms in a category $\mathcal{C}$. A *localization of* $\mathcal{C}$ *with respect to* S is a category $S^{-1}\mathcal{C}$ together with a functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that

1. $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$ for all $s \in S$; and
2. any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through q.

As usual, the localization of a category (if it exists) is unique up to unique equivalence.

Example A.31. Consider the category $\mathrm{Ch}(\mathcal{A})$ for an abelian category $\mathcal{A}$, and let S be the collection of chain homotopy equivalences. Then the universal property of the homotopy category implies that $\mathbb{K}(\mathcal{A})$ is in fact the localization $S^{-1}\mathrm{Ch}(\mathcal{A})$.

Definition. Let $\mathcal{A}$ be an abelian category. Let Q be the collection of all quasi-isomorphisms in $\mathbb{K}(\mathcal{A})$. The *derived category* $D(\mathcal{A})$ of $\mathcal{A}$ is defined to be the localization $Q^{-1}\mathbb{K}(\mathcal{A})$.

Similarly, we can define $D^{\geq 0}(\mathcal{A})$ (resp. $D^{\leq 0}(\mathcal{A})$) to be the localization $Q^{-1}\mathbb{K}^{\geq 0}(\mathcal{A})$ (resp. $Q^{-1}\mathbb{K}^{\leq 0}(\mathcal{A})$). We can also localize other full subcategories of $\mathbb{K}(\mathcal{A})$; for instance the category $\mathbb{K}^{+}(\mathcal{A})$ of all cochains in $\mathcal{A}$ bounded below to get $D^{+}(\mathcal{A})$, or the category $\mathbb{K}^{-}(\mathcal{A})$ of all cochains in $\mathcal{A}$ bounded above to get $D^{-}(\mathcal{A})$, or the category $\mathbb{K}^{b}(\mathcal{A}) = \mathbb{K}^{+}(\mathcal{A}) \cap \mathbb{K}^{-}(\mathcal{A})$ of all bounded cochains to get $D^{b}(\mathcal{A})$.

Remark A.32. If $\tilde{Q}$ is the collection of all quasi-isomorphisms in $\mathbb{K}(\mathcal{A})$, then the derived category $D(\mathcal{A})$ is naturally equivalent to the localization of $\mathrm{Ch}(\mathcal{A})$ at $\tilde{Q}$, and so one could equally well define it directly this way. However, in order to show existence, it is easier to explicitly describe the morphisms of $Q^{-1}\mathbb{K}(\mathcal{A})$ instead of directly working with $\tilde{Q}^{-1}\mathrm{Ch}(\mathcal{A})$.

We will sweep set-theoretic issues under the rug, though for sufficiently nice $\mathcal{A}$s they will not pose any significant obstruction. Also, the existence of Cartan-Eilenberg resolutions for a category with enough injectives (resp. projectives) allows one to construct $D^{\geq 0}(-)$ (resp. $D^{\leq 0s}(-)$) without worrying about set-theoretic issues.

Definition. A collection S of morphisms in an additive category $\mathcal{C}$ is a *multiplicative system* in $\mathcal{C}$ if it satisfies the following axioms:

- M1 S is closed under composition and contains all identity morphisms;
- M2 (Left Ore condition) if $t : X \to X'$ is in S and $g : X \to Y$ is an arbitrary morphism in $\mathcal{C}$, then there exists $Y' \in \mathcal{C}$ and there exist morphisms $f : X' \to Y'$ and $s : Y \to Y'$ such that

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\ X' & \xrightarrow{f} & Y' \end{array}$$

  commutes and that $s \in S$.
- M3 (Right Ore condition) if $s : Y \to Y'$ is in S and $f : X' \to Y'$ is an arbitrary morphism in $\mathcal{C}$, then there exists $X \in \mathcal{C}$ and there exist morphisms $g : X \to Y$ and $t : Y \to Y'$ such that the previous diagram commutes and that $t \in S$.
- M4 (Cancellation) For every pair of morphisms $f, g : X \to Y$ in $\mathcal{C}$, the following are equivalent:
  - there exists $s \in S$ with source Y such that $s \circ f = s \circ g$;
  - there exists $t \in S$ with target X such that $f \circ t = g \circ t$.

