Notes for Quals Prep

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Contents

Chapter 1

Algebraic Geometry

1.1 Varieties and Algebraic Constructions

1.1.1 Varieties

Given a field k, we have *affine space* $\mathbb{A}_k^n = k^n$ and *projective space* $\mathbb{P}_k^n$. The space of all *regular functions* on $\mathbb{A}_k^n$ is simply $k[x_1, ..., x_n]$, and this defines the Zariski topology on $\mathbb{A}_k^n$. On the other hand, homogeneous polynomials in $k[x_0, ..., x_n]$ defines the Zariski topology on $\mathbb{P}_k^n$.

Given ideal $I \subset k[x_1, ..., x_n]$, we have the closed set $Z(I) = \{P \in \mathbb{A}_K^n : f(P) = 0 \forall f \in I\}$. Similarly $Z(I)$ defines closed sets in $\mathbb{P}^n$ whenever I is a homogeneous ideal.

Definition. A set X is *irreducible* if for any expression $X = C_1 \cup C_2$ where C_1, C_2 are closed subsets of X, we must have either $C_1 = X$ or $C_2 = X$.

A subset A of a topological space X is irreducible if it is irreducible in the subspace topology.

Open subsets of irreducible sets are irreducible and dense. Also, the closure of irreducible subsets is irreducible as well. The image of an irreducible set under a continuous map is also irreducible.

Closed irreducible subsets (or open subsets of closed irreducible subsets) of affine or projective spaces are *varieties*. Every closed subset in affine or projective space can be uniquely written as a finite union of irreducible subsets with no inclusion relation.

Given a closed subset C of $\mathbb{A}_k^n$, write $I(C) \subset k[x_1, ..., x_n]$ for the ideal of functions vanishing on C. Similarly $I(C)$ is a homogeneous ideal for C a closed projective set.

A closed set C is irreducible (thus a variety) iff $I(C)$ is a prime ideal.

Theorem 1.1.1 (Hilbert Nullstellansatz). *If k is algebraically closed, then $I(Z(\mathfrak{a}))$ is the radical of $\mathfrak{a}$, for $\mathfrak{a}$ an ideal of $k[x_1, ..., x_n]$.*

1.1.2 Rings of Functions

Definition. A *sheaf* $\mathcal{F}$ on a topological space X is an assignment $U \mapsto \mathcal{F}(U)$ which to each open subset U of X assigns an abelian group $\mathcal{F}(U)$ (with $\mathcal{F}(\emptyset) = \{0\}$) along with group homomorphisms called *restriction maps* $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ (denoted by $\rho_{UV}(s) = s|_V$ for $s \in \mathcal{F}(U)$) where $V \subset U$, such that

- ρ_{UU} is the identity map
- $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$
- if $\{U_i\}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ such that $s_{U_i} = 0$ for all i, then $s = 0$ in $\mathcal{F}(U)$
- if $\{U_i\}$ is an open covering of U, and if given $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i and j, then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

Elements of $\mathcal{F}(U)$ are called *sections* on U, and sometimes we use the notation $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.

For a fixed point $P \in X$, the *stalk* $\mathcal{F}_P$ is the direct limit of abelian groups $\varinjlim \mathcal{F}(U)$ where U runs through all open neighbourhoods of P. The group $\mathcal{F}_P$ can be thought of as the set of equivalence classes $\langle U, s \rangle_P$ where U is an open neighbourhood of P and $s \in \mathcal{F}(U)$ such that $\langle U, s \rangle_P = \langle V, t \rangle_P$ iff $s|_{U \cap V} = t|_{U \cap V}$ (i.e. germs of local sections).

Given sheaves $\mathcal{F}$ and $\mathcal{G}$ over X, a map $\Phi : \mathcal{F} \to \mathcal{G}$ is a *sheaf morphism* if for all open subsets $U \subset X$, we have a group homomorphism $\Phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that Φ_U commutes with restriction maps (i.e. $\Phi_U(s)|_V =$

$\Phi_V(s|_V)$ for any $s \in \mathcal{F}(U)$). For each point $P \in X$, a sheaf morphism Φ induces the map $\Phi_P : \mathcal{F}_P \to \mathcal{G}_P$, $\langle U, s\rangle_P \mapsto \langle U, \Phi(s)\rangle_P$. A sheaf morphism Φ is injective (resp. surjective, bijective) iff Φ_P is injective (resp. surjective, bijective) on all local rings.

If $\mathcal{F}$ is a sheaf on X and there is a continuous map $f : X \to Y$, then we define the sheaf $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(U) := \mathcal{F}(f^{-1}(U))$ for any open set $U \subset Y$.

Definition. A *regular function* on a set X over k (where suppose $X \subset \mathbb{A}^n_k$ or $X \subset \mathbb{P}^n_k$) is a function $\varphi : X \to k$ such that for any point $P \in X$, there exists an open neighbourhood $U \subset X$ of P, there exist polynomials $f, g \in k[x_1, ..., x_n]$ ($k[x_0, x_1, ..., x_n]$ if X projective) such that $\varphi|_U - \frac{f}{g}$, where if $X \subset \mathbb{P}^n_k$ then f and g must be homogeneous polynomials of the same degree. Regular functions are necessarily continuous with X equipped with Zariski topology inherited from affine/projective space.

If X is a variety, we define the *sheaf of regular functions* $\mathcal{O}_X$ to be the sheaf where $\mathcal{O}_X(U)$ is the group of regular functions on the open subset U of X (pointwise operations), and where the restriction maps are usual restriction of functions.

A *morphism/regular map* of varieties $\varphi : X \to Y$ is a continuous map that induces a morphism of varieties $\varphi^\# : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ given by $\varphi^\#(s) = s \circ \varphi$. An *isomorphism* is an invertible morphism.

All of this is independent of the embedding of X (and Y) in affine/projective space.

Example 1.1.2. The space of *all* isomorphisms $\varphi : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is $\mathrm{PGL}(n, k) := GL(n + 1, k)/k^*$, where $k^* \hookrightarrow GL(n + 1, k)$ as diagonal matrices $\lambda \mapsto \lambda I_{n+1}$. More precisely, any automorphism $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ is of the form

$$\varphi(x_0 : x_1 : \cdots : x_n) = \left[\sum_{j=0}^{n} a_{0j}x_j : \sum_{j=0}^{n} a_{1j}x_j : \cdots : \sum_{j=0}^{n} a_{nj}x_j\right]$$

where $[(a_{ij})_{0\le i,j\le n}] \in \mathrm{PGL}(n, k)$.

The following are some important properties of regular functions on projective varieties.

Proposition 1.1.3. *If $\varphi : X \to Y$ is a morphism from an irreducible projective variety X to an affine variety Y, then φ is the constant map.*

Proposition 1.1.4. *If $\varphi : X \to Y$ is a morphism from an irreducible projective variety X and Y is any variety, then φ is a closed map.*

Definition. A *rational function* on a variety X over k is an equivalence class of pairs $\langle U, f\rangle$ where U is open in X and $f \in \mathcal{O}_X(U)$ such that $\langle U, f\rangle = \langle V, g\rangle$ iff $f|_{U\cap V} = g|_{U\cap V}$. Equivalently, a rational function is simply a regular function defined on a (dense) open set U. The space of rational functions is denoted by $K(X)$, and is a field.

Similarly, a *rational map* $f : X \to Y$ between varieties is an equivalence class of pairs $\langle U, f\rangle$ where U is open in X and $f : U \to X$ is a morphism such that $\langle U, f\rangle = \langle V, g\rangle$ iff $f|_{U\cap V} = g|_{U\cap V}$. Equivalently, a rational function is simply a regular function defined on a (dense) open set U.

Two varieties X and Y are said to be *birational* if there exist rational maps $f : X \to Y$ and $g : Y \to X$ such that $\langle X, g \circ f\rangle = \langle X, \mathrm{Id}_X\rangle$ and $\langle Y, f \circ g\rangle = \langle Y, \mathrm{Id}_Y\rangle$.

A rational function $\varphi : X \to Y$ is said to be *dominant* if $\varphi(X)$ is dense in Y.

Theorem 1.1.5. *Suppose $X \subset \mathbb{A}^n_k$ closed variety, and let $I(X)$ be the ideal of polynomials vanishing on X. Suppose k algebraically closed. Then*

1. $\mathcal{O}_X(X) \cong k[x_1, ..., x_n]/I(X)$.
2. *For each point $P \in X$, let $\mathfrak{m}_P \subset \mathcal{O}_X(X)$ be the ideal of global regular functions vanishing at P. Then $P \leftrightarrow \mathfrak{m}_P$ is a 1-1 correspondence between points of Y and maximal ideals of $\mathcal{O}_X(X)$.*
3. *For each $P \in X$, $\mathcal{O}_{X,P} \cong \mathcal{O}_X(X)_{\mathfrak{m}_P}$.*
4. $K(X) \cong \mathrm{Frac}(\mathcal{O}_X(X))$.
5. *For any non-zero $f \in \mathcal{O}_X(X)$, let $D(f) = \{P \in X : f(P) \neq 0\}$. Then, $\mathcal{O}_X(D(f))$ is the sub-ring of $K(X)$ consisting of fractions of the form g/f^n where $n \in \mathbb{Z}_{\ge 0}$, and $g \in \mathcal{O}_X(X)$.*

Theorem 1.1.6. *Suppose $X \subset \mathbb{P}^n_k$ closed, and let $I(X) \subset k[x_0, ..., x_n]$ be the homogeneous ideal of polynomials vanishing on X. Suppose k algebraically closed. Set $S(X) := k[x_0, ..., x_n]/I(X)$. Then*

1. $\mathcal{O}_X(X) = k$.

2. *For each point $P \in X$, let $\mathfrak{m}_P \subset S(X)$ be the ideal generated by homogeneous polynomials vanishing at P. Then $\mathcal{O}_{X,P}$ is the subring of the localization $S(X)_{\mathfrak{m}_P}$ containing those elements f/g ($f, g \in S(X), g \notin \mathfrak{m}_P$) such that f and g are homogeneous of the same degree.*

3. *$K(X)$ is the field of fractions of the form f/g where $f, g \in S(X)$, $g \neq 0$, such that f and g are homogeneous and of the same degree.*

4. *For any non-zero homogeneous $f \in \mathcal{O}_X(X)$, let $D(f) = \{P \in X : f(P) \neq 0\}$. Then, $\mathcal{O}_X(D(f))$ is the subring of $K(X)$ consisting of fractions of the form g/f^n where $n \in \mathbb{Z}_{\geq 0}$, $g \in \mathcal{O}_X(X)$, and $\deg g = n \deg f$.*

Corollary 1.1.6.1. *For any closed variety X and any point $P \in X$, $K(X)$ is the field of fractions of $\mathcal{O}_{X,P}$.*

These are all isomorphism-invariants of closed varieties. For closed projective varieties, $K(X)$ is in fact a birational invariant.

Theorem 1.1.7. *Suppose X is any closed variety and Y is a closed affine variety. Then there is a 1-1 correspondence between morphisms $\varphi : X \to Y$ and k-algebra homomorphisms $\varphi^\# : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Specifically, φ can be reconstructed from a k-algebra morphism $\varphi^\# : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ by picking a set of k-algebra generators $x_1, ..., x_n \in \mathcal{O}_Y(Y)$ (assuming $Y \subset \mathbb{A}^k_n$), and defining the map $\varphi : X \to Y$ by $\varphi(P) = (\xi_1(P), ..., \xi_n(P))$ where $\xi_i := \varphi^\#(x_i)$.*

Suppose X is also a closed variety.

1. *The sheaf morphism $\varphi^\# : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ is injective iff the k-algebra morphism $\varphi^\# : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is injective iff $\varphi(X)$ is dense in Y.*

2. *The sheaf morphism $\varphi^\# : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ is surjective iff the k-algebra morphism $\varphi^\# : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is surjective. Moreover, in this case φ defines a homeomorphism from X onto a closed subset of Y.*

3. *In particular, X and Y are isomorphic varieties iff $\mathcal{O}_X(X)$ and $\mathcal{O}_Y(Y)$ are isomorphic k-algebras.*

4. *The functor $X \mapsto \mathcal{O}_X(X)$ induces an arrow-reversing equivalence of categories between the categories of (closed) affine varieties over k and the category of finitely generated integral domains over k.*

Proposition 1.1.8. *For arbitrary X and Y (not necessarily closed) and a morphism $\varphi : X \to Y$, we have the following:*

1. *If φ is dominant, then the map $\varphi_P^\# : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$ is injective for all P (i.e. the sheaf map $\varphi^\# : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ is injective).*

2. *φ is an isomorphism iff φ is a homeomorphism and $\varphi_P^\# : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$ is an isomorphism of k-algebras for all points $P \in X$.*

Theorem 1.1.9. *For any varieties X and Y, there is a 1-1 correspondence between dominant rational maps $\varphi : X \to Y$ and k-algebra homomorphisms from $K(Y)$ to $K(X)$ given by $\langle U, \varphi \rangle \leftrightarrow K(Y) \ni \langle V, g \rangle \mapsto \langle \varphi^{-1}(V), f \circ \varphi \rangle$. Moreover, this correspondence is an arrow-reversing equivalence of categories between the category of varieties and dominant rational maps, and the category of finitely generated field extensions of k.*

Thus, X and Y are birationally equivalent iff there are open subsets $U \subseteq X$ and $V \subseteq Y$ such that U is isomorphic to V iff $K(X) \cong K(Y)$ as k-algebras.

Corollary 1.1.9.1. *The dimension of a variety is a birational invariant.*

Corollary 1.1.9.2. *Any irreducible variety X is birational to a projective hypersurface.*

Definition. A variety X of dimension n is *rational* if X is birational to $\mathbb{P}^n$ equivalently if $K(X) \cong k(x_1, ..., x_n)$ equivalently if X possesses an open subset U isomorphic to an open subset of $\mathbb{A}^n$.

Definition. A *finite* map $f : X \to Y$ of varieties (*not* necessarily a morphism) such that Y is covered by open affine subsets $\{V_i\}$ such that for each i, the open set $U_i := f^{-1}(V_i) \subset X$ is affine and such that $\mathcal{O}_X(U_i)$ is a finitely generated $\mathcal{O}_Y(V_i)$-*module* (via the pull-back map $\pi^* : \mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$).

Finite maps have finite fibers (i.e. $f^{-1}(y)$ is a finite set for all $y \in Y$). Finite morphisms furthermore are closed surjective maps. It is also clear that if $f : X \to Y$ is a finite map then X and Y must have the same dimension.

Notice that

Proposition 1.1.10. *Suppose $\pi_0 : X_0 \to Y_0$ is a morphism where X_0 and Y_0 are both closed sets of projective space. Let $Y \subset Y_0$ be any open subset, and set $X_0 := \pi_0^{-1}(Y_0)$. Let $\pi := \pi_0|_X : X \to Y$. Then π is a finite map if and only if π has finite fibers.*

Let $U_i \subset \mathbb{P}_k^n$ be the subset defined by $\{x_i \neq 0\}$ for $0 \leq i \leq n$. Then, U_i is isomorphic to $\mathbb{A}_k^n$, and the set of all U_i covers $\mathbb{P}_k^n$. Thus, any projective variety can be covered by open affine subsets. On the other hand, any affine variety can be taken as a subset of projective space; the closure of this subset in projective space is called the *projective closure* of the original affine variety.

1.1.3 Dimension

Definition. Given a topological space X, the *dimension* $\dim X$ is the supremum of all integers n such that there exists a chain of distinct irreducible subsets $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of X. By definition, the dimension is finite iff X is Noetherian vector space.

Examples and important facts:

1. The dimension is a birational, and thus an isomorphism, invariant.
2. $\dim \mathbb{A}^n = n$ and $\dim \mathbb{P}^n = n$.
3. A (closed) variety $X \subset \mathbb{P}^n$ or $X \subset \mathbb{A}^n$ has $\dim X = n - 1$ iff $X = Z(f)$ for an irreducible polynomial f (f homogeneous if X projective). Such a variety is a *hyper-surface*.
4. If $Y \subseteq X$ where both Y and X are closed irreducible varieties of the same kind (i.e. both affine or both projective), then $\dim Y \leq \dim X$ with equality iff $Y = X$.
5. The field $K(X)$ has transcendence degree $\dim X$ over k.
6. If $\mathfrak{a} \subset k[x_1, ..., x_r]$ is an ideal generated by r elements, then every irreducible component of $Z(\mathfrak{a})$ has dimension at least $n - r$.
7. $\dim P = 0$ for any point P.
8. $\dim Y = \dim \bar{Y}$ for any irreducible set Y.
9. A 1-dimensional variety is a *curve*, a 2-dimensional variety is a *surface*.

Definition. The *(Krull) dimension* of a Noetherian ring A is the maximal number n such that we have a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \mathfrak{p}_n$ of $n+1$ distinct prime ideals of A.

The *height* of a prime ideal $\mathfrak{p}$ is the maximal number n such that we have a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \mathfrak{p}_n = \mathfrak{p}$ of $n+1$ distinct prime ideals.

Proposition 1.1.11. *If X is an affine variety, then* $\dim X = \dim A(X)$. *If X is a projective variety and $S(X) := k[x_0, ..., x_n]/I(X)$, then* $\dim S(X) = \dim X + 1$.

Proposition 1.1.12. *If X is a variety and $P \in X$ arbitrary, then* $\dim \mathcal{O}_{X,P} = \dim X$.

Useful algebraic facts about heights and dimension:

1. If A is an integrally closed Noetherian domain, then $A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$ where the intersection is taken over all prime ideals of height 1.
2. If B is an integral domain and a finitely generated k-algebra, then $\dim B$ is the transcendence degree of the quotient field $\mathrm{Frac}(B)$ over k. Moreover, if $\mathfrak{p}$ is a prime ideal, then
$$\mathrm{height}(\mathfrak{p}) + \dim B/\mathfrak{p} = \dim B.$$
3. (Krull's Hauptidealsatz) If A is a Noetherian ring and $f \in A$ is neither a unit nor a zero divisor, then every minimal prime ideal $\mathfrak{p}$ containing f has height 1.
4. A Noetherian integral domain A is a UFD iff every prime ideal of height 1 is principal.
5. (Noether Normalization Theorem) If A is a finitely generated k-algebra, then there exists $d \in \mathbb{Z}_{\geq 0}$ and algebraically independent elements $y_1, ..., y_d$ such that the polynomial ring $k[y_1, ..., y_d]$ is a sub-ring of A, and moreover A is a finitely generated *module* over $k[y_1, ..., y_d]$. In fact, $d = \dim A$.

Proposition 1.1.13. *Suppose X and Y are varieties of dimension r, s respectively.*

- *If X and Y are closed subsets of $\mathbb{A}_k^n$, then every irreducible component of $X \cap Y$ has dimension $\geq r + s - n$.*

- *If X and Y are closed subsets of $\mathbb{P}^n_k$, then every irreducible component of $X \cap Y$ has dimension $\geq r+s-n$. Furthermore, if $r+s-n \geq 0$, then $X \cap Y \neq \emptyset$.*

- *In either case, if Y is a hypersurface not completely containing X, then equality holds.*

In order to compute the dimension of a variety, the following two facts are useful as well.

Theorem 1.1.14. *Suppose $\varphi : X \to \mathbb{P}^n$ is a morphism where X is a quasi-projective variety, and let $Y = \overline{\varphi(X)}$. For any $P \in X$, let $X_P := \varphi^{-1}(\varphi(P)) \subset X$ be the fiber of φ through P. Let $\mu(P) = \dim \mathcal{O}_{X_P,P}$. Then, $\mu : X \to \mathbb{Z}_{\geq 0}$ is an upper-continuous function, i.e. for any m the locus of points P such that $\mu(P) \geq m$ is closed in X.*

Moreover, if $X_0 \subset X$ is any irreducible component and $Y_0 \subset Y$ the closure of $\varphi(X_0)$, then

$$\dim(X_0) = \dim(Y_0) + \min_{P \in X_0} \mu(P).$$

Theorem 1.1.15. *If $\pi : X \to Y$ is a morphism with Y irreducible projective variety and X a closed subset of projective space, and if all fibers are irreducible of the same dimension d, then X is also irreducible.*

Example 1.1.16 (Fall 2020 Day 2)**.** Suppose $X \subset \mathbb{P}^n$ is an irreducible projective variety of dimension k, and let $\mathbb{G}(\ell, n)$ be the Grassmannian of ℓ-planes in $\mathbb{P}^n$ where $\ell < n-k$. Let $C(X) = \{L \in \mathbb{G}(\ell,n) : L \cap X \neq \emptyset\} \subset \mathbb{G}(\ell, n)$. We show that $C(X)$ is an irreducible variety, and we find its dimension.