Sweeping set-theoretic issues under the rug by assuming S and $\mathcal{C}$ are nice enough, we are now in a position to define $S^{-1}\mathcal{C}$ if S is a multiplicative system in the above sense. We take the objects of $S^{-1}\mathcal{C}$ to be the objects of $\mathcal{C}$, and for $X, Y \in S^{-1}\mathcal{C}$ we set

$$\mathrm{Hom}_{S^{-1}\mathcal{C}}(X, Y) := \operatorname*{colim}_{X' \in \mathcal{I}_X} \mathrm{Hom}_{\mathcal{C}}(X', Y)$$

where $\mathcal{I}_X$ is the category of objects X' equipped with morphisms $s : X' \to X$ such that $s \in S$. We define composition as follows:

Suppose $f \in \mathrm{Hom}_{S^{-1}\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{S^{-1}\mathcal{C}}(Y, Z)$. Then, there exists $s : X' \to X$ with $s \in S$ and there exists $t : Y' \to Y$ with $t \in S$ such that $f : X' \to Y$ and $g : Y' \to Z$. By the Ore property, there exists $X'' \in \mathcal{C}$ and morphisms $f' : X'' \to Y'$ and $s' : X'' \to X'$ with $s' \in S$ such that $t \circ f' = f \circ s'$. Then, $(s \circ s' : X'' \to X) \in S$, and we have the morphism $g \circ f' : X'' \to Z$. This then gives an element of $\mathrm{Hom}_{S^{-1}\mathcal{C}}(X, Z)$. Suppose we picked a different $\bar{X}''$ with different morphisms $\bar{f}' : \bar{X}'' \to Y'$ and $\bar{s}' : \bar{X}'' \to X'$ with $\bar{s}' \in S$ such that $t \circ \bar{f}' = f \circ \bar{s}'$. The ore condition then gives $W \in \mathcal{C}$ and maps $s'' : W \to X''$, $\bar{s}'' : W \to \bar{X}''$ such that $s' \circ s'' = \bar{s}' \circ \bar{s}'' : W \to X'$, with $s'' \in S$. Then we have a diagram

$$\begin{array}{ccc} W & \overset{f' \circ s''}{\underset{\bar{f}' \circ \bar{s}''}{\rightrightarrows}} & Y' \\ {\scriptstyle \bar{s}' \circ \bar{s}'' = s' \circ s''}\downarrow & & \downarrow{\scriptstyle t} \\ X' & \xrightarrow[f]{} & Y \end{array}$$

which computes. Thus $t \circ f' \circ s'' = f \circ s' \circ s'' = t \circ \bar{f}' \circ \bar{s}''$. The cancellation property then implies the existence of $\tilde{X} \in \mathcal{C}$ and a $\tilde{s} : \tilde{X} \to W$ such that $f' \circ s'' \circ \tilde{s} = \bar{f}' \circ \bar{s}'' \circ \tilde{s}$. It then follows that the pair $(s \circ s' : X'' \to X, g \circ f' : X'' \to Z)$ and $(s \circ \bar{s}' : \bar{X}'' \to X, g \circ \bar{f}' : \bar{X}'' \to Z)$ both correspond to the same element

$$(s \circ s' \circ s'' \circ \tilde{s} : \tilde{X} \to X, g \circ f' \circ s'' \circ \tilde{s} : \tilde{X} \to Z)$$

in the colimit defining $\mathrm{Hom}_{S^{-1}\mathcal{C}}(X, Z)$, where notice that $s \circ s' \circ s'' \circ \tilde{s} \in S$. Hence composition is well-defined. It is clear that $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X) \to \mathrm{Hom}_{S^{-1}\mathcal{C}}(X, X)$ since $\mathrm{id}_X \in S$.

Theorem A.33 (Gabriel-Zisman)**.** *The above description indeed makes $S^{-1}\mathcal{C}$ a category. Moreover, it is the localization of $\mathcal{C}$ at S.*

Proof. We need to show that the above composition so-defined is associative, and that $S^{-1}\mathcal{C}$ satisfies the universal property of localizations. In the diagram below, we can add in all of the dashed lines by the ore property, to get the following commutative diagram where all vertical arrows are in S.

$$\begin{array}{ccccccc}
W_2 & \overset{f_2}{\dashrightarrow} & X_1 & \overset{g_1}{\dashrightarrow} & Y' & \overset{h}{\longrightarrow} & Z \\
{\scriptstyle w_2}\downarrow & & \downarrow{\scriptstyle x'} & & \downarrow{\scriptstyle y} & & \\
W_1 & \overset{f_1}{\dashrightarrow} & X' & \overset{g}{\longrightarrow} & Y & & \\
{\scriptstyle w_1}\downarrow & & \downarrow{\scriptstyle x} & & & & \\
W' & \overset{f}{\longrightarrow} & X & & & & \\
{\scriptstyle w}\downarrow & & & & & & \\
W & & & & & &
\end{array}$$

It is then clear from this diagram that both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ correspond to the element $h \circ g_1 \circ f_2 : W_2 \to Z$ in the colimit defining $\mathrm{Hom}_{S^{-1}\mathcal{C}}(W, Z)$.

Now, the localization functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ is given by the identity on the class of objects, and on morphisms sends $f : X \to Y$ to the image of f in the colimit $\mathrm{Hom}_{S^{-1}\mathcal{C}}(X, Y)$. It is clear that q sends morphisms in S to isomorphisms in $S^{-1}\mathcal{C}$, since the inverse morphism of $s : X \to Y$ in $\mathrm{Hom}_{S^{-1}\mathcal{C}}(Y, X)$ is the image of id_X under $\mathrm{Hom}_{\mathcal{C}}(X, X) \to \mathrm{Hom}_{S^{-1}\mathcal{C}}(Y, X)$. The universal property is also easy to see from the universal property of the colimit. □

Remark A.34. Throughout, we are only ever using the fact that S is *right-multiplicative*. This is a far weaker condition. For S left multiplicative we then have

$$\mathrm{Hom}_{S^{-1}\mathcal{C}}(X, Y) = \mathrm{colim}_{Y' \in \mathcal{I}^Y} \mathrm{Hom}_{\mathcal{C}}(X, Y')$$

where $\mathcal{I}^Y$ is the (filtered) category of objects $Y' \in \mathcal{C}$ with a morphism $(Y \to Y') \in S$.

If S is multiplicative, then both definitions coincide.

Proposition A.35. *Let $\mathcal{A}$ be an abelian category with homotopy category $\mathbb{K}(\mathcal{A})$. Recall the cohomology functors $H^n : \mathbb{K}(\mathcal{A}) \to \mathcal{A}$. The set S of all quasi-isomorphisms (i.e. those morphisms f such that $H^n(f)$ is an isomorphism in $\mathcal{A}$ for all $n \in \mathbb{Z}$) is multiplicative.*

Proof. See [Ked, Proposition 10.3.3]. □

Definition. The *derived category* $D(\mathcal{A})$ is the localization of $\mathbb{K}(\mathcal{A})$ at quasi-isomorphisms. Similarly we can define the full subcategories $D^{\geq 0}(\mathcal{A})$, $D^{\leq 0}(\mathcal{A})$, and so on. Write $Q : \mathbb{K}(\mathcal{A}) \to D(\mathcal{A})$ for the canonical localization functor.

Since $H^n : \mathbb{K}(\mathcal{A}) \to \mathcal{A}$ takes quasi-isomorphisms to isomorphisms (by definition), it induces the n'th cohomology functor $H^n : D(\mathcal{A}) \to \mathcal{A}$.

Remark A.36. Rather than thinking of objects of $D(\mathcal{A})$ as honest cochain complexes, it is morally easier to think of them as equivalence classes of cochain complexes up to quasi-isomorphism. Quite a few functors are only defined up to quasi-isomorphism, for instance all derived functors.

Remark A.37. We also have the full subcategory $D^+(\mathcal{A})$ of all chains bounded below, i.e. the chains $C \in D^+(\mathcal{A})$ such that $H^n C = 0$ for all sufficiently negative n. We can similarly define the full subcategory $D^-(\mathcal{A})$ of all chains that are bounded above. These are equivalent in an obvious way to the subcategories consisting of those cochains C such that $C^n = 0$ for all sufficiently negative n, or all sufficiently positive n, respectively.