Indeed, consider the closed subset $Y = \{(p, L) \in \mathbb{P}^n \times \mathbb{G}(\ell,n) : p \in L \cap X\}$ of $\mathbb{P}^n \times \mathbb{G}(\ell, n)$. Let π_1 and π_2 denote the restriction to Y of the projection maps $\mathbb{P}^n \times \mathbb{G}(\ell,n) \to \mathbb{P}^n$ and $\mathbb{P}^n \times \mathbb{G}(\ell, n) \to \mathbb{G}(\ell, n)$. Then, notice that $\pi_1(Y) = X$. Moreover, for any fixed $p \in X$, the fibre $\pi_1^{-1}(p) \subset Y$ is isomorphic to the subset of $\mathbb{G}(\ell, n)$ of all planes L such that $p \in L$. Now, $\mathbb{G}(\ell, n)$ can be identified with $G(\ell+1, n+1)$ and p can be identified as a line in k^{n+1}. Thus $\pi_1^{-1}(p)$ is isomorphic to the subset of planes in $G(\ell+1, n+1)$ that contain a given fixed line, which in turn is isomorphic to the set of all ℓ-planes in k^n, which is precisely $G(\ell, n)$. Hence $\pi_1^{-1}(p)$ is an irreducible variety of dimension $\ell(n-\ell)$ for all $p \in X$. Since X is an irreducible projective variety of dimension k, it follows that Y is irreducible with dimension $\ell(n-\ell)+k$.

Finally, notice that the image of Y under π_2 is precisely $C(X)$, and moreover for any $L \in C(X)$ the set $\pi_2^{-1}(L)$ is isomorphic to $X \cap L$. We know that $L \mapsto \dim(\pi_2^{-1}(L)) = \dim(X \cap L)$ is an upper semi-continuous function from $C(X)$ to $\mathbb{Z}_{\geq 0}$, so that the subset U of $C(X)$ of all lines L such that $\dim(X \cap L) = 0$ is open in $C(X)$. Since $k + \ell < n$ we can find a line L such that $L \cap X$ has finitely many points so that $\dim(L \cap X) = 0$. Thus U is non-empty, and is thus a dense subset of $C(X)$. Thus we have found a dense open subset U of $C(X)$ such that $\pi_2^{-1}(L)$ has dimension 0 in Y for all $L \in U$, and so $\dim C(X) = \dim Y = k + \ell(n-\ell)$.

Minimal Generators of Ideals, and Dimension

Suppose Y is a variety of dimension r, embedded in either $\mathbb{A}^n$ or $\mathbb{P}^n$. Then, we know that $I(Y)$ must be generated be at least $n-r$ generators.

Definition. A (closed) projective variety Y of dimension r is a *strict complete intersection* if $I(Y)$ can be generated by $n-r$ elements. The variety Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n-r$ hypersurfaces.

It is a fact that strict complete intersections are also set-theoretic intersections. However, the converse is false.

Example 1.1.17 (Twisted Cubic Curve)**.** Let $Y = \{[t^3 : t^2u : tu^2 : u^3] : [t : u] \in \mathbb{P}^1\}$ be the *twisted cubic curve.* Then, $I(Y)$ cannot be generated by two elements; instead we have $I(Y) = \langle x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3\rangle$. However, Y is the intersection of the two hypersurfaces $Z(x_0x_2 - x_1^2)$ and $Z(2x_1x_2x_3 - x_2^3 - x_0x_3^2)$.

Example 1.1.18. All *linear varieties*, i.e. projective varieties whose ideal is generated by homogeneous linear polynomials are strict complete intersections. In other words, if $L \subset \mathbb{P}^n$ is a linear variety, then L has dimension r iff there exist $n-r$ linearly independent homogeneous linear polynomials $f_1, ..., f_{n-r} \in k[x_0, ..., x_n]$ such that

$$L = Z(f_1, ..., f_{n-r}).$$

1.1.4 Hilbert Polynomials, Degrees, and Intersection Multiplicities

We collect here some important and extremely useful properties of projective varieties, i.e. closed irreducible subsets of $\mathbb{P}^n$ for some n. Unless otherwise specified, all varieties will be projective varieties embedded in $\mathbb{P}^n$. Throughout, for ease of notation, let $S = S^{(n)} = k[x_0, ..., x_n]$.

Given a graded module M over S, we define M_ℓ to be the k-vector space of all elements of M_ℓ of degree ℓ (thus, for instance, S_ℓ is the vector space of all degree ℓ homogeneous polynomials in S). If we have an exact sequence of graded S-modules

$$0 \to M' \to M \to M'' \to 0,$$

then we get an exact sequence of k-vector spaces

$$0 \to M'_d \to M_d \to M''_d \to 0,$$

and thus

$$\dim_k M_d = \dim_k M'_d + \dim_k M''_d.$$

This is a useful computational fact.

Lemma-Definition. For a projective variety $Y \subset \mathbb{P}^n$ with (prime) ideal $I(Y)$, set $S(Y) := S/I(Y)$. Then, there exists a unique polynomial p_Y of degree $\dim Y$, called the *Hilbert Polynomial* of Y, such that for $\ell \in \mathbb{N}$ large enough we have

$$p_Y(\ell) = \dim_k S(Y)_\ell.$$

This polynomial *depends* on the embedding we choose of $Y \hookrightarrow \mathbb{P}^n$.

The function $h_Y(\ell) = \dim_k S(Y)_\ell$ is called the *Hilbert Function* of Y (with respect to the given embedding).

For example, the Hilbert polynomial of $\mathbb{P}^n$ is

$$p_{\mathbb{P}^n}(x) = \binom{x+n}{n} = \frac{(x+n)(x+n-1)\cdots(x+1)}{n!}.$$

If $f \in S^{(n)}$ is a homogeneous irreducible polynomial of degree d, then the Hilbert polynomial of the corresponding hypersurface $H = Z(f)$ is

$$p_H(x) = \binom{x+n}{n} - \binom{x-d+n}{n}.$$

Definition. The *degree* of a projective variety Y is $(\dim Y)!$ times the leading coefficient of the Hilbert Polynomial p_Y.

Some important facts about the degree:

1. The degree of a non-empty variety is always a positive integer.
2. $\deg \mathbb{P}^n = 1$. Also, the degree of a point is 1.
3. If $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are projective varieties of the same dimension r and $\dim(Y_1 \cap Y_2) < r$, then $\deg Y = \deg Y_1 + \deg Y_2$.
4. If $f \in S^{(n)}$ is a homogeneous irreducible polynomial of degree d, then the degree of the corresponding hypersurface $H = Z(f)$ is d as well.
5. If $Y \subset \mathbb{P}^n$ is a closed subset of projective space each of whose irreducible components have the same dimension, then Y has degree 1 iff Y is a *linear variety* of dimension r, i.e. a variety whose ideal is generated by exactly $n - r$ linearly independent linear homogeneous polynomials in $S^{(n)}$.

Definition. The *arithmetic genus* $p_a(X)$ of a projective variety X is the number $p_a(X) = (-1)^r(p_X(0) - 1)$, where p_X is the Hilbert Polynomial of X, and $r = \dim X$.

The arithmetic genus is an isomorphism invariant, even though the Hilbert polynomial is not.

Known that $p_a(\mathbb{P}^n) = 0$, and for H a hypersurface of degree d in $\mathbb{P}^n$, $p_a(H) = \binom{d-1}{n}$.

Example 1.1.19 (Fall 2020 Day 3)**.** Suppose $C \subset \mathbb{P}^3$ is an algebraic curve with Hilbert polynomial $p_C(m) = 3m + 1$ and Hilbert function h_C. Is it possible for $h_C(1) = 3$? What about $h_C(1) = 4$?

If $h_C(1) = 3$, then $S(C)_1$ has dimension three even though it is spanned by the images of x_0, x_1, x_2, x_3 in $S(C) = k[x_0, x_1, x_2, x_3]/I(C)$. Hence there exists a linear polynomial $\ell \in k[x_0, x_1, x_2, x_3]$ such that $C \subset Z(\ell)$. Thus C is contained in a hyperplane of $\mathbb{P}^3$, which implies that C is isomorphic to a curve in $\mathbb{P}^2$ (which is a hyper-surface). Since the leading coefficient of p_C is 3, it follows that C has degree 3, and so its genus is $\binom{3-1}{2} = 1$. However, the arithmetic genus can be calculated from the Hilbert polynomial via $(-1)^1(p_C(0) - 1)$, which is 0 if $p_C(m) = 3m + 1$. Thus no such curve can exist, i.e. $h_C(1) = 3$ is impossible.

Now for $h_C(1) = 4$, we consider the twisted cubic curve $C = Z(x_0x_2 - x_1^2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2)$. We have the exact sequence of graded modules given by

$$0 \to S(C) \xrightarrow{\times x_3} S(C) \to k[x_0, x_1, x_2, x_3]/(I(C) + \langle x_3 \rangle) \to 0,$$

where the first copy of $S(C)$ is twisted by increasing the degree of each component by 1. The degree d component of $k[x_0, x_1, x_2, x_3]/(I(C)+\langle x_3 \rangle) \cong k[x_0, x_1, x_2]/\left\langle x_0x_2 - x_1^2, x_2^2, x_1x_2 \right\rangle$ is spanned by $x_0^d, x_0^{d-1}x_1, x_0^{d-1}x_2$ for $d \geq 1$, and so always has dimension 3. Thus, taking degree d components in the exact sequence and comparing their k-dimensions, we get the recurrence $h_C(d) = h_c(d-1) + 3$ for all $d \geq 1$. Since $S(C)_0 = k$ has dimension 1, it follows that $h_C(1) = 4$ and $p_C(m) = 3m + 1$ for the twisted cubic curve C.

Example 1.1.20 (Fall 2019 Day 1)**.** Consider any field K. Suppose $X \subset \mathbb{P}^n_K$ is a variety with ideal $I(X) \subset K[Z_0, ..., Z_n]$ and homogeneous coordinate ring $S(X) = K[Z_0, ..., Z_n]/I(X)$. Prove that the degree of the Hilbert polynomial p_X is precisely $\dim X$. Also, for each $m \in \mathbb{N}$, find a variety $X \subset \mathbb{P}^n$ such that $h_X(m) \neq p_X(m)$.

Throughout, for a graded module M, we denote by M_d the degree d graded piece, and $M(k)$ by the graded module obtained from M by increasing the degree of each element of M by k so that $M(k)_d := M_{k-d}$. We proceed by induction on $\dim X$. If $\dim X = 0$, then X is just a single point p, where WLOG $p = [1 : p_1 : \cdots : p_n]$. Then $I(X) = \langle Z_i - p_iZ_0 : 1 \leq i \leq n \rangle$ and so $S(X) \cong K[Z_0]$. Since $\dim_K K[Z_0]_m = 1$ for all $m \in \mathbb{N}$, it follows that p_X has degree 0 (i.e. is a constant). Now suppose the degree of the Hilbert polynomial is the dimension of all varieties with dimension $< \dim X$. Since the intersection of all hyperplanes in $\mathbb{P}^n$ is $\emptyset$, there exists a linear polynomial $\ell \notin I(X)$, i.e. $X \not\subset Z(\ell)$. Then, $Y := Z(\ell) \cap X$ is a variety in $\mathbb{P}^n$ of dimension $\dim X - 1$. Thus, p_Y has degree $\dim X - 1$ by the induction hypothesis. Now, notice that $S(Y) = K[Z_0, ..., Z_n]/(I(X) + \langle \ell \rangle) = S(X)/\ell S(X)$. We have an exact sequence of graded $K[Z_0, ..., Z_n]$-modules

$$0 \to S(X)(1) \xrightarrow{\times \ell} S(X) \to S(X)/\ell S(X) \cong S(Y) \to 0$$

which, upon taking the m'th degree components only, induces the exact sequence $0 \to S(X)_{m-1} \to S(X)_m \to S(Y)_m \to 0$. This implies that $h_X(m) - h_X(m-1) = h_Y(m)$ for all $m \in \mathbb{N}$, and hence $p_X(m) - p_X(m-1) = p_Y(m)$. Since p_Y has degree $\dim X - 1$, it follows that p_X has degree $\dim X$.

We now give two examples, one where X is not irreducible and one where X is irreducible.

1. Suppose X is a dimension 0 closed subset of $\mathbb{P}^n$ with k points. Then $p_X(m) = k$ for all $m \in \mathbb{N}$. But $h_X(m)$ is at most the dimension of $K[Z_0, ..., Z_n]_m$, which is $\binom{n+m}{m}$. Thus, by taking k sufficiently large, we have $h_X(m) < k = p_X(m)$.

2. Let k be arbitrary, and set $F := Z_0^k - Z_1Z_2^{k-1} \in K[Z_0, ..., Z_n]$. Then F is clearly irreducible, so that $X = Z(F)$ is irreducible with ideal $I(X) = \langle F \rangle$. The exact sequence of graded $K[Z_0, ..., Z_n]$ modules

   $$0 \to K[Z_0, ..., Z_n](k) \xrightarrow{\times F} K[Z_0, ..., Z_n] \to K[Z_0, ..., Z_n]/\langle F \rangle = S(X) \to 0$$

   yields, via taking m'th components and then comparing degrees,

   $$h_X(m) = h_{\mathbb{P}^n}(m) - h_{\mathbb{P}^n}(m-k) = \begin{cases} \binom{n+m}{n} - \binom{n+m-k}{n} & m \geq k, \\ \binom{n+m}{n} & m < k \end{cases}.$$

   In particular,

   $$p_X(m) = \binom{n+m}{n} - \binom{n+m-k}{n} = \frac{1}{n!}\big((m+n)(m+n-1)\cdots(m+1) - (m+n-k)(m+n-1-k)\cdots(m+1-k)\big).$$

   Taking $k = m + n + 1$ for instance, we see that $h_X(m) = \binom{n+m}{n}$ while $p_X(m) = \binom{n+m}{n} - (-1)^n$.

Definition. Suppose Y is a projective variety and H a hypersurface not containing Y. Let Z be any irreducible component of $Y \cap H$, and let $\mathfrak{p} = I(Z)$ be the prime ideal corresponding to Z. The *intersection multiplicity* $i(Y, H; Z)$ of Y and H along Z is the *length* of the graded $S_\mathfrak{p}$-module $\big(S/(I(Y) + I(H))\big)_\mathfrak{p}$. Here, the length of an R-module M is the supremum of integers n such that there exists $n+1$ distinct R-submodules M_i such that

$$M_0 \subset M_1 \subset \cdots \subset M_n = M.$$

The following is a generalization of Bezout's theorem.

Theorem 1.1.21. *Suppose Y is a variety of dimension ≥ 1 in $\mathbb{P}^n$, and H a hypersurface not completely containing Y. Then,*

$$\sum_Z i(Y, H; Z) \deg Z = (\deg Y)(\deg H)$$

where the sum runs over all irreducible components Z of $Y \cap H$.

Example 1.1.22 (Spring 2020 Day 1 Qualifiers)**.** We show that any irreducible non-degenerate (i.e. not contained in any hyperplane) curve $C \subset \mathbb{P}^3$ of degree 3 is, after a linear change of coordinates on $\mathbb{P}^3$, the twisted cubic curve.

Indeed, fix any two distinct points P and Q on C. Let $R \in C\backslash\{P,Q\}$ be such that P,Q,R are collinear. Since C is non-degenerate, C cannot be a subset of the line through P,Q,R; thus we can pick a fourth point $S \in C$ such that S does not lie on the line P,Q,R. Then the four points determine a plane H. Since C is non-degenerate, $C \not\subset H$. Thus, by Bezout's Lemma, the number of intersection points of $C \cap H$ (any irreducible component of $C \cap H$ must be a point, since $\dim C = 1$) with multiplicity is equal to $(\deg C)(\deg H) = 3$. However, we have four distinct intersection points $P,Q,R,S \in C \cap H$, which is a contradiction. Thus R does not lie on the line PQ. Hence, PQ does not intersect C apart from the points P and Q.

Now, note that the family of planes containing the line PQ is parametrized by $\mathbb{P}^1$; indeed, if the line PQ is the zero locus of the two linear homogeneous polynomials $\ell_1, \ell_2 \in k[x_0,x_1,x_2,x_3]$, then H is a hyperplane containing PQ if and only if $H = Z(s\ell_1 + t\ell_2)$ for some $[s:t] \in \mathbb{P}^1$. Here, we have repeatedly used the fact that linear varieties are strict complete intersections. By Bezout's Lemma, any hyperplane H_λ ($\lambda \in \mathbb{P}^1$) containing P and Q must intersect C at a third unique point R_λ, and conversely any point $R \in C\backslash\{P,Q\}$ is the unique intersection point (apart from P and Q) of some hyperplane H_λ with C. Hence, we have a well-defined bijection $\phi : C\backslash\{P,Q\} \to \mathbb{P}^1$. This map and its inverse are both clearly morphisms, since the outputs are solutions to systems of polynomial equations one of which is linear. This map ϕ therefore defines the birational mapping $\phi : C \to \mathbb{P}^1$. Thus C is rational.

Now, C being rational implies that we can fix a birational map $\psi : \mathbb{P}^1 \to C$. Since $\mathbb{P}^1$ is a non-singular curve, it follows that ψ is a morphism. Thus we can write

$$\psi(x,y) = [F_0(x,y) : F_1(x,y) : F_2(x,y) : F_3(x,y)]$$

where $F_i \in k[x,y]$ are homogeneous. Since ψ is birational, it must be dominant. However, the image of projective space is always closed, and hence ψ is surjective. Since C is a curve of degree 3 while $\mathbb{P}^1$ is of degree 1, it follows that each of the F_i are cubic. Since C is non-degenerate, the F_i are linearly independent, and thus form a basis for the k-vector space of all degree 3 homogeneous polynomials in $k[x,y]$. Hence, after a linear change of coordinates on $\mathbb{P}^3$, we can send this basis to the basis x^3, x^2y, xy^2, y^3. Therefore C is a twisted cubic curve.

1.2 Types of Varieties and Constructions

1.2.1 Normal Varieties

Definition. A variety Y is *normal at a point* $P \in Y$ if $\mathcal{O}_{Y,P}$ is integrally closed (in its field of fractions). The variety is *normal* if it is normal at every point.

Some useful facts:

1. Every conic in $\mathbb{P}^2$ is normal.
2. If Y is affine, then Y is normal iff $\mathcal{O}_Y(Y)$ is integrally closed.
3. If $Y \subset \mathbb{P}^n$ is a projective variety and $S(Y) = k[x_0,...,x_n]/I(Y)$ is integrally closed, then Y is normal.
4. In general, the set of non-normal points of a variety forms a proper closed subset of the variety. Thus the set of normal points is dense.

1.2.2 Non-singular/Smooth Varieties

Definition. A Noetherian local ring A with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$ is *regular* if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim A$.

Definition. Suppose Y is a variety. If $P \in Y$ is such that the local ring $\mathcal{O}_{Y,P}$ is regular, then Y is said to be *non-singular* or *smooth* at the point P. A variety Y is *non-singular* or *smooth* if it is non-singular at every point.

Proposition 1.2.1. *Suppose X is a closed variety in $\mathbb{A}^n_k$ or $\mathbb{P}^n_k$, and suppose $X = Z(f_1,...,f_t)$ (here f_i homogeneous if X projective). Define abstractly the derivative of the polynomial $\frac{\partial f_i}{\partial x_j}$. The point $P \in X$ is non-singular iff the rank of the* Jacobian *matrix* $(\frac{\partial f_i}{\partial x_j}(P))$ *is* $n - \dim Y$.

As a corollary, the set of singular points of a variety is a proper closed subset of the variety.

Example 1.2.2 (Fall 2019 Day 2)**.** Let $\mathbb{P}^N$ be the space of non-zero homogeneous polynomials of degree d in $n+1$ variables over a field K, modulo multiplication by non-zero scalars. Let $U \subset \mathbb{P}^N$ be the set of all irreducible polynomials F such that $Z(F) \subset \mathbb{P}^n$ is a smooth curve. Prove that U is Zariski-open with irreducible complement $D := \mathbb{P}^N \setminus U$, and find the dimension of D.

Note that for F irreducible and $p \in Z(F)$, p is a non-smooth point of $Z(F)$ iff $\frac{\partial F}{\partial x_i}(p) = 0$ for all i. By Euler's Identity, we have $d \cdot F = \sum_{i=0}^n x_i \frac{\partial F}{\partial x_i}$, and so it follows that the locus of all non-smooth points of $Z(F)$ (for F irreducible) is precisely $Z(\frac{\partial F}{\partial x_i} : 0 \le i \le n)$. Thus $F \in U$ iff F is irreducible and $Z(\frac{\partial F}{\partial x_i} : 0 \le i \le n) = \emptyset$. However, if F is reducible, say $F = GH$ where $\deg G, \deg H \ge 1$, then $\frac{\partial F}{\partial x_i} = \frac{\partial G}{\partial x_i} H + G \frac{\partial H}{\partial x_i}$. Since $Z(G) \cap Z(H)$ is always non-empty as we are in projective space, it follows that we can find $p \in Z(G) \cap Z(H)$ so that $p \in Z(\frac{\partial F}{\partial x_i} : 0 \le i \le n)$. Therefore, if F is not irreducible then $Z(\frac{\partial F}{\partial x_i} : 0 \le i \le n) \neq \emptyset$, and hence $U = \{F \in \mathbb{P}^N : Z(\frac{\partial F}{\partial x_i} : 0 \le i \le n) = \emptyset\}$.

Now, let X be the (closed) subset of $\mathbb{P}^n \times \mathbb{P}^N$ given by

$$X = \{(p, F) : F \text{ reducible or } Z(F) \text{ not smooth at } p\} = \left\{(p, F) : p \in Z\left(\frac{\partial F}{\partial x_i} : 0 \le i \le n\right)\right\}.$$

If $\pi_1 : X \to \mathbb{P}^n$ is the projection onto the first coordinate, then for any $p \in \mathbb{P}^n$ the fibre $\pi_1^{-1}(p)$ is isomorphic (via $\pi_2|_{\pi^{-1}(p)}$) to the space of all $F \in \mathbb{P}^N$ such that $\frac{\partial F}{\partial x_i}(p) = 0$ for all i. However, notice that $\frac{\partial F}{\partial x_i}(p)$ is a homogeneous linear polynomial in the coefficients of F for a fixed p, and so the space of $F \in \mathbb{P}^N$ such that $\frac{\partial F}{\partial x_i}(p) = 0$ for all i is a linear variety generated by $n+1$ linearly independent generators. Hence $\pi_1^{-1}(p)$ has dimension $N - (n+1)$ for all $p \in \mathbb{P}^n$. It follows that X is an irreducible variety with dimension

$$\dim X = \dim \pi^{-1}(p) + \dim \mathbb{P}^n = (N - n - 1) + n = N - 1.$$

Next, notice that $\pi_2(X) = D$, and since X is an irreducible projective variety, it follows that D is closed and irreducible. Hence U is Zariski-open with irreducible complement D. It remains to calculate the dimension of D. Notice that for any $F \in D$, the fibre $\pi_2^{-1}(F)$ is isomorphic to $Z(\frac{\partial F}{\partial x_i} : 0 \le i \le n) \subset \mathbb{P}^n$. Since there exist irreducible F with exactly one non-smooth point (for instance, take any projective cone of dimension $n-1$ in $\mathbb{P}^n$), and since $F \mapsto \dim \pi_2^{-1}(F)$ is upper-semi continuous, there exists a dense open subset D_0 of D such that the fibres of $\pi_2|_{\pi_2^{-1}(D_0)} : \pi_2^{-1}(D_0) \to D_0$ are all zero dimensional, and thus π_2 is a finite regular map. Hence

$$\dim D = \dim D_0 = \dim X = N - 1.$$

1.2.3 Grassmannians

Given an n-dimensional vector space V, let the *Grassmannian* $G(m, V)$ be the set of all m-dimensional linear subspaces of V. We define a variety structure on $G(m, V)$ as follows. Suppose $W \in G(m, V)$. If $x_1, ..., x_m$ is any basis of W, then

$$x_1 \wedge x_2 \wedge \cdots \wedge x_m \in \bigwedge^m V \backslash \{0\}.$$

If $y_1, ..., y_m$ is any other basis, and if A is the change of basis matrix $\underline{y} = \underline{x}A$, then

$$y_1 \wedge \cdots \wedge y_m = (\det A) x_1 \wedge x_2 \wedge \cdots \wedge x_m \in \bigwedge^m V \backslash \{0\}.$$

Hence, attached to a m-dimensional linear subspace W of V, there is an intrinsically determined element of $\varphi(W) \in \mathbb{P}(\bigwedge^m V)$. Here, if X is a vector space, then the *projectivization* $\mathbb{P}(X)$ *of* X is the set $(X - 0)/k^*$. The upshot is that we have a well-defined map

$$\varphi : G(m, V) \to \mathbb{P}(\textstyle\bigwedge^m V) \cong \mathbb{P}^{\binom{n}{m} - 1}.$$

This map is in fact injective, since if $\omega \in \mathrm{Im}(\varphi)$, then

$$W = \{v \in V : v \wedge \omega = 0\}.$$

The map φ is called the *Plücker Embedding* of $G(m, V)$. Moreover, any identification $\mathbb{P}(\bigwedge^m V) \cong \mathbb{P}^N$ ($N = \binom{n}{m} - 1$) allows us to attach a point of $\mathbb{P}^N$ to any element of $G(m, V)$; these are the *Plücker coordinates*.

Since $v \wedge \omega = 0$ iff $\omega = v \wedge \phi$ for some $\phi \in \bigwedge^{m-1} V$, it follows that $\omega \in \mathrm{Im}(\varphi)$ is totally decomposable (i.e. $\omega = v_1 \wedge \cdots \wedge v_m$ for some linearly independent $v_i \in V$) iff the linear map $w_\omega : V \to \bigwedge^{m+1} V$, $v \mapsto v \wedge \omega$ has rank $n - m$. This map cannot have rank $< n - m$ for any $\omega \in \bigwedge^m V$, and so

$$\mathrm{Im}(\varphi) = \{[\omega] \in \mathbb{P}(\textstyle\bigwedge^m V) : \mathrm{rank} w_\omega \le n - m\}.$$

This is a determinantal (closed) projective subvariety of $\mathbb{P}(\bigwedge^m V) \cong \mathbb{P}^N$.

To make it more concrete, take $V = k^n$, and let e_i be the standard basis vectors. Then, the Grassmannian variety $G(m,n)$ is covered by open affine subsets U_I where $I \subset \{1,2,...,n\}$ is any subset of size m, and where

$$U_{\{i_1,...,i_m\}} = \left\{ W \in G(m,n) : W = \operatorname{span}_k(v_1,...,v_m) \text{ where } v_t = e_{i_t} + \sum_{j \notin I} a_{tj} e_j \right\}.$$

Notice that $U \cong \mathbb{A}^{m(n-m)}$ via the map $W \mapsto (a_{tj})_{1\leq t\leq m, j\notin I}$. Hence $\dim G(m,V) = m(\dim V - m)$.

Finally, note that a k-dimensional subspace of k^n is the same thing as a $k-1$-dimensional plane in $\mathbb{P}^{n-1}$. Thus, the Grassmannian space $\mathbb{G}(k,n)$ of k-dimensional planes in $\mathbb{P}^n$ is a variety, and in fact is the same as $G(k+1,n+1)$.

1.2.4 Flag Varieties

Throughout we fix a vector space V of dimension n over a field k.

Definition. A *flag* is a finite sequence $W_\bullet = (W_1,...,W_m)$ of subspaces of V such that $\{0\} \subset W_1 \subset \cdots \subset W_m = V$. The *size* or *signature* of the flag is the m-tuple $(\dim W_1, ..., \dim W_{m-1}, \dim W_m = n)$. Any sequence $\pi = (n_1,...,n_m)$ of positive integers such that $0 < n_1 < n_2 < \cdots < n_m = n$ is the size of some flag of V.

A *complete flag* is a flag with size $[1,n] = (1,2,...,n)$. If we fix a basis $\{e_i\}$ for V, then $W_i = \operatorname{Span}_k\{w_j : 1 \leq j \leq i\}$ is a complete flag, called the *standard complete flag.*

Given a flag size π, the *flag variety* $\mathbb{F}_\pi(V)$ (or $\mathbb{F}(\pi, V)$) of flags of size π on V is the set of all flags on V with flag size π. Using incidence varieties, it can be shown that $\mathbb{F}_\pi(V)$ is an irreducible subvariety of

$$\mathbb{G}(n_1,V) \times \mathbb{G}(n_2,V) \times \cdots \times \mathbb{G}(n_m,V)$$

where $\pi = (n_1,...,n_m)$. The dimension of $\mathbb{F}_\pi(V)$ is

$$\sum_{1\leq i<j\leq m} (n_j - n_{j-1})(n_i - n_{i-1}) = \sum_{j=1}^{m} (n_j - n_{j-1}) n_{j-1} = \sum_{i=1}^{m} (n - n_i)(n_i - n_{i-1})$$

where we set $n_0 := 0$. In particular, the complete flag variety $\mathbb{F}_{[1,n]}(V)$ has dimension $\frac{n(n-1)}{2}$.

A linear map $T \in \operatorname{End}(V)$ is said to *stabilize* a flag $W_\bullet$ if for all $1 \leq i \leq m$, we have $T(W_i) \subseteq W_i$. The group $GL(V)$ acts on $\mathbb{F}_\pi(V)$ via $W_\bullet \mapsto T(W_1) \subset \cdots \subset T(W_m)$, and in fact acts transitively. If we fix a basis $\{e_i\}$ for V so that $V \cong k^n$ and $GL(V) \cong GL_n(k)$, then the stabilizer of the flag $W_\bullet \in \mathbb{F}_\pi(k^n)$ obtained by deleting some of the intermediate subspaces of the standard complete flag is the group of invertible lower block triangular matrices where the dimensions of each block on the diagonal are $n_i - n_{i-1}$. In particular, the stabilizer of the standard complete flag is the group of all invertible lower triangular matrices.

Example 1.2.3 (Fall 2021 Day 3)**.** Let $\mathbb{P}^{n^2-1}$ be the variety of all non-zero $n \times n$ complex matrices modulo scalars, and let X be the set of $[A] \in \mathbb{P}^{n^2-1}$ where A is nilpotent. Prove that X is a closed irreducible subvariety of $\mathbb{P}^{n^2-1}$ and find its dimension.

For any $A \in \mathbb{P}^{n^2-1}$, consider the characteristic polynomial

$$\chi(A) = \det(TI - A) = T^n + c_{n-1}(A)T^{n-1} + \cdots + c_1(A)T + c_0(A) \in \mathbb{C}[\mathbb{P}^{n^2-1}][T],$$

where $\mathbb{C}[\mathbb{P}^{n^2-1}]$ denotes the graded ring of all polynomials on $\mathbb{P}^{n^2-1}$. A quick calculation shows that $\chi(\lambda A)(T) = \lambda^n \chi(A)(T/\lambda)$, which implies that $c_{n-i}(\lambda A) = \lambda^i c_{n-1}(A)$ for all $0 \leq i \leq n-1$ and for all $A \in \mathbb{P}^{n^2-1}$. Hence $c_{n-i} \in \mathbb{C}[\mathbb{P}^{n^2-1}]$ is homogeneous of degree i. By Cayley-Hamilton, we know that $[A] \in X$ iff $\chi(A) = T^n$ iff $[A] \in Z(c_0,...,c_{n-1})$. Hence

$$X = Z(c_0,...,c_{n-1})$$

is a Zariski-closed subset of $\mathbb{P}^{n^2-1}$.

We now consider the complete flag variety $\mathcal{F}$ on $\mathbb{C}^n$. Define the incidence variety

$$\Lambda = \{(A, W_\bullet) \in X \times \mathcal{F} : A \text{ stabilizes } W_\bullet\}.$$

Let $\pi : \Lambda \to \mathcal{F}$ be the projection map. Then this map is surjective since for any complete flag $W_\bullet$ we can easily define a nilpotent matrix A such that $A(W_i) = W_{i-1}$. Now, the group $GL(n,\mathbb{C})$ acts on X by conjugation and on $\mathcal{F}$, and moreover each $g \in GL(n,\mathbb{C})$ is an automorphism of X and $\mathcal{F}$ respectively. It thus acts as automorphisms on Λ. Notice that the induced action of $GL(n,\mathbb{C})$ on the fibres of π is transitive, since any complete flag $W_\bullet$ can be sent to the standard complete flag $E_\bullet$ by some $g \in GL(n, \mathbb{C})$, and nilpotent stabilizers A of $W_\bullet$ are in 1-1 correspondence with nilpotent stabilizers gAg^{-1} of $E_\bullet$. Therefore all the fibres of π are isomorphic. However, the nilpotent stabilizers of $E_\bullet$ are precisely the nilpotent upper triangular matrices, i.e. upper triangular matrices with zeros on the entire diagonal. The vector space of such matrices is $\binom{n}{2}$, and so the projectivization has dimension $\binom{n}{2} - 1$. Hence each fibre of π has dimension $\binom{n}{2} - 1$. Since $\mathcal{F}$ is itself an irreducible variety of dimension $\binom{n}{2}$, it then follows that Λ is irreducible with dimension $2\binom{n}{2} - 1 = n^2 - n - 1$.

Now let $\rho : \Lambda \to X$ be the projection onto X. Again, ρ is surjective, since for any nilpotent matrix A we can complete the partial flag $(A^{n-1}(\mathbb{C}^n), ..., A(\mathbb{C}^n), \mathbb{C}^n)$ (after removing zero spaces) by inserting intermediary subspaces. In particular, as Λ is irreducible, it follows that X is irreducible.

Moreover, if A is a nilpotent matrix of rank $n-1$, then its Jordan block form is the maximal nilpotent Jordan block $J_{n,0}$. Since the only flag that $J_{N,0}$ stabilizes is the standard complete flag $E_\bullet$, it follows that the fibre $\rho^{-1}([A])$ over $[A]$ is a singleton. Since the set of all matrices in $\mathbb{P}^{n^2-1}$ with rank $\geq n-1$ is an open subset of $\mathbb{P}^{n^2-1}$, and since $n \times n$ nilpotent matrices cannot have rank n, it follows that the set of all $[A] \in X$ that have rank $n-1$ is an open (and thus dense) subset. Since ρ has singleton fibres over an open dense subset of X, it follows that

$$\dim X = \dim \Lambda = n^2 - n - 1.$$

1.2.5 Blowing Up

We first construct the blow up of $\mathbb{A}^n$ at the point $O = (0, 0, ..., 0)$. The product $\mathbb{A}^n \times \mathbb{P}^{n-1}$ is a quasi-projective variety. Let $x_1, ..., x_n$ be coordinates on $\mathbb{A}^n$ and $y_1, ..., y_n$ coordinates on $\mathbb{P}^{n-1}$. The *blowing up of* $\mathbb{A}^n$ *at the point* O is then the irreducible variety

$$\tilde{\mathbb{A}^n} = Z(x_i y_j - x_j y_i : i, j = 1, 2, ..., n).$$

Let $\varphi : \tilde{\mathbb{A}^n} \hookrightarrow \mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$ be the composition of the inclusion and projection onto the first factor. Then, φ gives an isomorphism between $\tilde{\mathbb{A}^n} \backslash \varphi^{-1}(O)$ and $\mathbb{A}^n \backslash O$. Secondly, $\varphi^{-1}(O) \cong \mathbb{P}^{n-1}$.

If Y is an arbitrary closed subvariety of $\mathbb{A}^n$ passing through O, then the *blowing up of* Y *at the point* O is the variety

$$\tilde{Y} := \overline{\varphi^{-1}(Y - O)},$$

where the closure occurs in $\tilde{\mathbb{A}^n}$. The restriction $\varphi|_{\tilde{Y}} = \varphi : \tilde{Y} \to Y$ is a surjective morphism, and induces an isomorphism of $\tilde{Y} \backslash \varphi^{-1}(O)$ to $Y - O$ so that φ is a birational map between $\tilde{Y}$ and Y.

Often, the set $\varphi^{-1}(O) \subset \tilde{Y}$ is called the *exceptional divisor* of the blow up, and $\tilde{Y}$ the *proper transform* of Y.

Example 1.2.4. Let $Y = Z(y^2 - x^3 - x^2) \subset \mathbb{A}^2$. Then $\tilde{\mathbb{A}^2}$, the blow up of $\mathbb{A}^2$, is the subset of $\mathbb{A}^2 \times \mathbb{P}^1$ given by $Z(xu - ty)$ (where t, u are coordinates on the $\mathbb{P}^1$ factor). To evaluate $\tilde{Y}$, we study Y on each open affine subset U_i. If $t \neq 0$, then set $t = 1$. The closure of $\tilde{Y} \cap \{t \neq 0\}$ is cut out by the equations $y^2 - x^3 - x^2 = 0$ and $y - xu$ in $\mathbb{A}^3$. This has two irreducible components, namely the line $\{x = y = 0\}$ and the curve $Z(u^2 - x - 1, y - xu)$. Since $\tilde{Y}$ is the pre-image of $Y - O$ under φ, we ignore the first line, and consider the closure of the second curve, namely $Z(u^2 - x - 1, y - xu)$. Thus

$$\tilde{Y} \cap \{t \neq 0\} = \{(u^2 - 1, u(u^2 - 1), u) : u \in k\}.$$

On the other hand, if $u \neq 0$, then

$$\overline{\varphi^{-1}(Y \backslash 0) \cap \{u \neq 0\}} = \overline{(x, y, t) : x = ty, (x, y) \neq (0, 0), y^2 = x^2(x + 1)} = \{(x, y, t) : x = ty, 1 = t^3 y + t^2, t \in k^*\}.$$

Therefore, as a variety in $\mathbb{A}^2 \times \mathbb{P}^1$, we have

$$\tilde{Y} = \{(u^2 - 1, u^3 - u; 1 : u) : u \in k\}$$

To blow up at another point P, translate linearly to send P to O.

Example 1.2.5 (Fall 2021 Day 1)**.** Consider the following varieties (the convention adopted is that varieties are not necessarily irreducible) in the affine plane $\mathbb{A}^2_{\mathbb{C}}$:

$$X_1 = Z(x^2 - 1), \quad X_2 = Z(x^2 - y), \quad X_3 = Z(x^2 - y^2), \quad X_4 = Z(x^2 - y^3), \quad X_5 = Z(x^2 - y^4).$$

Prove that these five varieties are pairwise non-isomorphic.

Notice first that $x^2 - y^{i-1}$ is irreducible iff i is even. It follows that each of X_1, X_3, X_5 are not isomorphic to each of X_2, X_4. Moreover, by finding the gradient of the defining polynomial, notice that X_1 and X_2 are smooth curves whereas X_3, X_4, X_5 have a singular point at $P = (0, 0)$. Since smoothness is preserved under isomorphism, we have proves that all pairs of distinct X_i, X_j are non-isomorphic save for the pair X_3, X_5. To distinguish them, we blow up the singularity at P.

Let $\tilde{\bullet}$ denote the blow up at P of a variety in $\mathbb{A}^2$. Then, we have $\tilde{\mathbb{A}}^2 = \{(x, y; u : v) \in \mathbb{A}^2 \times \mathbb{P}^1 : xv = yu\}$. Let $\varphi : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ and $\pi : \tilde{\mathbb{A}}^2 \to \mathbb{P}^1$ denote the projection onto the first and second coordinate respectively. Then, letting $k = 1, 2$, we have

$$\begin{aligned}
\tilde{X}_{2k+1} &= \overline{\varphi^{-1}(X_{2k+1} - P)} = \overline{\{(x, y; u : v) : y \neq 0, x^2 = y^{2k}, xv = yu\}} \\
&= \overline{\{(x, y; u : v) : y \neq 0, x = y^k, xv = yu\}} \cup \overline{\{(x, y; u : v) : y \neq 0, x = -y^k, xv = yu\}} \\
&= \overline{\{(y^k, y; u : v) : y \neq 0, x = y^k, y^{k-1}v = u\}} \cup \overline{\{(-y^k, y; u : v) : y \neq 0, -y^{k-1}v = u\}} \\
&= \{(y^k, y; y^{k-1} : 1)\} \cup \{(y^k, y; -y^{k-1} : 1)\}.
\end{aligned}$$

Here, we use the convention that $0^0 = 1$. Thus, setting $E = \varphi^{-1}(P)$, we see that $\pi(\tilde{X}_3 \cap E) = \{[1 : 1], [-1 : 1]\}$ whereas $\pi(\tilde{X}_5 \cap E) = \{[0 : 1]\}$. Hence X_3 and X_5 are not isomorphic as well.