Suppose from now on that $\mathcal{A}$ has enough injectives. Then, we can describe $D^+(\mathcal{A})$ (the category of all cochains homotopy equivalent to cochains C such that $C^n = 0$ for all $n < 0$).

Lemma A.38. *If $I \in \mathbb{K}(\mathcal{A})$ is a complex of injective objects, bounded below, then for any complex $X \in D(\mathcal{A})$ the canonical morphism*

$$\mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(X, I) \to \mathrm{Hom}_{D(\mathcal{A})}(X, I)$$

is an isomorphism.

Proposition A.39. *Suppose $\mathcal{A}$ has enough injectives. For each complex $X \in D^+(\mathcal{A})$ there is an injective complex I bounded below quasi-isomorphic to X. Moreover, such an I is uniquely determined up to homotopy.*

Definition. The above injective complex I quasi-isomorphic to X is called an *injective resolution* of X.

Proof. Suppose WLOG that $C^n = 0$ for $n < 0$. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $C^\bullet$, with the differentials given by $d_v^{p,q} : I^{p,q} \to I^{p,q+1}$ and $d_h^{p,q} : I^{p,q} \to I^{p+1,q}$. Define

$$I^n := \bigoplus_{p+q=n} I^{p,q}, \quad \text{and} \quad d_I^n := \sum_{p+q=n} (d_v^{p,q} + d_h^{p,q}).$$

By definition, we are given maps $C^n \to I^{n,0}$, which then obviously extend to maps $C^n \to I^n$. A quick argument with the spectral sequence corresponding to the double complex $E_{0,\to}^{\bullet,\bullet} = I^{\bullet,\bullet}$ shows that $E_{2,\to}^{p,q} = 0$ if $q > 0$ and $E_{2,\to}^{p,0} = H^pC$, and hence

$$H^nC = H^nI.$$

Hence the map $C^\bullet \to I^\bullet$ is a quasi-isomorphism. Using the previous lemma, we see that two injective resolutions for C are canonically isomorphic in $D(\mathcal{A})$. □

Combining the previous lemma and the previous proposition, we get the following result.

Proposition A.40. *Suppose $\mathcal{A}$ has enough injectives. The injective resolution functor $D^+(\mathcal{A}) \to \mathbb{K}(\mathcal{A})$ represents an equivalence of categories between $D^+(\mathcal{A})$ and the full subcategory of $\mathbb{K}(\mathcal{A})$ formed of injective complexes bounded below.*

A.6 Derived Functor

Consider an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}, \mathcal{B}$. Suppose also that $\mathcal{A}$ has enough injectives. By defining it component-wise, we can extend this to an additive functor

$$F : D(\mathcal{A}) \to D(\mathcal{B})$$

which takes homotopies to homotopies. Thus we get a naturally induced functor

$$F : \mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathcal{B}).$$

If F is moreover exact, then it sends quasi-isomorphisms to quasi-isomorphisms, and so we get an induced functor

$$F : D(\mathcal{A}) \to D(\mathcal{B}).$$

If F is only left or right exact, then this is not possible. However, it is possible to define a *derived functor* associated to F.

Definition. The *(total) right derived functor* (if it exists) of $F : \mathcal{A} \to \mathcal{B}$ is a functor $\mathrm{R}F : D(\mathcal{A}) \to D(\mathcal{B})$ equipped with a natural transformation $\alpha : Q \circ F \Rightarrow \mathrm{R}F \circ Q$ of functors from $\mathbb{K}(\mathcal{A}) \to D(\mathcal{B})$, satisfying the following universal property: If $G : D(\mathcal{A}) \to D(\mathcal{B})$ is another functor, then the natural transformation α induces a bijection

$$\mathrm{Hom}(RF, G) \to \mathrm{Hom}(Q \circ F, G \circ Q), \quad \beta \mapsto (\beta \circ Q) \circ \alpha.$$

The following diagrams are useful:

$$\begin{array}{ccc} \mathbb{K}(\mathcal{A}) & \xrightarrow{F} & \mathbb{K}(\mathcal{B}) \\ {\scriptstyle Q}\downarrow & \overset{\alpha}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle Q} \\ D(\mathcal{A}) & \underset{\mathrm{R}F}{\dashrightarrow} & D(\mathcal{B}) \end{array} \qquad \begin{array}{ccccc} & & Q \circ F & & \\ & \overset{\alpha}{\swarrow\!\!\swarrow} & & \searrow & \\ \mathrm{R}F \circ Q & & \underset{\beta \circ Q}{\Longrightarrow} & & G \circ Q \end{array}$$

Remark A.41. We can define (total) left-derived functors $\mathrm{L}F$ similarly.