1.2.6 Cone over a Variety

Consider the morphism $\pi : \mathbb{A}^{n+1} - \{(0, 0, ..., 0)\} \to \mathbb{P}^n$. If Y is a closed subset of $\mathbb{P}^n$, then

$$C(Y) := \pi^{-1}(Y) \cup \{(0, 0, ..., 0)\}$$

is an affine variety in $\mathbb{A}^{n+1}$. In fact, the ideal of $C(Y)$ in $k[x_0, ..., x_n]$ is simply the ideal of Y as a homogeneous ideal in $k[x_0, ..., x_n]$, i.e. we can simply forget about the graded structure on $k[x_0, ..., x_n]$. Some facts about cones:

1. $C(Y)$ is irreducible iff Y is irreducible. In this case $\dim C(Y) = \dim Y + 1$, unless Y is a linear variety.
2. Suppose Y is non-singular closed projective variety, and let $X = C(Y)$. Then the point $O = (0, ..., 0)$ is a singular point on X. If $P \in X - O$, then P is a singular point of X iff $\pi(P)$ is a singular point of Y.
3. In the same setup as above, suppose also that Y is non-singular so that the only singular point of X is O. Let $\tilde{X}$ be the blow up of X at O, and let $\varphi : \tilde{X} \to X$ be the canonical birational morphism. Then $\tilde{X}$ is non-singular. Moreover, $\varphi^{-1}(O) \cong Y$.

More generally, by carrying out some linear translation, we can take the vertex of the cone to be any point in $\mathbb{A}^{n+1}$.

1.2.7 d-Uple Embedding

Suppose $n, d > 0$ are given, and let $N = \binom{n+d}{n} - 1$. Let $M_0, ..., M_N$ be all the monomials of degree d in the $n + 1$ variables $x_0, ..., x_n$. Then, the map $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ given by

$$\rho_d(P) = [M_0(P) : M_1(P) : \cdots : M_N(P)]$$

embeds $\mathbb{P}^n$ onto a closed subset of $\mathbb{P}^N$ (i.e. $\rho_d : \mathbb{P}^n \to \mathrm{Im}(\rho_d)$ is an isomorphism of varieties, with $\mathrm{Im}(\rho_d)$ with the induced variety structure from $\mathbb{P}^N$). This is the *d-uple embedding of* $\mathbb{P}^n$. The image of ρ_d is in fact the zero locus of the kernel of the map

$$k[y_0, ..., y_n] \to k[x_0, ..., x_n], y_i \mapsto M_i.$$

The Hilbert polynomial of the d-uple embedding is

$$\binom{dx + n}{n}$$

so that the degree of the d-uple embedding is d^n.

We record here two important examples:

1. The *twisted cubic curve* in $\mathbb{P}^3$, given by $X = \{[t^3 : t^2u : tu^2 : u^3] : [t, u] \in \mathbb{P}^1\}$. It is the 3-uple embedding of $\mathbb{P}^1$.

2. The 2-uple embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ is called the *Veronese Surface*. Sometimes, the d-uple embedding is also called the *Veronese map*. The Veronese surface is an example of a *determinantal variety*, in the following sense: Suppose $y_0, ..., y_5$ are the coordinate functions on $\mathbb{P}^5$. Then, the Veronese surface is the locus of points P such that the following matrix

$$\begin{pmatrix} y_0 & y_3 & y_4 \\ y_3 & y_1 & y_5 \\ y_4 & y_5 & y_2 \end{pmatrix}$$

has rank 1 at P (here, the map ρ_2 is given by $\rho_2(x_0 : x_1 : x_2) = [x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2]$). In fact all 2-uple embeddings of projective space are examples of determinantal varieties. If the 2-uple embedding ρ_2 of $\mathbb{P}^n$ sends $[x_0, ..., x_n]$ to $[Z_{ij}] \in \mathbb{P}^N$ where $Z_{ij} = x_ix_j$, then the image of ρ_2 is the zero locus of all 2×2 minors of the $(n+1) \times (n+1)$ symmetric matrix whose (i, j)'th entry (for $i \leq j$) is $Z_{i-1,j-1}$.

Under the d-uple mapping, the image of a variety $Y \subset \mathbb{P}^n$ is a sub-variety of $\mathbb{P}^N$.

Example 1.2.6. Let $Y \subset \mathbb{P}^2$ be the curve given by the polynomial $x_0^3 + x_1^3 + x_2^3$. If we want to embed it as a sub-variety of the Veronese surface, then notice that Y can equivalently be written as the zero locus of the polynomials

$$x_0^4 + x_0x_1^3 + x_0x_2^3, x_0^3x_1 + x_1^4 + x_1x_2^3, x_0^3x_2 + x_1^3x_2 + x_2^4.$$

If $\rho_2(x_0 : x_1 : x_2) = [x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2]$, then notice that the image $\rho_2(Y)$ is the intersection of $\rho_2(\mathbb{P}^2)$ with

$$Z(y_0^2 + y_1y_3 + y_2y_4, y_0y_3 + y_1^2 + y_2y_5, y_0y_4 + y_1y_5 + y_2^2).$$

1.2.8 Segre Embedding

Suppose $r, s > 0$. Define the map of sets $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^n$ where $n := (r+1)(s+1) - 1$ given by

$$\psi([x_0 : \cdots : x_r], [y_0 : \cdots : y_s]) = [x_iy_j]$$

where $\{x_i\}$ are coordinates on $\mathbb{P}^r$ and $\{y_j\}$ are coordinates on $\mathbb{P}^s$. This is clearly an injective map. Endowing the image of ψ with the Zariski topology from $\mathbb{P}^n$, we induce the structure of a projective variety onto $\mathbb{P}^r \times \mathbb{P}^s$. If z_{ij} are coordinates on $\mathbb{P}^n$ ($0 \leq i \leq r, 0 \leq j \leq s$) so that the image of ψ is the zero locus of the kernel of the map

$$k[z_{ij}] \to k[x_0, ..., x_r, y_0, ..., y_s], \quad z_{ij} \mapsto x_iy_j,$$

then the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ is the zero locus of all polynomials of the form

$$z_{ij}z_{k\ell} - z_{i\ell}z_{kj}.$$

Thus the Segre embedding is also another example of a determinantal variety.

Example 1.2.7 (Quadric Hypersurface)**.** The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ is a quadric hypersurface in $\mathbb{P}^3$ whose defining equation is $xw = yz$. Notice that the quadric hypersurface has two families of lines, one coming from the first factor of $\mathbb{P}^1$ and one coming from the second factor.

The Segre embedding allows us to define the product of two varieties. Indeed, if $X \subseteq \mathbb{P}^r$ and $Y \subseteq \mathbb{P}^s$ are any two varieties (either open or closed in their respective ambient projective space), then we define the variety structure on $X \times Y$ to be the one induced by the image of $X \times Y$ under the Segre embedding. If X and Y are irreducible, then $X \times Y$ are irreducible as well. Moreover, the product so defined is in fact the product in the category of varieties, i.e. if $\varphi : Z \to X$ and $\psi : Z \to Y$ are any two morphisms, then there exists a unique morphism $\Phi : Z \to X \times Y$ such that $\pi_X \circ \Phi = \varphi$ and $\pi_Y \circ \Phi = \psi$.

Proposition 1.2.8. *If $Y \subset \mathbb{P}^m$ and $Z \subset \mathbb{P}^n$ are any projective varieties with ideals $I(Y)$ and $I(Z)$ respectively, so that $Y \times Z \subset \mathbb{P}^{mn+m+n}$, then*

$$S(Y \times Z) \cong \bigoplus_{d \geq 0} S(Y)_d \otimes_k S(Z)_d$$

where recall that for a (closed) projective variety $X \subset \mathbb{P}^N$ the graded ring $S(X)$ is

$$k[x_0, ..., x_N]/I(X).$$

Corollary 1.2.8.1. *If the Hilbert polynomial of $Y \subset \mathbb{P}^r$ and $Z \subset \mathbb{P}^s$ is p_Y, p_Z respectively, then the Hilbert polynomial of $Y \times Z \subset \mathbb{P}^{rs+r+s}$ is $p_{Y \times Z} = p_Y p_Z$.*

Corollary 1.2.8.2. *The degree of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ is $\binom{r+s}{r}$.*

Proposition 1.2.9. *If Y and Z are any projective varieties of dimension r and s respectively, then the arithmetic genus of $X \times Y$ is*

$$p_a(Y \times Z) = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z).$$

1.2.9 Projecting from a Point

Embed $\mathbb{P}^n$ as a hyperplane in $\mathbb{P}^{n+1}$, and fix $P \in \mathbb{P}^{n+1} \backslash \mathbb{P}^n$. Define the map $\varphi : \mathbb{P}^{n+1} - P \to \mathbb{P}^n$ which maps the point Q to the intersection point of the unique line through P and Q with $\mathbb{P}^n$. Then φ is a morphism, called the *projection from* P; indeed if $P = [p_0 : p_1 : \cdots : p_{n+1}]$ and we assume that $\mathbb{P}^n = \{x_{n+1} = 0\} \subset \mathbb{P}^{n+1}$ then

$$\varphi(x_0 : x_1 : \cdots : x_{n+1}) = [p_{n+1}x_0 - p_0x_{n+1} : p_{n+1}x_1 - p_1x_{n+1} : \cdots : p_{n+1}x_n - p_nx_{n+1}].$$

If $X \subset \mathbb{P}^{n+1}$ is any variety not containing P, then the restriction $\varphi|_X : X \to \mathbb{P}^n$ is called the *projection of* X *into* $\mathbb{P}^n$. The dimension of $\varphi(X)$ is the same as the dimension of X, unless X is a cone with vertex at P.

1.3 Tangent Spaces, Tangent Cones, Dual Varieties and Singularities

1.3.1 Tangent Spaces

Definition. Suppose X is any variety, and $P \in X$. The *Zariski Cotangent Space* $T_P^*(X)$ is the vector space

$$T_P^*(X) = \mathfrak{m}_P/\mathfrak{m}_P^2$$

where $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{X,P}$, i.e. the ideal of germs of functions at P vanishing at P.

The *Zariski Tangent Space* $T_P(X)$ is the dual to this vector space.

Some important facts:

1. We have $\dim_k T_P(X) \geq \dim X$ for all $P \in X$, where (by definition) equality holds iff P is a non-singular point of X.

2. Any morphism $f : X \to Y$ induces the map $f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ (this map is sometimes denoted by f_P^*). We thus have an induced map $T_{f(P)}^*(Y) \to T_P^*(X)$, and hence we have a map $T_P(X) \to T_{f(P)}(X)$. This map is denoted by $df_P : T_P(X) \to T_{f(P)}(Y)$. Given any element $\alpha \in T_P(X)$ (which, by definition, is a linear functional on $\mathfrak{m}_P/\mathfrak{m}_P^2$), we have the linear functional $df_P(\alpha) \in T_{f(P)}(Y)$ which takes an element $[a] \in T_{f(P)}^*(Y)$ to
$$df_P(\alpha)([a]) := \alpha([a \circ f]).$$
In other words, $df_P(\alpha) = \alpha \circ f^\#$.

3. We have $T_P(\mathbb{A}_k^n) = k^n$ for all $P \in \mathbb{A}_k^n$. To be precise, note that $\mathfrak{m}_P = \langle x_i - P_i : 1 \leq i \leq r\rangle$ and $\mathfrak{m}_P^2 = \langle (x_i - P_i)(x_j - P_j) : 1 \leq i, j \leq r\rangle$. Thus,
$$T_P^*(\mathbb{A}^n) = \operatorname{span}_k(x_i - P_i : 1 \leq i \leq r).$$
Linear functionals of $T_P^*(\mathbb{A}^n)$ are in one-to-one correspondence with points k^n via the evaluation map; more precisely, a linear functional $\alpha \in T_P(\mathbb{A}^n)$ corresponds to evaluating at a point $Q + P \in k^n$. Thus we have $T_P(\mathbb{A}^n) = k^n$.

Lemma 1.3.1. *Suppose $X \hookrightarrow \mathbb{A}^n$, and let $X = Z(f_1, ..., f_t)$ with $f_i \in k[x_1, ..., x_n]$. Then at any point $P \in X$, we have an embedding $T_P(X) \hookrightarrow T_P(\mathbb{A}^n)$, so that we can identify $T_P(X)$ as a subspace of $T_P(\mathbb{A}^n) = k^n$. This subspace of k^n corresponds to the null space of the $t \times n$ matrix $(\frac{\partial f_i}{\partial x_j}(P))_{i,j}$, and is called the* affine tangent space of X at P.

Proof. Indeed, the inclusion map $\iota : X \to \mathbb{A}^n$ yields the embedding $d\iota_P : T_P(X) \to T_P(\mathbb{A}^n)$. Since X is defined by the f_i, it follows that $\alpha \in \operatorname{Im}(d\iota_P)$ iff $\alpha(\bar{f_i}) = 0$, where $\bar{f_i}$ is the image of the polynomial $f_i \in k[x_1, ..., x_n]$ in $\mathfrak{m}_{P,\mathbb{A}^n}/\mathfrak{m}_{P,\mathbb{A}^n}^2$. This implies that $\bar{f_i}$ is simply the linear part of f_i when we decompose it into a sum of homogeneous polynomials in $x_i - P_i$ of different degrees. Hence

$$\bar{f_i} \equiv \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(P) \cdot (x_j - P_j) \pmod{\mathfrak{m}_{P,\mathbb{A}^n}^2}.$$

Since $T_P(\mathbb{A}^n)$ can be identified with k^n via evaluation at $Q + P$, it follows that $Q \in T_P(\mathbb{A}^n)$ iff

$$\sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(P) Q_j = 0$$

for all i. This yields the result. □

Example 1.3.2 (Fall 2018 Day 3)**.** Let K be an algebraically closed field of characteristic zero. Consider the curve $C = \{(t^3, t^4, t^5) : t \in K\} \subset \mathbb{A}^3$. Prove that no neighbourhood of $(0, 0, 0)$ in C can be embedded into $\mathbb{A}^2$.

Suppose $f \in I(C)$, and write $f(x, y, z) = a + b_1x + b_2y + b_3z + q$ where $q \in \langle x, y, z\rangle^2 \subset K[x, y, z]$. Then, we have $a + b_1t^3 + b_2t^4 + b_3t^5 + q(t^3, t^4, t^5) = 0$ in $k[t]$. Since monomials in q have minimum degree of 2, $q(t^3, t^4, t^5)$ is divisible by t^6. Hence, comparing coefficients of smaller powers of t, we see that $a, b_1, b_2, b_3 = 0$. It follows that $I(C) \subseteq \langle x, y, z\rangle^2$. In particular, $\frac{\partial f}{\partial x}(0, 0, 0) = 0 = \frac{\partial f}{\partial y}(0, 0, 0) = \frac{\partial f}{\partial z}(0, 0, 0)$ for all $f \in I(C)$, which implies that the affine tangent space $T_{(0,0,0)}(C)$ to C at $(0, 0, 0)$ in $\mathbb{A}^3$ is 3-dimensional. However any embedding of a neighbourhood of $(0, 0, 0)$ in C into $\mathbb{A}^2$ must in particular induce an injection $T_{(0,0,0)}(C) \hookrightarrow T_p(\mathbb{A}^2)$ for some $p \in \mathbb{A}^2$. This is however impossible since $T_p(\mathbb{A}^2)$ is only 2-dimensional.

Proposition 1.3.3 (Generalization of Sard's Theorem)**.** *Suppose $f : X \to Y$ is a surjective morphism of varieties over a field k of characteristic 0. Then there exists open $U \subset Y$ such that for any non-singular $P \in f^{-1}(U)$, the map df_P is surjective.*

Theorem 1.3.4. *Suppose $\pi : X \to Y$ is a finite morphism. Then π is an isomorphism iff π is bijective and $d\pi_P : T_P(X) \to T_{\pi(P)}(Y)$ is an injection for all $P \in X$.*

In particular, suppose $\pi_0 : X_0 \to \mathbb{P}^n$ is any morphism. Let $U \subset \mathbb{P}^n$ be an arbitrary open subset, and set $X := \pi_0^{-1}(U) \subset X_0$ and $\pi := \pi_0|_X : X \to U$. If π is one-to-one (in particular, has finite fibers so is a finite map) and $d\pi_P : T_P(X) \to T_{\pi(P)}\mathbb{P}^n$ is an injection for all $P \in X$, then π is an isomorphism of X with its image.

Notice here no mention is made of non-singular points, i.e. the varieties need not be singular.

Definition. Suppose $X \subset \mathbb{P}^n$ is a closed set. Fix $P \in X$. Then, the *projective tangent space of X at P*, sometimes denoted by $\mathbb{T}_P(X)$, can be described equivalently as follows:

1. If $\mathbb{A}^n \cong U \subset \mathbb{P}^n$ is an open affine neighbourhood of P, then $\mathbb{T}_P(X)$ is the projective closure of $T_P(X \cap U) \subset A^n \cong k^n$ in $\mathbb{P}^n$.
2. If $X = Z(f_1, ..., f_t)$ with each f_i homogeneous, then $\mathbb{T}_P(X)$ is the zero locus of the linear homogeneous polynomials $\sum_{j=0}^n \frac{\partial f_i}{\partial x_j}(P) \cdot x_j$.
3. If $\tilde{X} \subset \mathbb{A}^{n+1}$ is the cone over X, then $\mathbb{T}_P(X)$ is the subspace of $\mathbb{P}^n$ corresponding to the Zariski tangent space $T_{\tilde{P}}\tilde{X} \subset k^{n+1}$ where $\tilde{P} \in k^{n+1}\backslash 0$ is any non-zero point lying over P.

Definition. Given a (closed) variety $X \subset \mathbb{P}^n$ of dimension k, the assignment $P \mapsto \mathbb{T}_P(X) \in \mathbb{G}(k, n)$ is a rational map $\mathcal{G} = \mathcal{G}_X : X \to \mathbb{G}(k, n)$ called the *Gauss map*; here, the domain of definition of $\mathcal{G}$ is simply the (open) set of non-singular points of X. Thus if X is non-singular, $\mathcal{G}$ is a morphism.

The closure of the image of $\mathcal{G}_X$ is called the *Gauss image* or the *variety of tangent planes to X*, and is denoted by $\mathcal{T}X$.

Example 1.3.5. If $X = Z(f) \subset \mathbb{P}^n$, then $\mathcal{G}_X : X \to \mathbb{G}(n-1, n) \cong (\mathbb{P}^n)^*$ is simply the map

$$\mathcal{G}_X(P) = \left[\frac{\partial f}{\partial x_0}(P) : ... : \frac{\partial f}{\partial x_n}(P)\right].$$

Definition. Suppose $X \subset \mathbb{P}^n$ a closed variety. The *tangential variety of X* is

$$TX = \bigcup_{\Lambda \in \mathcal{T}(X)} \Lambda$$

where the union occurs in $\mathbb{P}^n$.

1.3.2 Tangent Cones

Definition. Consider $P \in X$, where X is a variety. Let $\mathcal{O}_P$ be the ring of germs of functions at P with maximal ideal $\mathfrak{m}_P$. Then the graded sub-ring $B = \bigoplus_{i \geq 0} \mathfrak{m}_P^i/\mathfrak{m}_P^{i+1}$ is a quotient of the graded sub-ring

$$A = \bigoplus_{i \geq 0} \mathrm{Sym}^i(\mathfrak{m}_P/\mathfrak{m}_P^2) =: \mathrm{Sym}^*(T_P^*(X)).$$

Notice that the graded sub-ring B is just $\mathcal{O}_P$ with the filtration defined by $\mathfrak{m}_P$, while A is the ring of regular functions on $T_P(X)$. Let $I \subset A$ be the homogeneous ideal of A such that $B = A/I$.

The *tangent cone* $TC_P(X)$ is the zero locus in $T_P(X)$ of the homogeneous ideal I.

Proposition 1.3.6. *Suppose $X \subset \mathbb{A}^n$ with $P \in X$. Suppose $I(X) = \langle f_1, ..., f_t \rangle$, and suppose that at P we expand $f_i(X+P)$ into homogeneous terms. Let* $\text{in}(f_i)$ *be the* initial term *of f_i at P, i.e. the homogeneous part of lowest degree in $f_i(X+P)$. Then, $TC_P(X) = Z(\text{in}(f_i) : 1 \le i \le t)$.*

The tangent cone gives finer detail about the nature of the singularity at a point as compared to the tangent space.

Example 1.3.7. Let $X = Z(y^2 - x^3)$ and $Y = Z(y^2 - x^2 - x^3)$ be varieties in $\mathbb{A}^2$. Let $P = (0,0)$. Then $T_P(X) = \mathbb{A}^2 = T_P(Y)$. However, $TC_P(X) = Z(y^2)$ is simply the x-axis, whereas $TC_P(Y) = Z(y^2 - x^2)$ is the union of two distinct lines $\{y = x\}$ and $\{y = -x\}$.

Definition. The *multiplicity* of $P \in X$ on a k-dimensional variety X is $(k-1)!$ times the leading coefficient of the Hilbert Polynomial of the graded ring $\bigoplus_{i \ge 0} \mathfrak{m}_P^i / \mathfrak{m}_P^{i+1}$, i.e. it is $(k-1)!$ times the leading coefficient of the unique polynomial $p \in \mathbb{Q}[t]$ such that for all sufficiently large $i \in \mathbb{N}$, we have

$$p(i) = \dim_k \mathfrak{m}_P^i / \mathfrak{m}_P^{i+1}.$$

Example 1.3.8. Consider $P = (0,0)$ on the variety $X = Z(y^2 - x^3)$ in $\mathbb{A}^2$. For any $i \ge 2$ note that

$$\mathfrak{m}_P^i / \mathfrak{m}_P^{i+1} = \langle x^{j-i} y^j : 0 \le j \le i \rangle \,/\, \langle f \in k[x,y]/I(X), \deg f = i+1 \rangle = \text{span}_k\{x^i, x^{i-1}y\},$$

where notice that for $j \ge 2$, we have $x^{i-j}y^j \equiv x^{i-j+3}y^{j-2} \pmod{I(X)}$ with $x^{i-j+3}y^{j-2} \in \mathfrak{m}_P^{i+1}$, and thus the Hilbert polynomial is simply the constant polynomial 2. Thus the multiplicity of X at P is 2.

Proposition 1.3.9. *If $X = Z(f)$ is a hypersurface in $\mathbb{A}^n$, and $P \in Z(f)$, then the multiplicity of P is the degree of* $\text{im}(f)$.

Proposition 1.3.10. *$P \in X$ is non-singular iff the multiplicity of P on X is 1.*

1.3.3 Curve Singularities

Recall that the completion $\hat{\mathcal{O}}$ of a Noetherian local ring $\mathcal{O}$ with maximal ideal $\mathfrak{m}$ is the inverse limit $\lim_{\leftarrow} \mathcal{O}/\mathfrak{m}^n$. Here are some important algebraic properties of completion.

1. $\hat{\mathcal{O}}$ is a local ring with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{\mathcal{O}}$, and there is a natural injective homomorphism $\mathcal{O} \hookrightarrow \hat{\mathcal{O}}$.
2. If B is a local ring that is an $\mathcal{O}$-algebra finitely generated as an $\mathcal{O}$-module, and if $\mathfrak{n}$ is the maximal ideal of B above $\mathfrak{m}$, then $\hat{B}$ (completion with respect to $\mathfrak{n}$) is isomorphic to $B \otimes_{\mathcal{O}} \hat{\mathcal{O}}$.
3. $\dim \mathcal{O} = \dim \hat{\mathcal{O}}$.
4. $\mathcal{O}$ is regular iff $\hat{\mathcal{O}}$ is regular.
5. (Cohen Structure Theorem) If $\mathcal{O}$ is complete regular local ring of dimension n containing a field, then $\mathcal{O} \cong k[[x_1, ..., x_n]]$, the ring of formal power series over the residue field $k = \mathcal{O}/\mathfrak{m}$ of $\mathcal{O}$.
6. Suppose $f \in k[[x,y]]$ is written as $f = f_r + f_{r+1} + \cdots$ with f_m homogeneous of degree m, and if $f_r = g_s h_t$ with g_s homogeneous of degree s and h_t homogeneous of degree t and with g_s, h_t not sharing any linear factors, then there are formal power series $h = h_t + h_{t+1} + \cdots$ and $g = g_s + g_{s+1} + \cdots$ in $k[[x,y]]$ such that $f = gh$. This implies that the completion of $k[x,y]/\langle f \rangle$ is isomorphic to $k[[x,y]]/\langle gh \rangle$.

Definition. Suppose we have two closed affine sets X and Y and points $P \in X$ and $Q \in Y$. The points P and Q are said to be *analytically isomorphic* if the completion of $\mathcal{O}_{X,P}$ is isomorphic to the completion of $\mathcal{O}_{Y,Q}$.

If $P \in X$ and $Q \in Y$ are analytically isomorphic, then:

1. $\dim X = \dim Y$; in fact, a converse holds: if P and Q are non-singular with $\dim X = \dim Y$, then P and Q are analytically isomorphic.
2. multiplicity of P at X is equal to the multiplicity of Q at Y.

Now suppose X is a curve (i.e. $\dim X = 1$) and $P \in X$ such that $\dim T_P(X) = 2$. We have the following types of curve singularities:

Node This singularity is analytically isomorphic to the origin in the curve $xy = 0$ (equivalently, the curve $y^2 = x^2$) in $\mathbb{A}^2$. The tangent cone $TC_P(X)$ is the union of two distinct lines.

Cusp This singularity is analytically isomorphic to the origin in the curve $y^2 = x^3$ in $\mathbb{A}^2$. The tangent cone $TC_P(X)$ is a single line.

Tacnode This singularity is analytically isomorphic to the origin in the curve $y^2 = x^4$ in $\mathbb{A}^2$.

Oscnode This singularity is analytically isomorphic to the origin in the curve $y^2 = x^6$ in $\mathbb{A}^2$.

Ramphoid Cusp This singularity is analytically isomorphic to the origin in the curve $y^2 = x^5$ in $\mathbb{A}^2$.

1.3.4 Dual Varieties

Definition. A *tangent hyperplane* to a variety $X \subset \mathbb{P}^n$ is any hyperplane H such that $T_P(X) \subset H$ for some non-singular point $P \in X$.

Notice that hyperplanes of $\mathbb{P}^n$ can be considered as elements of the *dual projective space* $(\mathbb{P}^n)^* = \mathbb{G}(n-1, n)$.

Definition. The closure in $(\mathbb{P}^n)^*$ of the locus of all tangent hyperplanes to a variety $X \subset \mathbb{P}^n$ is called the *dual variety of* X and is denoted by $X^* \subset (\mathbb{P}^n)^*$.

To evaluate the dimension of a dual variety, we can use *incidence varieties.*
Consider the closure Φ of the set

$$\{(P, H) : P \in X \backslash \mathrm{Sing}(X), H \supset \mathbb{T}_P(X)\}.$$

Let $\pi_1 : \Phi \to X \subset \mathbb{P}^n$ be the projection on the first factor. The fibre $\pi_1^{-1}(P)$ consists of the set of all (P, H) such that $H \supset \mathbb{T}_P(X)$. Identifying H with its homogeneous linear polynomial, we see that the set of such H corresponds to the set of all linear homogeneous polynomials vanishing on $\mathbb{T}_P(X)$. By fixing a basis for $\mathbb{T}_P(X)$, this gives us a system of $\dim X + 1$ linearly independent equations for the coefficients of such linear homogeneous polynomials. Hence, every fibre of π_1 is irreducible and of dimension $n - \dim X - 1$ (since linear varieties are strict complete intersections). Moreover, the closure of the image of π_1 is simply X which has dimension $\dim X$. Hence, it follows that Φ has dimension $(n - \dim X - 1) + \dim X = n - 1$. By considering the projection $\pi_2 : \Phi \to (\mathbb{P}^n)^*$, it follows that X^* has dimension at most $n-1$, and will have dimension exactly $n - 1$ if for every $H \in X^*$ the fibre $\pi_2^{-1}(H)$ has dimension 0, i.e. consists only of finitely many points. (We have repeatedly used here the theorem towards the end of the section on dimensions.)

Example 1.3.11. Suppose X is a projective variety, and consider the projective closure $\overline{C(X)} =: \tilde{X}$ of the affine cone $C(X)$. Then, any $H \in \tilde{X}^*$ contains $\mathbb{T}_{\tilde{P}}(\tilde{X})$ for all non-zero points $\tilde{P}$ in the fibre of a point $P \in X$. In particular, it follows that $\tilde{X}^* \cong X^*$.

Definition. Dual varieties whose dimension is *not* $n - 1$ are called *deficient.*
The *deficiency* of the dual of X, written $\delta(X^*)$, is the number $n - 1 - \dim(X^*)$.

Thus a cone is deficient.

Example 1.3.12. Consider the Segre embedding $\psi : \mathbb{P}^r \times \mathbb{P}^s \hookrightarrow \mathbb{P}^n$ $(n = rs + r + s)$, and let X denote the image of the Segre embedding in $\mathbb{P}^n$. We calculate the dual X^* of the Segre embedding, and show that X^* is deficient if $m \neq n$.

Recall that we can write the coordinates on $\mathbb{P}^n$ as z_{ij} with $0 \le i \le r$ and $0 \le j \le s$. In this case, X is given by the zero locus of all equations of the form $z_{ij}z_{k\ell} - z_{i\ell}z_{kj}$. Let us compute the tangent spaces to X. Let $P = [P_{ij}] = \psi(A, B)$, so that $P_{ij} = A_iB_j$. Suppose $A_{i_0} = 1$ and $B_{j_0} = 1$. Then,

$$\begin{aligned}\mathbb{T}_P(X) &= Z\big(P_{k\ell}z_{ij} + P_{ij}z_{k\ell} - P_{kj}z_{i\ell} - P_{i\ell}z_{kj} \;:\; 0 \le i, k \le r, 0 \le j, \ell \le s\big)\\ &= Z\big(A_kB_\ell z_{ij} + A_iB_jz_{k\ell} - A_kB_jz_{i\ell} - A_iB_\ell z_{kj} \;:\; 0 \le i, k \le r, 0 \le j, \ell \le s\big).\end{aligned}$$

However, notice that

$$\begin{aligned}A_kB_\ell z_{ij} + A_iB_jz_{k\ell} - A_kB_jz_{i\ell} - A_iB_\ell z_{kj} &= A_kB_\ell(z_{ij} + A_iB_jz_{i_0j_0} - A_iz_{i_0j} - B_jz_{ij_0}) + A_iB_j(z_{k\ell} + A_kB_\ell z_{i_0j_0} - A_kz_{i_0\ell} - B_\ell z_{kj_0})\\ &\quad - A_kB_j(z_{i\ell} + A_iB_\ell z_{i_0j_0} - A_iz_{i_0\ell} - B_jz_{ij_0}) - A_iB_\ell(z_{kj} + A_kB_jz_{i_0j_0} - A_kz_{i_0j} - B_jz_{kj_0}),\end{aligned}$$

and thus

$$\mathbb{T}_P(X) = Z\big(z_{ij} + A_iB_jz_{i_0j_0} - A_iz_{i_0j} - B_jz_{ij_0} : 0 \le i \le r, 0 \le j \le s, i \neq i_0, j \neq j_0\big).$$

Since this is a linear variety, it is a strict complete intersection, so that $\dim \mathbb{T}_P(X) = n - rs = r + s$. Hence X is non-singular. Now suppose $H = Z(f)$ is a hyperplane in X^* where f is a linear homogeneous polynomial, say $f = \sum_{i,j} c_{ij}z_{ij}$. Then $(P, H) \in \pi_2^{-1}(H)$ iff $f(P) = 0$ and

$$f \in \langle z_{ij} + P_{ij}z_{i_0j_0} - P_{ij_0}z_{i_0j} - P_{i_0j}z_{ij_0} : 0 \le i \le r, 0 \le j \le s, i \neq i_0, j \neq j_0\rangle =: \mathfrak{a}_p$$

where $P_{i_0j_0} = 1$ iff $f(P) = 0$ and

$$\sum_i \left(\sum_j c_{ij} P_{i_0 j} \right) z_{ij_0} + \sum_j \left(\sum_i c_{ij} P_{ij_0} \right) z_{i_0 j} \equiv \left(\sum_{i,j} c_{ij} P_{ij} \right) z_{i_0 j_0} \pmod{\mathfrak{a}_p}$$

iff $\sum_j c_{ij} P_{i_0 j} = 0$ for all $i \neq i_0$, $\sum_i c_{ij} P_{ij_0} = 0$ for all $j \neq j_0$,

$$\sum_{i,j,i\neq i_0, j\neq j_0} c_{ij} P_{ij} = c_{i_0 j_0}, \quad \text{and} \sum_{i,j} c_{ij} P_{ij} = 0.$$

It follows that if we represent $(c_{ij}) = C \in \mathbb{P}(M_{r\times s}(k))$, then this system is equivalent to solving $A^T C = 0$ and $CB = 0$ ($A \in \mathbb{P}^r$ and $B \in \mathbb{P}^s$), where we assume that at least one solution is known. If $r \neq s$, and if there is one solution, then there will be infinitely many solutions (assuming k is an infinite field). In particular, $\dim \pi_2^{-1}(H) > 0$ for all H, and hence $\dim X^* < n - 1$.

Here we list two important theorems on dual varieties.

Proposition 1.3.13. *Suppose $X \subset \mathbb{P}^n$ closed variety and $X^* \subset (\mathbb{P}^n)^*$ its dual. Suppose $\Phi_X \subset \mathbb{P}^n \times (\mathbb{P}^n)^*$ and $\Phi_{X^*} \subset (\mathbb{P}^n)^* \times \mathbb{P}^n$ are the incidence varieties corresponding to X and X^* respectively. Then, under the identification $\mathbb{P}^n \times (\mathbb{P}^n)^* = (\mathbb{P}^n)^* \times \mathbb{P}^n$, we have $\Phi_{X^*} = \Phi_X$. In particular, $(X^*)^* = X$.*

Theorem 1.3.14 (Landman's Theorem)**.** *Suppose $X \subset \mathbb{P}^n$ is a non-singular variety with deficient dual. Then $\delta(X^*) \equiv \dim X \pmod 2$.*

1.4 Projective Curves

From now on, we consider a *curve* to be a irreducible variety/quasi-variety of $\mathbb{P}^n_k$ of dimension 1. A *complete curve*, or equivalently, a *projective curve* is a curve that is also a closed subset of $\mathbb{P}^n_k$. A *non-singular curve* is a curve that is a non-singular variety.

1.4.1 Bi-rational Classification of Projective Curves

Lemma 1.4.1. *Suppose A is a Noetherian local domain of dimension one with maximal ideal $\mathfrak{m}$, then the following conditions are equivalent: (i) A is a discrete valuation ring, (ii) A is integrally closed, (iii) A is a regular local ring, (iv) $\mathfrak{m}$ is a principal ideal.*

In particular, notice that if X is a non-singular irreducible curve and $P \in X$, then $\mathcal{O}_{X,P}$ is a regular Noetherian local domain of dimension 1. Thus $\mathcal{O}_{X,P}$ satisfies all of the above conditions. In particular, it is a discrete valuation ring. We denote this valuation by v_P.

A *projective curve* is a closed irreducible dimension 1 subset of projective space. A *function field of dimension n over k* is a field of transcendence degree n over k.

Proposition 1.4.2. *Suppose X is a non-singular projective curve, U an open subset of X, and Y any projective variety. If $\varphi : U \to Y$ is a morphism, then there exists a unique morphism $\bar{\varphi} : X \to Y$ extending φ.*

Theorem 1.4.3.

1. *The categories of (i) non-singular projective curves and dominant morphisms, (ii) quasi-projective curves (i.e. open subset of a closed projective curve) and dominant rational maps, and (iii) function fields of dimension 1 over k and k-algebra homomorphism, are all equivalent. The functor from (i) to (iii) is given by $Y \mapsto K(Y)$, the field of rational functions on a curve Y. The functor from (i) to (ii) is to simply take closures.*
2. *Every curve is bi-rationally equivalent to a non-singular projective curve.*
3. *Every non-singular quasi-projective curve is isomorphic to an open subset of a non-singular projective curve, i.e. if a projective curve is non-singular on a non-empty (dense) open set, then it is non-singular everywhere.*
4. *For any function field K of dimension 1 over k, there exists a uniquely determined non-singular projective curve X (up to isomorphism) such that $K(X) = K$ and such that there is a one-to-one correspondence between points of X and discrete valuation rings $(\mathcal{O}, v)$ such that $\mathcal{O}$ is a sub-ring of K with $k\backslash\{0\} \subset \mathcal{O}^*$. Conversely, if X is projective non-singular curve, then X has the above DVR-point correspondence with its function field $K(X)$.*

Corollary 1.4.3.1. *If Y is a curve bi-rationally equivalent to $\mathbb{P}^1$, then Y is either isomorphic to $\mathbb{P}^1$, or Y is a closed affine curve (i.e. $Y \cong$ closed variety of some affine n-space) isomorphic to an open subset of $\mathbb{A}^1$ and such that $A(Y)$ is a unique factorization domain.*

Proposition 1.4.4. *Suppose $f : X \to Y$ is a non-constant morphism where X is a projective non-singular curve over k and Y any curve over k. Then f is a surjective finite morphism, and Y is projective.*

1.4.2 Line Bundles

Definition. If X is a variety, then an *invertible sheaf* aka. a *line bundle* on X is a sheaf $\mathcal{L}$ on X satisfying the following properties:

- For each open $U \subset X$, $\mathcal{L}(U)$ is a $\mathcal{O}_X(U)$-module.
- The module operations are compatible with restriction, i.e. if $f \in \mathcal{O}_X(U)$, $s \in \mathcal{L}(U)$, and $V \subset U$ is open, then $(f \cdot s)|_V = f|_V \cdot s|_V$.
- For every open affine subset U of X, the sheaf $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_X|_U$ as $\mathcal{O}_X|_U$-modules, i.e. for all open subsets $V \subset U$, the $\mathcal{O}_X(V)$-module $\mathcal{L}(V)$ is a rank 1 $\mathcal{O}_X(V)$-module.

Note: here we do not distinguish between invertible sheaves and line bundles, though technically they are slightly different things. A line bundle is in fact a variety with a projection map and local trivializations and so on, whereas an invertible sheaf is the sheaf of regular functions on the line bundle. We do not need to distinguish between them here, so that whenever the term line bundle is used we actually mean an invertible sheaf.

Obviously $\mathcal{O}_X$ is a line bundle on X. It is known that, if X is a variety over k, then $\Gamma(X, \mathcal{L})$ is a finite-dimensional vector space over k for any line bundle $\mathcal{L}$.

Lemma-Definition. Suppose X is a variety with sheaf of regular functions $\mathcal{O}_X$.

1. If $\mathcal{L}$ and $\mathcal{L}'$ are line bundles on X, then their *tensor product* $\mathcal{L} \otimes \mathcal{L}'$ is the line bundle $\mathcal{M}$ such that for any open subset $U \subset X$, $\mathcal{M}(U)$ is the space of functions $s : U \to \bigsqcup_{P \in U} \mathcal{L}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{L}'_P$ where for any $P \in U$, there exists open neighbourhood $V \subset U$ of P and there exists an element $s_V \in \mathcal{L}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{L}'(V)$ with $s(Q) = s_V(Q)$ for all $Q \in V$ (the tensor product of modules). The stalks $\mathcal{M}_P$ of $\mathcal{M}$ are simply $\mathcal{L}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{L}'_P$.

   In particular, we define $\mathcal{L}^n$ to be the n'th tensor power $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$.
2. If $\mathcal{L}$ is a line bundle, then the *dual* $\mathcal{L}^*$ is the line bundle such that for each open subset $U \subset X$, we have $\mathcal{L}^*(U) = (\mathcal{L}(U))^*$ is the $\mathcal{O}_X(U)$-module of $\mathcal{O}_X(U)$-module homomorphisms from $\mathcal{L}(U)$ to $\mathcal{O}_X(U)$.

Let $\text{Pic}(X)$ be the set of all ($\mathcal{O}_X$-isomorphism classes) of line bundles on X. Then, $\text{Pic}(X)$ is a group with group operation $\otimes$, the identity line bundle given by $\mathcal{O}_X$, and the inverse given by the dual. This group $\text{Pic}(X)$ is called the *Picard Group* of X.