Remark A.42. The universal property of right derived functors is an example of a *right Kan extension*.

Here, and throughout, we will always embed $A \hookrightarrow \mathbb{K}(\mathcal{A})$ as a full subcategory by taking $X \in \mathcal{A}$ to the cochain $X^\bullet$ given by $X^0 = X$ and $X^n = 0$ for $n \neq 0$.

Definition. A complex $X \in \mathcal{A}$ is a *dg-injective complex* if X^n is injective for all n, and for all acyclic complexes I every chain map $I \to X$ is null-homotopic.

Example A.43. All injective complexes that are bounded above or below are dg-injective complexes.

Theorem A.44. *If $\mathcal{A}$ has enough injectives and F is an additive covariant left-exact functor, then $\mathrm{R}F$ exists when restricted to a functor $RF : D^+(\mathcal{A}) \to D(\mathcal{B})$.*

If any of the following conditions hold:

- *$\mathcal{A}$ and $\mathcal{B}$ are the category of abelian sheaves on a site (recall that regular Mod_R can also considered as an abelian sheaf on some site); or*
- *for any quasi-isomorphism $f : X \to Y$ between dg-injective complexes X and Y, the cochain map $Q \circ F(f)$ is an isomorphism in $D(\mathcal{B})$;*

then $\mathrm{R}F$ exists as a functor on the entire category $D(\mathcal{A})$, i.e. $\mathrm{R}F : D(\mathcal{A}) \to D(\mathcal{B})$ is well-defined.

The composition of functors

$$\mathbb{K}^+(\mathcal{A}) \xrightarrow{Q} D^+(\mathcal{A}) \xrightarrow{\mathrm{R}F} D(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}$$

for all n is a universal δ-functor, i.e. any short exact sequence of complexes

$$0 \to X \to Y \to Z \to 0$$

yields a long exact sequence

$$\cdots \to R^{n-1}F(Z) \to R^nF(X) \to R^nF(Y) \to R^nF(Z) \to R^{n+1}F(X) \to \cdots$$

in a universal way. In fact, when restricted to the full subcategory $\mathcal{A} \subset \mathbb{K}^+(\mathcal{A})$, the above δ-functor is precisely the n'th right-derived functor R^nF described in the context of (universal) δ-functors.

Remark A.45. The full proof is incredibly hard, and also not very important. For $D^+(\mathcal{A})$, the proof uses a Cartan-Eilenberg resolution. In fact, let $C \in D^+(\mathcal{A})$; then if I is an injective resolution of C it follows that

$$\mathrm{R}F(C) = F(I).$$

More generally, a full additive subcategory $\mathcal{I}$ of $\mathcal{A}$ is said to be *F-injective* if the following conditions hold:

- every object of $\mathcal{A}$ is a subobject of an object in $\mathcal{I}$;
- if $0 \to A' \to A \to A'' \to 0$ is exact in $\mathcal{A}$, and A' and A are in $\mathcal{I}$, then A'' is also in $\mathcal{I}$;
- if $0 \to A' \to A \to A'' \to 0$ is exact in $\mathcal{A}$, and $A' \in I$, then

  $$0 \to FA' \to FA \to FA'' \to 0$$

  is exact in $\mathcal{B}$.

If $\mathcal{A}$ has enough injectives, then the subcategory of all injectives is also F-injective. Recall that an object is *F-acyclic* if $R^nF(X) = 0$ for all $n > 0$. The full subcategory of all F-acyclic objects in $\mathcal{A}$ is F-injective.