Definition. A *rational* section of a line bundle $\mathcal{L}$ is a section of $\mathcal{L} \otimes \mathcal{K}_X$, where $\mathcal{K}_X$ is the sheaf such that $\mathcal{K}_X(U) = K(X)$ for all open sets U of X. Thus, a rational section of $\mathcal{L}$ can be thought of as a regular section of $\mathcal{L}$ possibly multiplied with rational functions on X (which may have poles).

Equivalently, a rational section is a regular section of $\mathcal{L}|_U$ for some open subset U of X.

An important class of line bundles on a projective variety X are the *twisting sheaves* $\mathcal{O}_X(d)$ where $d \in \mathbb{Z}$ (twisting sheaves also depend on an embedding $X \hookrightarrow \mathbb{P}^n$). Suppose $X \subseteq \mathbb{P}^n$, and let $I(X) \subset k[x_0, ..., x_n]$ be the ideal of X. Consider the graded ring $S(X)$. For any $P \in X$, let $\mathfrak{m}_P \subset S(X)$ be the ideal generated by homogeneous polynomials vanishing at P. Recall that $\mathcal{O}_{X,P} = S(X)_{(\mathfrak{m}_P)}$ is the ring of all ratios f/g where $f, g \in S(X)$ are homogeneous of the same degree, and $g \notin \mathfrak{m}_P$. Set $\mathcal{O}_X(d)_P$ to be the $\mathcal{O}_{X,P}$-module of all ratios f/g where $f, g \in S(X)$ are homogeneous with $\deg f - \deg g = d$, and $g \notin \mathfrak{m}_P$. We now define the sheaf $\mathcal{L}$ as follows: for any open subset $U \subset X$, let $\mathcal{L}(U)$ be the set of functions s from U to $\bigsqcup_{P \in U} \mathcal{O}_X(d)_P$ such that for each $P \in U$, there exists an open neighbourhood $V \subset U$ of P such that $s = f/g$ where $f, g \in S(X)$ with $\deg f - \deg g = d$, and for all $Q \in V$ we have $g \notin \mathfrak{m}_Q$ and $s(Q) = f/g \in \mathcal{O}_X(d)_P$. This is in fact a line-bundle, called the *twisted sheaf* $\mathcal{O}_X(d)$. In particular, $\mathcal{O}_X(1)$ is called the *twisting sheaf of Serre*.

More generally, if $\mathcal{L}$ is a line bundle on X, then $\mathcal{L}(d) := \mathcal{L} \otimes \mathcal{O}_X(d)$ is the *twist of $\mathcal{L}$ by d*. We have the following properties:

1. The local rings $\mathcal{O}_X(d)|_P$ for $P \in X$ is precisely the $\mathcal{O}_{X,P}$-module $\mathcal{O}_X(d)|_P$ defined above, i.e. the $\mathcal{O}_{X,P}$-module of all ratios f/g where $f, g \in S(X)$ are homogeneous with $\deg f - \deg g = d$, and $g \notin \mathfrak{m}_P$.

2. $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(m+n)$.
3. If $f \in k[x_0, ..., x_n]$ is homogeneous of degree d, we get a morphism of sheaves $\mathcal{O}_X(n) \to \mathcal{O}_X(n+d)$, $s \mapsto f \cdot s$. This morphism is usually *not* an isomorphism.
4. The space $\Gamma(D(f), \mathcal{O}_X(d))$ is the $\mathcal{O}_X(D(f))$-module of fractions g/f^m here $g \in S(X)$ is homogeneous and $\deg -m \deg f = d$. In particular, $\Gamma(X, \mathcal{O}_X(d))$ is the k-vector space of all degree d elements in $S(X)$.
5. There are no global (regular) sections of $\mathcal{O}_X(d)$ for $d < 0$. However, the space of global *rational* sections is spanned by $x_0^{-d}, ..., x_n^{-d}$.

The important example is $\mathcal{O}_{\mathbb{P}^n}(d)$. In fact, it is known that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, or more precisely,

$$\mathrm{Pic}(\mathbb{P}^n) = \{\mathcal{O}_{\mathbb{P}^n}(d) : d \in \mathbb{Z}\}.$$

Definition. Suppose X is a variety and $\mathcal{L}$ a line bundle on X. We say that $\mathcal{L}$ is *generated by global sections* if there exists a set of elements $\{s_i\} \subset \Gamma(X, \mathcal{L})$ such that for every $P \in X$, the images $\{s_i|_P\}$ of the s_i in the stalk $\mathcal{L}_P$ generate the stalk $\mathcal{L}_P$ as an $\mathcal{O}_{X,P}$-module.

1.4.3 Divisors

Basic Definitions

Proposition 1.4.5. *If X is a complete non-singular curve and Y any curve, and if $f : X \to Y$ is a non-constant morphism, then f is a finite morphism, $K(X)$ is a finite extension field of $K(Y)$, and Y is also complete.*

Definition. A *divisor* D on a curve X is a formal finite integral linear combination of the form $D = \sum_{P \in S} n_P P$ where $S \subset X$ is finite and $n_P \in \mathbb{Z}$. This set S of all $P \in X$ such that $n_P \neq 0$ is the *support* $\mathrm{Supp}(D)$ of the divisor D.

If $U \subset X$ is open, then we set $D|_U := \sum_{P \in U} n_P P$.

The divisor D is *effective*, written $D \geq 0$, if $n_P \geq 0$ for all $P \in \mathrm{Supp}(D)$.

The *degree* of the divisor D is

$$\deg D := \sum_{P \in \mathrm{Supp}(D)} n_P.$$

The group of divisors on a variety X is denoted $\mathrm{Div}(X)$.

Recall that for a curve C, the local rings $\mathcal{O}_{X,P}$ are discrete valuation rings. Let v_P denote the valuation on the DVR $\mathcal{O}_{X,P}$.

Definition. Given $P \in X$, a function $f \in K(X) \backslash \{0\}$ is said to have a *zero at P of order $v_P(f)$* if $v_P(f) > 0$ (i.e. $f \in \mathfrak{m}_{X,P}$), and a *pole at P of order $-v_P(f)$* if $v_P(f) < 0$.

It is known that $\{P \in X : v_P(f) \neq 0\}$ is a finite set for all $f \in K(X) \backslash \{0\}$, so that the *divisor of f*

$$\mathrm{div}(f) := \sum_{P \in X} v_P(f) P$$

is well-defined. We thus have a group homomorphism $\mathrm{div} : K(X) \backslash \{0\} \to \mathrm{Div}(f)$, such that $k \subset \ker(\mathrm{div})$.

Two divisors D and D' are said to be *linearly equivalent*, written $D \sim D'$, if there exists $f \in K(X)$ such that $D' - D = \mathrm{div}(f)$. The set of equivalence classes of linearly equivalent divisors forms a group (in fact, is the co-kernel of the div homomorphism), denoted by $\mathrm{Cl}(X)$.

Definition. Suppose $\varphi : X \to Y$ is a finite morphism of curves. The *degree* $\deg \varphi$ of φ is the degree of the finite field extension $[K(X) : K(Y)]$.

Definition. Suppose $\varphi : X \to Y$ is a finite morphism of non-singular curves. For any $Q \in Y$, let $t \in \mathcal{O}_{Y,Q}$ be a *local parameter at Q*, i.e. an element such that $v_{Y,Q}(t) = 1$. If $\varphi(P) = Q$, the integer $v_{X,P}(\varphi^\#(t)) \in \mathbb{N}$ is independent of the choice of local parameter at Q, and is called the *ramification index* e_P of φ at the point P. If $e_P > 1$ then f is *ramified* at P and Q is a *branch point* of f. If $e_P = 1$, then f is *unramified* at P. If $\mathrm{char} k = 0$ or if $\mathrm{char} k$ does not divide e_P, then the ramification at P is *tame*. Otherwise, it is said to be *wild*.

We can now define a group homomorphism $\varphi^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ as follows. For any point $Q \in Y$, we set

$$\varphi^*(Q) = \sum_{P \in \varphi^{-1}(Q)} e_P P,$$

and extend φ^* linearly to all of $\mathrm{Div}(Y)$. One checks that φ^* preserves linear equivalence (since $\varphi^* \circ \mathrm{div} = \mathrm{div} \circ \varphi^\#$), so that we actually get a group homomorphism $\varphi^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$.

We have the following facts:

1. If $\varphi : X \to Y$ is a finite morphism of non-singular curves, then for any divisor D on Y we have $\deg(f^*D) = (\deg f)(\deg D)$.
2. $\deg(\mathrm{div}(f)) = 0$ for all $f \in K(X)^*$. Thus, $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$ is a surjective group homomorphism.

Example 1.4.6. A projective non-singular curve X is rational (i.e. birational to $\mathbb{P}^1$) iff there exist two distinct points P and Q such that $P \sim Q$. In such a case, as X is a closed non-singular subset of projective space, if X is birational to $\mathbb{P}^1$ then it must be isomorphic to $\mathbb{P}^1$.

If X is indeed rational, so that $X \cong \mathbb{P}^1$, then given any distinct points P and Q we have the rational function $f(x, y) = \frac{P_1 x - P_0 y}{Q_1 x - Q_0 y}$, where clearly $\mathrm{div}(f) = P - Q$. On the other hand, if we can find distinct points P and Q such that $P - Q = \mathrm{div}(f)$ for some $f \in K(X)^*$, then we have the well-defined morphism $f : X - Q \to U_0 \subset \mathbb{P}^1$. This can be extended by setting $f(Q) = [1 : 0]$. Since X is projective non-singular, and f is non-constant, it follows that f is finite and the image $f(X)$ is closed in $\mathbb{P}^1$. Thus f is surjective. Now, note that $\deg[1 : 0] = 1 = \deg Q$. Since $f^{-1}([1 : 0]) = \{Q\}$ (since for all other points $R \in X - Q$, we have $f \in \mathcal{O}_{X,R}$ so that $f(R) \in k$ makes sense), it follows that $\deg f = 1$. Hence $K(X) = K(\mathbb{P}^1)$, and so X is bi-rational, and thus isomorphic, to $\mathbb{P}^1$.

Divisors and Line Bundles

Definition. Given a divisor D, define the sheaf $\mathcal{O}_X(D)$ by setting

$$\mathcal{O}_X(D)(U) := \{0\} \cup \{f \in K(X)^* : (\mathrm{div} f + D)|_U \geq 0\}.$$

Since for any $f \in \mathcal{O}_X(U)$, we have $(\mathrm{div} f)|_U \geq 0$, it follows that $\mathcal{O}_X(D)(U)$ is indeed an $\mathcal{O}_X(U)$-module, so that $\mathcal{O}_X(D)$ is a line bundle.

Suppose s is a rational section of a line bundle $\mathcal{L}$ on a non-singular curve X. Suppose $\mathcal{U}$ is an open cover of X such that for any $U \in \mathcal{U}$, $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_X|_U$ as $\mathcal{O}_X|_U$-modules. Let $\varphi_U : \mathcal{L}|_U \to \mathcal{O}_X|_U$ be the $\mathcal{O}_X|_U$-module isomorphism. Then φ_U sends s to a rational function on U. We set $\mathrm{ord}_P(s) = \mathrm{ord}_P(\varphi_U(s|_U))$ for all $P \in U$, where $\varphi_U(s|_U)$ is a rational function.

$$\mathrm{div}(s) := \sum_{P \in X} \mathrm{ord}_P(s) P.$$

For instance, the divisor of x_0^{-1} (a rational section of $\mathcal{O}_{\mathbb{P}^1}(-1)$) has divisor $-[0 : 1] \in \mathrm{Div}(\mathbb{P}^1)$.

Theorem 1.4.7. *The map* $\mathrm{Cl}(X) \to \mathrm{Pic}(X)$*,* $D \mapsto \mathcal{O}_X(D)$*, is an isomorphism of groups. The inverse sends* $\mathcal{L}$ *to the divisor class of* $\mathrm{div}(s)$ *for* any *rational section* s *of* $\mathcal{L}$*.*

Moreover, given $\mathcal{L} = \mathcal{L}(D_0) \in \mathrm{Pic}(X)$*, there is a 1-1 correspondence between equivalence classes in* $(\Gamma(X, \mathcal{L}) - 0)/k^*$ *(i.e. the projective vector space of* regular *global sections) and* effective *divisors linearly equivalent to* D_0 *given by*

$$\Gamma(X, \mathcal{L}) \backslash \{0\} \ni s \mapsto \mathrm{div}(s) \in \mathrm{Div}(X),$$

where $\mathrm{div} s = \mathrm{div} t$ *iff* $s = \lambda t$ *for some* $\lambda \in k^*$*.*

Definition. The *complete linear system* $|D|$ of a divisor D is the set of all effective divisors linearly equivalent to D. Since we have the bijection between $|D|$ and $(\Gamma(X, \mathcal{O}_X(D)) - 0)/k^*$ whenever $|D|$ is non-empty, we can endow $|D|$ the structure of a k-vector space in such a way that $\dim_k |D| = \dim_k \Gamma(X, \mathcal{O}_X(D)) - 1$.

A *linear system* is then simply any k-vector subspace of the complete linear system $|D|$.

The dimension of $\Gamma(X, \mathcal{L})$ is denoted by $h^0(\mathcal{L})$. In particular, the dimension $h^0(\mathcal{O}_X(d))$ of $\Gamma(X, \mathcal{O}_X(D))$ is denoted by $\ell(D)$, so that $\dim_k |D| = \ell(D) - 1$.

Lemma 1.4.8. *The function* ℓ *is monotonically increasing, i.e. if* D *is a non-zero divisor and* P *a point, then* $\ell(D - P) \leq \ell(D)$*.*

Proof. Notice that $\Gamma(X, \mathcal{O}_X(D - P))$ and $\Gamma(X, \mathcal{O}_X(D))$ are both sub-spaces of $K(X)$, and so it suffices to show that $\Gamma(X, \mathcal{O}_X(D - P)) \subseteq \Gamma(X, \mathcal{O}_X(D))$, i.e. if $f \in K(X)^*$ satisfies $\mathrm{div}(f) + D - P \geq 0$ then $\mathrm{div}(f) + D \geq 0$. This is obvious. □

Lemma 1.4.9. *If* $\deg D < 0$ *then* $\ell(D) = 0$*. If* $\ell(D) \neq 0$ *and* $\deg D = 0$*, then* $D \sim 0$ *i.e.* $\mathcal{L} \cong \mathcal{O}_X$*.*

Lemma 1.4.10. *D is linearly equivalent to an effective divisor iff* $\ell(D) > 0$*.*

The Canonical Sheaf

Recall that for each $P \in X$ and each regular function f, we have a well-defined element $d_P f \in T^*_P(X) \in \mathfrak{m}_P/\mathfrak{m}_P^2$.

Definition. The *sheaf of differential forms* Ω_X over a non-singular variety X is defined to be the $\mathcal{O}_X$-module such that $\Omega_X(U)$ is the set of all maps $\omega : U \to \bigsqcup_{P\in U} \mathfrak{m}_P/\mathfrak{m}_P^2$ such that for every $P \in U$, there exists a neighbourhood $V \subset U$ of P and there exist functions $f_1, ..., f_t, g_1, ..., g_t \in \mathcal{O}_X(U)$ such that for every $Q \in V$, we have

$$\omega(Q) = \sum_{i=1}^{t} f_i d_Q g_i.$$

Clearly $\Omega_{X,P} = \mathfrak{m}_P/\mathfrak{m}_P^2 = T^*_P(X)$.

We have a well-defined sheaf morphism $d : \mathcal{O}_X \to \Omega_X$ which sends $f \in \mathcal{O}_X(U)$ to $df \in \Omega_X(U)$. On each open set U, this map $d : \mathcal{O}_X(U) \to \Omega_X(U)$ is a *k-derivation*, i.e. a k-linear map such that $d(fg) = fdg+gdf \in \Omega_X(U)$.

Proposition 1.4.11. *If X is a non-singular variety and $P \in X$, then there exists an affine neighbourhood $U \subset X$ of P such that $\Omega_X(U)$ is a free $\mathcal{O}_X(U)$-module of rank* $\dim X$.

Moreover, if the local parameters $u_1, ..., u_r \in \mathfrak{m}_P$ form a basis for $\mathfrak{m}_P/\mathfrak{m}_P^2$, then in a small enough neighbourhood V of P such that all of the u_i are well-defined, a $\mathcal{O}_X(V)$-basis for $\Omega_X(V)$ is given by $du_1, ..., du_r$.

Definition. If X is a non-singular variety of dimension r, then $\bigwedge^r \Omega_X$ (the sheaf defined by $(\bigwedge^r \Omega_X)(U) := \bigwedge^r(\Omega_X(Y)))$ is a line bundle called the *canonical sheaf* ω_X of X.

In particular, if X is a non-singular curve, then $\omega_X = \Omega_X$. Moreover, under the divisor-line bundle correspondence given previously, any divisor in the linear equivalence class of divisors corresponding to ω_X is called a *canonical divisor*, and is usually denoted by K_X.

Example 1.4.12. Let us compute the canonical sheaf of $\mathbb{P}^n$. Consider the open affine set U_i. WLOG, fix $i = 0$. Then, we have the local parameters $y_1 := (x_1 - P_1x_0)/x_0, ..., y_n := (x_n - P_nx_0)/x_0$ which form a basis for $\mathfrak{m}_P/\mathfrak{m}_P^2$ for all $P \in U_0$ (where $P = [1 : P_1 : \cdots : P_n]$). Since any function in $\mathcal{O}_X(U_i)$ can be written as a polynomial in $y_1, ..., y_n$, we see easily that for any regular function f defined on some open set V of U_i, the k-derivation d sends f to

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} dy_i.$$

Thus $\Omega_{\mathbb{P}^n}(U_0)$ is spanned as an $\mathcal{O}_X(U_0)$-module by the dy_i. Hence,

$$\omega_X(U_0) = \bigwedge^n \Omega_{\mathbb{P}^n}(U_0) = \mathcal{O}_X(U_0) \cdot dy_1 \wedge \cdots \wedge dy_n.$$

Now consider U_1, and suppose $P = [1 : P_1 : \cdots : P_n] \in U_0 \cap U_1$ ($P_1 \neq 0$). Then we have the local parameters $z_j = (P_1x_j - P_jx_1)/x_1$ for all $j \neq 1$. Thus for $j \neq 0, 1$, we have $(z_j + P_j)x_1 = P_1x_j = P_1(y_j + P_j)x_0$ so that

$$z_j = -P_j + P_1\frac{x_0}{x_1}(y_j + P_j) = -P_j + P_1\frac{y_j + P_j}{y_1 + P_1}.$$

Thus

$$dz_j = P_1\frac{1}{y_1 + P_1}dy_j - P_1\frac{y_j + P_j}{(y_1 + P_1)^2}dy_1.$$

Also, $z_0 = P_1(x_0/x_1) - 1 = \frac{P_1}{y_1+P_1} - 1$ so that $dz_0 = -\frac{P_1}{(y_1+P_1)^2}dy_1$. Hence,

$$\begin{aligned} dz_0 \wedge dz_2 \wedge \cdots \wedge dz_n &= -\frac{P_1}{(y_1 + P_1)^2}dy_1 \wedge \bigwedge_{j=2}^{n} \left(P_1\frac{1}{y_1 + P_1}dy_j - P_1\frac{y_j + P_j}{(y_1 + P_1)^2}dy_1\right) \\ &= -\frac{P_1^n}{(y_1 + P_1)^{n+1}}dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n = -\frac{P_1^n x_0^{n+1}}{x_1^{n+1}}dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n. \end{aligned}$$

Thus, we see that the $\mathcal{O}_{\mathbb{P}^n}(U_i)$-module is spanned by $(1/x_i)^{n+1}d(x_0s/x_i) \wedge \cdots \wedge d(\hat{x_i}/x_i) \wedge \cdots \wedge d(x_n/x_i)$, so that as a graded module it is simply $\mathcal{O}_{\mathbb{P}^n}(-n-1)(U_i)$. Therefore, $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

Example 1.4.13. We can also calculate the canonical divisor K for $\mathbb{P}^1$. It suffices to find a divisor on $\mathbb{P}^1$ corresponding to $\mathcal{O}_{\mathbb{P}^1}(-2)$. One way to compute it is to notice that the map $\mathrm{Cl}(X) \to \mathrm{Pic}(X)$ is an isomorphism of groups, so that the divisor corresponding to $\mathcal{O}_{\mathbb{P}^1}(-2)$ is -2 times a divisor corresponding to $\mathcal{O}_{\mathbb{P}^1}(1)$. However, the sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$ has a global section $x \in k[x, y] = S(\mathbb{P}^1)$. Since $\mathrm{div}(x) = [0 : 1]$, it then follows that

$$K_{\mathbb{P}^1} = -2[0 : 1].$$

As any two points of $\mathbb{P}^1$ are linearly equivalent, we have $[K_{\mathbb{P}^1}] = \{-P - Q : P, Q \in \mathbb{P}^1\}$.

We compute it another way. Recall that $\Omega_{\mathbb{P}^1}$ by definition is the sheaf of differentials. Thus, we just need to pick any (meromorphic) global differential form. For instance, take $\omega = d(x_0/x_1) = d(x_0/x_1 - \lambda)$, where $x_0/x_1 - \lambda$ is a local parameter at $[\lambda, 1]$. This has no zeros or poles anywhere on U_1. However, under a change of coordinates we have $\omega = d(\frac{1}{x_1/x_0}) = -\frac{1}{(x_1/x_0)^2}d(x_1/x_0)$. Since $v_{[1:0]}(\frac{1}{(x_1/x_0)^2}) = -2$, we have a degree 2 pole at $[1:0]$. Hence $K = -2[1:0]$.

Example 1.4.14. Let $X \subset \mathbb{P}^n$ be any smooth hyper-surface with defining function $f \in k[x_0, ..., x_n]$. Let $i : X \to \mathbb{P}^n$ be the inclusion morphism. The cotangent bundle Ω_X is determined by

$$0 \to \mathcal{O}_X(-d) \xrightarrow{\varphi \mapsto d(f\cdot\varphi)} i^*\Omega_{\mathbb{P}^n} \xrightarrow{d\varphi \mapsto d(\varphi|X)} \Omega_X \to 0.$$

The canonical bundle is $\mathcal{O}_X(d - n - 1)$.

Theorem 1.4.15 (Adjunction Formula)**.** *Suppose X is a projective non-singular variety with a line bundle $\mathcal{L}$. Suppose $s \in \Gamma(X, \mathcal{L})$ is such that $Y = \{s = 0\}$ is a non-singular sub-variety of X, then $\omega_Y = (\omega_X \otimes \mathcal{L})|_Y$.*

Here, restricting by a curve Y (or more generally a divisor D), essentially acts by multiplying by the divisor of Y (or D).

Intersection Divisors

Definition. Suppose X is a curve embedded in $\mathbb{P}^n$, and suppose H is some hyper-surface of $\mathbb{P}^n$ not containing X. Define the *intersection divisor* $X \cdot H$ to be

$$X \cdot H = \sum_{P \in X \cap H} i(X, H; P) \cdot P.$$

Clearly, by Bezout's Lemma, we have

$$\deg(X \cdot H) = (\deg X)(\deg H).$$

We give a cleaner way to compute $i(X, H; P)$ if $H = Z(F)$ with F homogeneous of degree d. Since $X \not\subset H$, the polynomial F is not identically zero on X. Cover X by the open affine sets $X \cap U_i$. On $X \cap U_i$ we have by definition that $x_i \neq 0$, so that for each $P \in X \cap U_i$ we can consider the element $F/x_i^d \in \mathcal{O}_{X,P}$. Define $v_{X,P}(F) = v_P(F/x_i^d)$ where v_P is the valuation on $\mathcal{O}_{X,P}$. If $P \in U_i \cap U_j$ then $\frac{F/x_i^d}{F/x_j^d} = (x_j/x_i)^d \in \mathcal{O}_{X,P}^*$, so that $v_P(F/x_i^d) = v_P(F/x_j^d)$. Thus $v_{X,P}(F)$ is well-defined. We claim that $i(X, H; P) = v_{X,P}(F)$.

Indeed, if $\mathbb{P}^n$ is the ambient projective space and if we set $S = k[x_0, ..., x_n]$, then recall that $i(X, H; P)$ is the length of the $S_{\mathfrak{m}_P}$-module $(S/(\langle F \rangle + I(X)))_{\mathfrak{m}_P}$. WLOG suppose $P \in U_0$ with $P_0 = 1$. Since we work in U_0, we can set $y_i := x_i/x_0$ so that we actually need to find the length of the $k[y_1, ..., y_n]_{\langle y_i - P_i : 1 \leq i \leq r \rangle}$-module

$$((k[y_1, ..., y_n]) / \langle f, I(X \cap U_0) \rangle)_{\langle y_i - P_i : 1 \leq i \leq r \rangle}.$$

Here we denote $f = F(1, y_1, ..., y_n)$. This is equal to the length of the $\mathcal{O}_{X,P}$-module $\mathcal{O}_{X,P}/\langle f \rangle$. Since $\mathfrak{m}_{X,P}$ is a principal ideal and $f \in \mathfrak{m}_{X,P}^v \backslash \mathfrak{m}_{X,P}^{v+1}$ where $v := v_{X,P}(F) = v_P(f)$ (we set $\mathfrak{m}_{X,P}^0 := \mathfrak{O}_{X,P}$), it follows that $\langle f \rangle = \mathfrak{m}_{X,P}^v$. The length $i(X, H; P)$ of the $\mathcal{O}_{X,P}$-module $\mathcal{O}_{X,P}/\mathfrak{m}_{X,P}^v$ is precisely v, as required.

Lemma 1.4.16. *If H and H' are two hypersurfaces of the same degree, then $X \cdot H \sim X \cdot H'$.*

Proof. Suppose $H = Z(F)$ and $H' = Z(F')$ with F and F' are homogeneous of degree d and both not identically zero on X. Then, $F'/F \in K(X)$ so that for each $P \in X$ with $P_i \neq 0$, we have

$$v_P(F'/F) = v_P(F'/x_i^d) - v_P(F/x_i^d) = i(X, H'; P) - i(X, H; P).$$

Thus,

$$\operatorname{div}(F'/F) = \sum_{P \in X} v_P(F'/F)P = \sum_{P \in X} i(X, H'; P)P - \sum_{P \in X} i(X, H; P)P = X \cdot H' - X \cdot H.$$

As $F'/F \in K(X)$, it follows that $X \cdot H' \sim X \cdot H$. ☐

Corollary 1.4.16.1. *Let L be any hyperplane in X (i.e. hypersurface of degree 1), and let H be a hypersurface of degree d. Then $X \cdot H \sim dX \cdot L$.*

Definition. The *hyperplane section divisor* of a curve X is any divisor linearly equivalent to the intersection divisor for some hyperplane. Note that the hyperplane section divisor technically depends on the embedding $X \hookrightarrow \mathbb{P}^n$ chosen.

Proposition 1.4.17. *Let D be any hyperplane divisor of a curve $X \subset \mathbb{P}^n$. Then, $\mathcal{L}(D) \cong \mathcal{O}_X(1)$. Therefore $\mathcal{O}_X(1)$ is sometimes called the hyperplane sheaf, for obvious reasons.*

Proof. Let $D = X \cdot H$ where $H = Z(\ell)$, $\ell \in k[x_0, ..., x_n]$ linear. Since $X \not\subset H$, ℓ is a well-defined element of $S(X)$. Thus ℓ is a global section of $\mathcal{O}_X(1)$. Now, notice that $\text{div}\ell = X \cdot H = D$ by definition. It follows that $\mathcal{L}(D) = \mathcal{O}_X(1)$. □

1.4.4 Riemann-Roch and Applications

From here onwards, unless otherwise specified we assume all curves are non-singular projective curves over an algebraically closed field k.

Definition. For a projective variety X, the *arithmetic genus* $p_a(X)$ of X is $(-1)^{\dim X}(P_X(0) - 1)$ where P_X is the Hilbert polynomial of X. The *geometric genus* $p_g(X)$ of X is the dimension of the k-vector space $\Gamma(X, \omega_X)$, where ω_X is the canonical sheaf of X. It is clear that the geometric genus is a birational invariant.

If X is a non-singular curve, these two numbers coincide, and is called the *genus* $g = g(X)$ of X.

Remark 1.4.18. If $X \subset \mathbb{P}^2$ is an irreducible dimension 1 projective variety that has singular points, then the arithmetic and geometric genus do not coincide, but there is a formula linking them. Recall that if a curve X is blown up at a point, then the resulting blow up $\tilde{X}$ is birational to X. Thus the geometric genus coincides. On the other hand, if we blow up X at a singularity of multiplicity m to get $\tilde{X}$, then the arithmetic genus reduces by $\frac{1}{2}m(m-1)$. Hence,

$$p_g(X) = p_a(X) - \sum_i \frac{1}{2} m_i(m_i - 1)$$

where the sum is taken over all singularities with multiplicities m_i encountered upon repeatedly blowing up X at singularities until we reach a non-singular curve. In particular, if X is singular then $p_g(X) < p_a(X)$. If all of the singularities of X are ordinary double points, then $p_a(X) - p_g(X)$ is the number of such singularities.

Example 1.4.19. We compute the dimension $h^0(\mathcal{O}_X(n))$, where $\mathcal{O}_X(n)$ is a twisting sheaf. Notice that $\Gamma(X, \mathcal{O}_X(n)) = S(X)_n$, where $S(X)_n$ is the n'th degree graded piece of $S(X)$. Therefore, it follows by definition $h^0(\mathcal{O}_X(n)) = h_X(n)$ where h_X is the Hilbert function of X. The Hilbert polynomial of X is linear with constant term $1 - g$, and linear coefficient the degree of X.

Now, let ℓ be any linear function such that $X \not\subset Z(\ell)$, or equivalently, $\ell \notin I(X)$. Then, ℓ gives a regular global section of $\mathcal{O}_X(1)$. Notice that $\text{div}(\ell)$ is precisely the intersection divisor of X with $Z(\ell)$. Bezout's Theorem then implies that $\deg \text{div}(\ell) = \deg X$, and so the degree of any divisor corresponding to $\mathcal{O}_X(1)$ is simply $\deg X$. It then follows that the degree of any divisor corresponding to $\mathcal{O}_X(n)$ is $n \cdot \deg X$.

Proposition 1.4.20 (Genus-Degree Formula)**.** *If $X \subset \mathbb{P}^2$ is a curve of degree d, then* $g(X) = \frac{1}{2}(d-1)(d-2)$

Recall that for a divisor D, we have $\ell(D) = \dim_k \Gamma(X, \mathcal{O}_X(D))$.

Theorem 1.4.21 (Riemann-Roch)**.** *Suppose D is a divisor on a curve X. Then*

$$\ell(D) - \ell(K - D) = \deg D + 1 - g(X).$$

Corollary 1.4.21.1. *We have $\ell(K_X) = g(X)$ and $\deg K_X = 2g(X) - 2$.*

Proof. We have by definition

$$\ell(K_X) = \dim_k \Gamma(X, \mathcal{O}_X(K_X)) = \dim_k \Gamma(X, \omega_X) = p_g(X) = g(X).$$

Setting $D = K$ in the Riemann-Roch theorem, noting that $\ell(0) = \dim_k \Gamma(X, \mathcal{O}_X) = \dim kk = 1$, we have

$$g(X) - 1 = \deg K_X + 1 - g(X)$$

and hence $\deg K_X = 2g(X) - 2$. □

Recall the lemma that if $\deg(D) < 0$ then $\ell(D) = 0$, and moreover if $\ell(D) > 0$ and $\deg(D) = 0$ then $D \sim 0$. This motivates the following definition.

Definition. A divisor D is *special* if $\ell(K - D) > 0$ with *index of speciality* $\ell(K - D)$. Otherwise, D is *non-special.*

If $\deg D > 2g(X) - 2$, then D is non-special.

Example 1.4.22. Suppose X has genus g and degree d. Let D_n denote the divisor corresponding to $\mathcal{O}_X(n)$ for $n \geq 1$ (note that $D_{m+n} = mD_n$ for all $m, n \in \mathbb{N}$). Then, for all $n > (2g-2)/d$, we have $\deg D_n = n \cdot d > 2g-2$, and so D_n is non-special for all $n \geq 1$. By Riemann-Roch, we have

$$h_X(n) = h^0(\mathcal{O}_X(n)) = \deg D_n + 1 - g = dn + 1 - g = p_X(n)$$

(where h_X, p_X are the Hilbert function and the Hilbert polynomial of X respectively). Hence the Hilbert function and polynomial coincide for all $n > (2g-2)/d$. We also see more generally that $\ell(K_X - D_n) = h_X(n) - p_X(n)$ for all $n \in \mathbb{Z}$, and so $\ell(K_X - D_n) = h^0\big(\omega_X \otimes \mathcal{O}_X(-n)\big)$ is a measure of the deviation of p_X from h_X.

Example 1.4.23 (Fall 2019 Day 3)**.** Let $X \subset \mathbb{P}^3$ be a smooth curve of degree 5 and genus 2. We show that there exists a unique quadric surface Q and a line L (not necessarily unique) such that $X \cup L$ is the complete intersection of Q with a cubic surface.

Note first that X is not contained in any plane since it violates the degree-genus formula. Consider the restriction map $\rho : \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_X(2)$. Now, we know that $h_{\mathbb{P}^3}(2) = \binom{3+2}{2} = 10$, while by Riemann-Roch (noting that the degree of the divisor of $\mathcal{O}_X(2)$ is $10 > 2 = \deg K_X$) we have $h_X(2) = 10 + 1 - 2 = 9$. Hence ρ has a non-zero kernel, and so there exists a degree 2 homogeneous polynomial f such that $f|_X \equiv 0$, or equivalently, $X \subset Z(f)$. If f is reducible so that it has a linear factor, then X would be contained in a plane which is impossible. Thus $Q := Z(f)$ is a quadric surface containing X. If Q' is another quadric surface containing X, then X is an irreducible component of $Q \cap Q'$ of degree 5, contradicting Bezout's Theorem which says that the total degree of $Q \cap Q'$ is 4. Hence Q is unique.

Similarly, the restriction map $\mathcal{O}_{\mathbb{P}^3}(3) \to \mathcal{O}_X(3)$ has non-empty kernel since $h_{\mathbb{P}^3}(3) = \binom{3+3}{3} = 20$ while, by Riemann-Roch, we have $h_X(3) = 15 + 1 - 2 = 14$. Since any element in this kernel has to be irreducible as X is not contained in any plane, we can find a cubic surface S containing X. As both S and Q are closed irreducible projective varieties of dimension 2, they cannot be contained in one another. Thus $S \cap Q$ is a dimension 2 intersection with an irreducible component C. Finally, by Bezout's Theorem again, it follows that $S \cap Q = C \cup L$ for some line L.

Example 1.4.24 (Fall 2021 Day 2)**.** Suppose $C \subset \mathbb{P}^3$ is a smooth irreducible non-degenerate curve of degree 4. If $g(C) = 0$, show that C is contained in a quadric. If $g(C) = 1$, then show that C is the intersection of two quadrics. Show also that $g(C) \geq 2$ is impossible.

Consider the restriction map $\rho : \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_C(2)$. We have $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$. On the other hand, $\deg \mathcal{O}_C(2) = 2 \deg C = 8$ and $\deg K_C = 2g(C) - 2 \leq 0$ for $g(C) = 0, 1$. Hence, by Riemann-Roch, we have

$$h^0(\mathcal{O}_C(2)) = 8 + 1 - g(C) \leq 9.$$

Thus there exists a quadratic homogeneous polynomial f on $\mathbb{P}^3$ such that $f|_C \equiv 0$, or equivalently, $C \subset Z(f)$. If f were reducible, then C would be contained in a plane contradicting non-degeneracy. Hence C is contained in an irreducible quadric surface if $g(C) \leq 1$. If moreover $g(C) = 1$, then $h^0(\mathcal{O}_C(2)) = 8$, and so we can find linearly independent irreducible quadratic homogeneous polynomials f_1, f_2 such that $C \subset Z(f_1) \cap Z(f_2)$. By linear independence of f_1, f_2, it follows that $Z(f_1) \not\subset Z(f_2)$ and vice versa. By Bezout's theorem, noting that the degree of $Z(f_1) \cap Z(f_2)$ is $\deg f_1 \cdot \deg f_2 = 4 = \deg C$, C is the only irreducible component of $Z(f_1) \cap Z(f_2)$. Hence $C = Z(f_1) \cap Z(f_2)$ is the intersection of two irreducible quadric surfaces.

Consider $P \in C$, and consider the projection map $\pi_P : \mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2$. This is a rational map from C onto some plane curve C'. Now, if L is a line intersecting C', then the closure of $\pi_P^{-1}(L)$ is a hyperplane intersecting C, and the degree of this intersection is 4 (since $\deg C = 4$). Moreover, P lies on this intersection. By choosing P and the projection appropriately, it follows that a general line L intersects C' at three points (the fourth point of intersection of the closure of $\pi_P^{-1}(L)$ is P itself). Hence C' has degree 3, and thus has arithmetic genus 1 by the degree-genus formula. Since the geometric genus is at most the arithmetic genus, it follows that $p_g(C') \leq 1$.

We now claim that π_P is a birational map. Suppose that there exist two points $Q_1, Q_2 \in C'$ such that $\pi_P^{-1}(Q_1), \pi_P^{-1}(Q_2)$ are not singletons. Let L_i be the line joining P and Q_i; then $\pi_P^{-1}(Q_i) = L_i \cap C$. Now, as L_1, L_2 are two lines intersecting at a point (P), there exists a hyperplane H containing both L_1 and L_2. Then, $\pi_P^{-1}(Q_1), \pi_P^{-1}(Q_2) \subset H \cap C$ and $P \in H \cap C$. Thus, $H \cap C$ contains at least five points, namely P as well as all points in the fibre $\pi_P^{-1}(Q_i)$. However, Bezout's Theorem implies that $\deg(H \cap C) = \deg C = 4$, which is a contradiction. Hence there exists at most one fibre $\pi_P^{-1}(Q)$ of π_P that contains more than one point. Since the fibre is a zero dimensional closed subset of C, it has finitely many points. Hence, $\pi_P : C \backslash (\{P\}) \cup \pi_P^{-1}(Q) \to C' \backslash Q$ is an isomorphism, or in other words, π_P is a birational map from C to C'. However, the geometric genus is a birational invariant, and hence $g(C) = p_g(C) = p_g(C') \leq 1$. Therefore $g(C) \geq 2$ is impossible.

Example 1.4.25. Consider the non-singular curve $X = Z(F)$ on $\mathbb{P}^2$ where F is homogeneous of degree d and so that $d \neq 0$ in k. We compute the canonical sheaf and canonical divisor. Consider first on U_0. Let $y_1 = x_1/x_0$ and $y_2 = x_2/x_0$ and set $f(y_1, y_2) := F(1, y_1, y_2) = F(1, x_1/x_0, x_2/x_0)$. Define $V_i^0 = \{\frac{\partial f}{\partial x_1} \neq 0\}$. Since F is non-singular, it follows that V_1^0 and V_2^0 cover U_0. Consider the differential form $\omega = \frac{1}{\partial f/\partial y_2} dy_1$ on V_2^0. Since $f(y_1, y_2)$ is identically zero on $X \cap U_0$, it follows that

$$\frac{\partial f}{\partial y_1} dy_1 + \frac{\partial f}{\partial y_2} dy_2 = 0.$$

Hence we can define ω on V_1^0 as well, by

$$\omega = -\frac{1}{\partial f/\partial y_1} dy_2.$$

Notice now that

$$\frac{\partial f}{\partial y_1} = \frac{\partial F}{\partial x_1}\left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right) = \frac{1}{x_0^{d-1}} \frac{\partial F}{\partial x_1} \quad \text{and} \quad \frac{\partial f}{\partial y_2} = \frac{1}{x_0^{d-1}} \frac{\partial F}{\partial x_2}.$$

Summing up, we have defined a nowhere zero regular differential form ω on $U_0 \cap V_j$ by $\omega = (-1)^j \frac{x_0^{d-1}}{\partial F/\partial x_j} d(x_i/x_0)$ where $\{i, j\} = \{1, 2\}$, and where $V_j := \{\frac{\partial F}{\partial x_j} \neq 0\}$.