If $\mathcal{I}$ is an F-injective subcategory of $\mathcal{A}$, then for every $X \in D^+(\mathcal{A})$ there exists $I \in D^+(\mathcal{A})$ with a quasi-isomorphism $X \to I$ such that $I^n \in \mathcal{I}$ for all $n \in \mathbb{Z}$; in such a case, FI is a well-defined object in $D^+(\mathcal{A})$, and we have

$$\mathrm{R}F(X) = FI.$$

Remark A.46. In almost all cases, we will only really need to consider RF on the category $D^+(\mathcal{A})$.

Definition. The objects $H^n(\mathrm{R}F) : D(\mathcal{A}) \to \mathcal{A}$ are called the *(classical) nth right derived functors of F*, and are denoted by R^nF.

The previous proposition tells us that this notion coincides with the previous notion of right derived functors coming from universal δ-functors.

Remark A.47. It should be noted that $\mathrm{R}F$ contains strictly more information than the classical derived functors R^nF, since it tells us about the existence of quasi-isomorphisms between certain (not necessarily canonical) cochain complexes.

Lemma A.48. *If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are left-exact functors and $\mathcal{A}$ has enough injectives, then we have a canonical spectral sequence*

$$R^q G \circ R^p F(X) \Rightarrow H^{p+q}\big(RG \circ RF(X)\big)$$

for all $X \in D^+(\mathcal{A})$.

Remark A.49. This shows how to construct a spectral sequence (for computation purposes) given the composition of two derived functors. Many classical statements giving spectral sequences are in fact proving some statement about a composition of total derived functors and then taking nth cohomology (which loses information!).

Remark A.50. The condition that $X \in D^+(\mathcal{A})$ is simply a mild boundedness condition just so that the spectral sequence actually converges. It is a sufficient, but certainly not necessary, condition.

Theorem A.51 (Grothendieck Spectral Sequence)**.** *If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ are additive left-exact functors such that for every injective object $M \in \mathcal{A}$, the objects $R^n F(M)$ is G-acyclic. Then,*

$$\mathrm{R}(G \circ F) = (\mathrm{R}G) \circ (\mathrm{R}F).$$

This formulation of the Grothendieck spectral sequence is in fact strictly stronger than the previous one. Not only does it tell us that the cohomology of the total complex associated to the double complex $R^q G \circ R^p F(X)$ are the same as the cohomology of $R(G \circ F)(X)$, it in fact tells us that these two complexes are quasi-isomorphic. This is a strictly stronger condition since it is very easy to write down two complexes that are not quasi-isomorphic whose cohomologies coincide.

Remark A.52. It is not true in general that $\mathrm{R}(G \circ F) = (\mathrm{R}G) \circ (\mathrm{R}F)$!

We conclude this section with a description of derived homs. Fix an abelian category $\mathcal{A}$. For $X, Y \in \mathrm{Ch}(\mathcal{A})$ with Y bounded below, consider

$$\mathrm{Hom}^r(X, Y) := \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X, Y[r])$$

where recall that $Y[r]^n = Y^{r+n}$ and $d^n_{Y[r]} = (-1)^r d_Y^{n+r}$. We can define a differential $d : \mathrm{Hom}^r(X, Y) \to \mathrm{Hom}^{r+1}(X, Y)$ given by

$$df := d_Y \circ f - (-1)^r f \circ d_X$$

for $f \in \mathrm{Hom}^r(X, Y)$. This yields a bifunctor

$$\mathrm{Hom}^\bullet : \mathbb{K}(\mathcal{A})^{op} \times \mathbb{K}^+(\mathcal{A}) \to \mathbb{K}(\mathrm{Ab}).$$

If $\mathcal{A}$ has enough injectives, we then get a corresponding (total) right derived functor

$$\mathrm{RHom} : D(\mathcal{A})^{op} \times D^+(\mathcal{A}) \to D(\mathrm{Ab}).$$

We have canonical isomorphisms

$$\mathrm{Hom}_{D(\mathcal{A})}(X, Y) \cong H^0\big(\mathrm{RHom}(X, Y)\big) \quad \text{and} \quad H^r\big(\mathrm{RHom}(X, Y)\big) \cong \mathrm{Hom}_{D(\mathcal{A})}(X, Y[r])$$

for all $X \in D(\mathcal{A})$ and all $Y \in D^+(\mathcal{A})$.

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