It remains to study ω on $\mathbb{P}^2 \backslash U_0$. Due to symmetry, we consider U_1 only. In this case, $d(x_0/x_1) = -\frac{x_0^2}{x_1^2} d(x_1/x_0)$, so that on $V_2 \cap U_1$

$$\omega = x_0^{d-1} \frac{1}{\partial F/\partial x_2} d(x_1/x_0) = -x_0^{d-3} x_1^2 \frac{1}{\partial F/\partial x_2} d(x_0/x_1) = -\left(\frac{x_0}{x_1}\right)^{d-3} \frac{x_1^{d-1}}{\partial F/\partial x_2} d(x_0/x_1).$$

Since on U_1

$$\frac{1}{x_1^{d-1}} \frac{\partial F}{\partial x_0} d(x_0/x_1) + \frac{1}{x_1^{d-1}} \frac{\partial F}{\partial x_2} d(x_2/x_1) = 0,$$

we also have on $U_1 \cap V_0$

$$\omega = \left(\frac{x_0}{x_1}\right)^{d-3} \frac{x_1^{d-1}}{\partial F/\partial x_0} d(x_2/x_1).$$

Thus, we see that on U_1 the form ω has a zero of order $d-3$ in $X \cap U_1 \backslash U_0$. Similarly ω is zero of order $d-3$ on $X \cap U_2 \backslash U_0$. Hence, the canonical divisor is

$$K_X = (d-3) \sum_{P \in X \backslash U_0} P.$$

In the language of intersection divisors, we thus that $K_X = (d-3)(X \cdot H_0)$ where $H_0 = Z(x_0)$. By Bezout's theorem, the degree of $X \cdot H_0$ is equal to $\deg X = d$. Therefore, we see that

$$\deg K_X = d(d-3) = (d-1)(d-2) - 2 = 2\binom{d-1}{2} - 2.$$

From the genus-degree formula, we have $g(X) = \binom{d-1}{2}$, so that $\deg K_X = 2g(X) - 2$ as expected.

Corollary 1.4.25.1. *For a curve X, we have $X \cong \mathbb{P}^1$ iff $g(X) = 0$.*

Proof. A simple computation checks that $g(\mathbb{P}^1) = 0$. Now suppose $g(X) = 0$. Then $\deg K = -2$. Take $D = P - Q$. Then, $\deg(K - D) < 0$ so that $\ell(K - D) = 0$. Riemann-Roch implies that $\ell(D) = 0 + 1 - g(X) = 1$. However, $\deg D = 0$, and so $\ell(D) > 0$ implies that $D \sim 0$, i.e. $P \sim Q$. As P and Q were arbitrarily distinct, it follows that any two points are linearly equivalent. Therefore, X is birational, and thus isomorphic to $\mathbb{P}^1$. $\square$

Corollary 1.4.25.2. *If D is an effective divisor on a curve X of genus g, then $\ell(D) \leq \deg D + 1$, where equality holds iff $D = 0$ or $g = 0$.*

Proof. If $D = 0$ equality is obvious from the fact that $\ell(0) = 1$. If $g = 0$, then $\deg K = -2$ so that any effective divisor D is non-special. In this case, we have $\ell(D) = \deg D + 1 - g = \deg D + 1$.

Now suppose D is a non-zero divisor on a curve X with genus 1. Then $\deg K = 0$ so that $\deg(K - D) < 0$ (as $D \neq 0$). Thus D is non-special, and hence once again we have $\ell(D) = \deg D$.

Finally, suppose D is a non-zero effective divisor on X with $g \geq 2$. We claim in this case that $\ell(K) > \ell(K - P)$ for any $P \in X$. If we establish this, then the monotonicity of ℓ implies that $\ell(K - D) < \ell(K) = g(X)$ for all non-zero effective divisors D, which would imply

$$\ell(D) = \deg D + 1 + \ell(K - D) - g < \deg D + 1.$$

Thus it remains to prove that $\ell(K-P) < \ell(K)$ for all $P \in X$. We have, by Riemann-Roch again,

$$\ell(K) - \ell(K-P) = g - \ell(K-P) = \deg P + 1 - \ell(P) = 2 - \ell(P) \geq 0.$$

If $\ell(P) = 2$, then there exists a non-constant function $f \in K(X)^*$ such that $\operatorname{div}(f) + P \geq 0$. Hence $v_P(f) \geq -1$, where $v_Q(f) \geq 0$ for all $Q \in X - P$. As f is non-constant, there exists $Q \in X$ such that $v_Q(f) \neq 0$. Since $\deg \operatorname{div}(f) = 0$, it follows that $v_P(f) = -1$ and that there exists $Q \in X$ such that $\operatorname{div} f = Q - P$. Hence $P \sim Q$. This however implies that $X \cong \mathbb{P}^1$ contradicting $g(X) \geq 2$. $\square$

Lemma-Definition. A curve X is *elliptic* if it satisfies any of the following three equivalent conditions: (i) $g(X) = 1$; (ii) $K_X \sim 0$; and (iii) $\omega_X \cong \mathcal{O}_X$.

Proof. Equivalence of (ii) and (iii) is obvious. For the equivalence of (i) and (ii) notice that $\deg K_X = 0$ but $\ell(K_X) = 1 > 0$ so that $K_X \sim 0$. $\square$

Proposition 1.4.26. *An elliptic curve X is an abelian group variety, i.e. X is an abelian group such that the operations are all morphisms. Moreover, as a group we have $X \cong \ker \deg \subset \operatorname{Cl}(X)$.*

Proof. Fix a point $P_0 \in X$. We claim that we have a bijection $X \to \ker \deg \subset \operatorname{Cl}(X)$ given by $P \mapsto P - P_0$. This map is clearly well-defined. It is injective since if $P - P_0 \sim Q - P_0$, then $P \sim Q$ so that $X \cong \mathbb{P}^1$. However, X is an elliptic curve so that $g(X) = 1 \neq 0 = g(\mathbb{P}^1)$. We now check surjectivity. Let $D \in \ker \deg$. Applying Riemann-Roch to the divisor $D + P_0$, and noting that $\deg(K - D - P_0) = -1 < 0$, we have

$$\ell(D + P_0) = 1 + 1 - g(X) = 1.$$

Thus there is an effective divisor in the linear equivalence class of $D + P_0$. However, an effective divisor of degree 1 is simply a point P. Hence $D + P_0 \sim P$ or $D \sim P - P_0$. $\square$

Definition. A curve X is said to be *hyper-elliptic* if $g(X) \geq 2$ and if there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree 2.

Proposition 1.4.27. *If X is a curve of genus 2, then X is hyper-elliptic.*

Proof. We claim that the canonical sheaf ω_X has no base-points. Indeed, since $\ell(K) = 2$ and $\deg K = 2$, the complete linear system $|K|$ is 1-dimensional, so that there is an effective divisor $D \sim K$. Now, $|K|$ is base-point free iff $\ell(K - P) = \ell(K) - 1 = 1$ for all $P \in X$. By the Riemann-Roch Theorem applied to P, we have

$$\ell(P) - \ell(K - P) = \deg P + 1 - g(X) = 0.$$

So that $\ell(K - P) = \ell(P)$. As $g(X) = 2$ and P is a non-zero effective divisor, we have $\ell(P) \leq \deg P = 1$. Hence $\ell(P) = 1$ so that $\ell(K - P) = 1$ as required.

Now K having no base points implies that ω_X is generated by global sections. Since $\dim_k |K| = 1$, it follows that there is a non-constant $f \in K(X)^*$ such that $K + \operatorname{div}(f) \geq 0$. We have the morphism $\varphi : X \to \mathbb{P}^1$ given by $\varphi(P) = [1 : f(P)]$ (where if f has a pole at P, then we set $\varphi(P) = [0 : 1]$). It is a finite morphism, since $\varphi^{-1}(a)$ is a dimension 0 sub-variety of X, so that $\varphi : X \to \mathbb{P}^1$ has finite fibres.

It remains to compute the degree of φ. Since $\operatorname{div}_0(f) \sim K$ so that $\deg \operatorname{div}_0(f) = \deg K = 2$, it follows from $\varphi^*([1 : 0]) = \operatorname{div}_0(f)$ that $\deg \varphi = 2$. Therefore X is hyper-elliptic. $\square$

Example 1.4.28 (Fall 2020 Day 1)**.** Suppose X is a smooth projective curve of genus g. Fix a point $P \in X$. We show there exists a non-constant rational function f which is regular everywhere except for a pole of order $\leq g + 1$ at P.

Let K be the canonical divisor for X. For a given divisor D, let $\mathcal{L}(D)$ denote the corresponding line bundle, and let $\ell(D) = \dim_k \Gamma(X, \mathcal{L}(D))$. Then $\ell(D) \geq 0$ for all divisors D. By the Riemann-Roch Theorem applied to the divisor $(g + 1)P$, it follows that

$$\ell\big((g+1)P\big) = \ell(K - (g+1)P) + (g+1) + 1 - g = \ell(K - (g+1)P) + 2 \geq 2.$$

In particular, this implies that there exists a non-constant rational function f on X with $\operatorname{div}(f) + (g+1) \cdot P \geq 0$, i.e. there exists a non-constant rational function f on X that is regular everywhere except for a pole of order at most $g + 1$ at P.

Example 1.4.29 (Fall 2018 Day 2)**.** Consider a curve $C \subset \mathbb{P}^2$. Then by adjunction $K_C = \mathcal{O}_C(d-3)$. If $d \geq 4$, suppose $\pi : C \to \mathbb{P}^1$ a degree 2 morphism (i.e. C hyperelliptic of degree 4). Then π can be considered as a rational function in $K(C)$, and (after possibly some non-trivial automorphism of $\mathbb{P}^1$) we have $h^0(\mathcal{O}_C(p+q)) \geq 2$ for any $p, q \in C$ in a general fibre of π. Since $\deg C \geq 4$, two points impose independent conditions on $|K_C|$ since $K_C = \mathcal{O}_C(d-3)$ where $d-3 \geq 1$, and so any polynomial on C with zeros at p and q must be divisible by the linear polynomial of the line connecting p and q, and this reduces the dimension by 2. This implies that $\ell(K_C - p - q) = \ell(K_C) - 2 = g - 2$. By Riemann-Roch however, we have $\ell(p+q) = 1$, a contradiction. If $\deg C \geq 5$, then $d - 3 \geq 2$, and so three points impose linearly independent conditions by considering the equation of the plane connecting these three points.

Example 1.4.30 (Spring 2018 Day 2)**.** Show that any genus 2 curve contains a divisor with positive degree not linearly equivalent to an effective divisor.

Riemann-Roch states that $\ell(D) - \ell(K_C - D) = \deg D - 1$. Suppose $\deg D \geq 2$; then RHS is positive and so $\ell(D) > 0$ and so D is (linearly equivalent to) an effective divisor. Which means to solve the question we need to consider degree 1 divisors. So we need to consider something like $p + q - r$. Suppose $D = p + q - r$ is linearly equivalent to an effective divisor, which since the degree is 1, must be a point. Thus we have $p + q - r - s \sim 0$. Thus there exists a morphism $\varphi : C \to \mathbb{P}^1$ of degree 2 with zeros at p and q and poles at r and s. Thus C must be hyper-elliptic (if not, we are done). Otherwise choose another point p' and consider $D' = p' + q - r$. If $D' \sim D$, then $p \sim p'$ and so $C \cong \mathbb{P}^1$ (contradicting $g_C = 2$). Thus D' is not linearly equivalent to D. If D' is again linearly equivalent to a point, then we choose a new point and keep going. Each such new non-effective divisor yields a new degree 2 morphisms $C \to \mathbb{P}^1$, and so we have infinitely many different degree 2 morphisms from C to $\mathbb{P}^1$. This is however is impossible because reasons.

1.4.5 Hurwitz-Riemann Formula and Applications

Recall the notion of ramification and ramification indices of a finite morphism of curves $f : X \to Y$ (again, here a curve means a dimension 1 irreducible closed non-singular subset of projective space, with k algebraically closed).

Definition. A morphism $f : X \to Y$ is *separable* if it induces a separable field extension $K(Y) \hookrightarrow K(X)$.

We construct a ramification divisor as follows. Suppose ω_X and ω_Y are the canonical line bundles on curves X and Y respectively, and suppose we have a finite separable morphism of curves $f : X \to Y$. Fix $P \in X$, and let $Q = f(P)$. Let u be a local parameter at P (i.e. an element of $\mathcal{O}_{X,P}$ that is a basis for $\mathfrak{m}_{X,P}/\mathfrak{m}_{X,P}^2$), and let t be a local parameter at Q. Then dt generates the free $\mathcal{O}_{Y,Q}$-module $\omega_{Y,Q}$ while du generates $\omega_{X,P}$.

Now, the morphism f yields a map $\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ for all $P \in X$. Since $\omega_{Y,f(P)}$ is a $\mathcal{O}_{Y,f(P)}$-module, we have the module

$$\omega_{Y,f(P)} \otimes_{\mathcal{O}_{Y,f(P)}} \mathcal{O}_{X,P}.$$

As f is finite separable, it is known that f induces an injection $f^* : \omega_{Y,Q} \otimes_{\mathcal{O}_{Y,Q}} \mathcal{O}_{X,P} \hookrightarrow \omega_{X,P}$. In particular, there exists a unique element $g \in \mathcal{O}_{X,P}$ such that $f^*dt = g \cdot du$. Here, we abuse notation slightly by writing 'dt' for the element $dt \otimes 1 \in \omega_{Y,Q} \otimes_{\mathcal{O}_{Y,Q}} \mathcal{O}_{X,P}$. This g is sometimes denoted by dt/du.

The *ramification divisor* of f is the effective divisor

$$R = \sum_{P \in X} v_P\left(\tfrac{dt}{du}\right) \cdot P.$$

It is known that $v_P(dt/du) \geq e_P - 1$ for all ramified points P, with equality iff f is tamely ramified at P. In most cases in the quals, we can assume tame ramification.

Locally around p on C, a finite separable morphism $f : C \to C'$ 'looks' like $z \mapsto z^{e_P}$.

Lemma 1.4.31. *Suppose $f : X \to Y$ is a finite morphism. Then, for any $Q \in Y$, we have*

$$\sum_{P \in f^{-1}(Q)} e_P = \deg f.$$

Proof. This is clear from the definition of f^* and the fact that $\deg(f^*Q) = \deg(f) \cdot \deg Q$. □

Theorem 1.4.32 (Hurwitz-Riemann Formula)**.** *Suppose $f : X \to Y$ is a finite separable morphism of curves. If K_X, K_Y are the canonical divisors of X and Y respectively, then*

$$K_X \sim f^*K_Y + R.$$

In particular, taking degrees, we have

$$2g(X) - 2 = (\deg f)\big(2g(Y) - 2\big) + \deg R.$$

If f has only tame ramification, then $\deg R = \sum_{P \in X}(e_P - 1)$.

Before looking at applications, let us briefly make note of the purely inseparable case, i.e. the case where $\mathrm{char}(k) = p$ and $K(X)$ is a purely inseparable field extension of $K(Y)$ (i.e. for any $\alpha \in K(X)$, there exists a power q of p such that $\alpha^q \in K(Y)$).

Proposition 1.4.33. *If $f : X \to Y$ is a purely inseparable finite morphism of curves, then $g(X) = g(Y)$ and f is a composition of k-linear* Frobenius morphisms.

*Here, given a curve X with $K(X)$ of characteristic p, consider the curve X_p corresponding to the field $K(X_p)$ of p'th roots of elements of $K(X)$ (in some fixes algebraic closure of $K(X)$). The finite morphism $F : X_p \to X$ corresponding to the inclusion $K(X) \hookrightarrow K(X)^{1/p} = K(X_p)$ of degree p is called the k-*linear Frobenius morphism. *It is known that the k-linear Frobenius morphism is ramified everywhere with ramification index p.*

Let us now list some applications of the Hurwitz-Riemann formula.

Example 1.4.34. If $f : X \to Y$ is any finite morphism of curves, then $g(X) \geq g(Y)$.

Example 1.4.35. The degree of the ramification divisor R is always an even number.

Example 1.4.36. Suppose C is the smooth projective curve associated to the affine plane curve $C_0 = Z(y^2 - x^4 - 1)$. Naively, we take $y^2z^2 = x^4 + z^4$ and use degree-genus formula to get genus of 3. This is wrong since $y^2z^2 = z^4 + y^4$ is singular at $[0 : 1 : 0]$. Thus, we use Riemann-Hurwitz Formula. We consider the more general case of C being associated to $y^2 = f(x)$, $f \in k[x]$.

We have the map $x : C_0 \to \mathbb{A}^1$, and upon taking projective closures, we get a map $f : C \to \mathbb{P}^1, [x : y : z] \mapsto [x : z]$, which is a double cover since a fibre over x_0 corresponds to solutions of $y^2 = f(x_0)$. This map f ramifies over $y = 0$, and if the curve looks like $y^2 = f(x)$, then $\deg f$ is added to ramification terms. Moreover, over '∞' (i.e. $[0 : 1]$), then in an open neighbourhood functions look like $k((t))[y]/(y^2 - f(t^{-1}))$. If $\deg f$ is even, then there is no field extension here, and so infinity is not ramified. If $\deg f$ is odd, then we need to take a degree 2 extension, and so there is ramification with $e_\infty = 2$. Hence $\deg R = \deg f$ if $\deg f$ is even, and $\deg R = \deg f + 1$ if $\deg f$ is odd. Hence, the genus is given by

$$2g(C) - 2 = -2 \cdot 2 + \deg f + \delta_{\deg f \equiv 1 \,(\mathrm{mod}\, 2)}.$$

Therefore

$$g(C) = \left\lceil \frac{\deg f}{2} \right\rceil - 1.$$

Example 1.4.37 (Spring 2013 Day 3)**.** Consider curve C_0 cut out by $y^3 - 3y = x^5$, and we have a map $x : C_0 \to \mathbb{A}^1$, and thus a map $\pi : C \to \mathbb{P}^1$ extending x. Part b of the question shows that $y^3 - 3y - a$ has repeated roots iff $a = \pm 2$, where if $a = 2$ then -1 is a repeated root and if $a = -2$ then 1 is a repeated root (both are multiplicity 2). Thus, the morphism π ramifies over affine points where $y^3 - 3y = x^5$ has repeated solutions in y, which is true whenever $x^5 = \pm 2$. Each such solution contributes $2 - 1 = 1$ (since repeated root has multiplicity 1). At infinity, we have $y^3 \sim x^5$ and so $y \sim x^{5/3}$, and so is ramified at ∞. More precisely, local functions on C around ∞ is $\mathbb{C}((t))[y]/\left\langle y^3 - 3y - t^{-5}\right\rangle$, where $y^3 - 3y - t^{-5}$ is irreducible element of $\mathbb{C}((t))[y]$ (this is elementary). Since this is a cubic irreducible, the field extension is a degree 3 extension, and so π is ramified over ∞ with multiplicity 3, and so contribution is 2. Hence the ramification divisor is

$$2 \cdot [\infty] + \sum_{x, x^5 = \pm 2} [x].$$

The degree of the sum is 10 and the infinity point gives 2, and so $\deg R = 12$. Riemann-Hurwitz yields $2g_C - 2 = 3(0 - 2) + 12$, and solving yields $g_C = 4$.

Theorem 1.4.38 (Classification of Curves of Genus 2)**.** *Suppose k is algebraically closed with characteristic $\neq 2$. Then, there is a 1-1 correspondence between isomorphism classes of curves of genus 2 over k, and triples of distinct elements $\beta_1, \beta_2, \beta_3$ ($\neq 0, 1$) of k modulo the action of the symmetric group S_6, where the action is given as follows:*

Suppose $\beta_1, \beta_2, \beta_3 \in k \backslash \{0, 1\}$ and σ is any element of S_6. Consider the sequence $(0, 1, \infty, \beta_1, \beta_2, \beta_3)$ in $\mathbb{P}^1$ (where we embed k as U_0 in $\mathbb{P}^1$, and denote by ∞ the point $[0 : 1]$). There exists a unique automorphism $\varphi \in PGL(1)$ of $\mathbb{P}^1$ such that $\varphi(\sigma 0) = 0, \varphi(\sigma 1) = 1$, and $\varphi(\sigma\infty) = \infty$. We then define the action of σ on $(\beta_1, \beta_2, \beta_3) \in (k \backslash \{0, 1\})^3$ by

$$\sigma(\beta_1, \beta_2, \beta_3) = (\varphi(\sigma\beta_1), \varphi(\sigma\beta_2), \varphi(\sigma\beta_3)).$$

Proof. Suppose X is a curve of genus 2. Then, the canonical divisor K_X defines a degree 2 finite morphism $f : X \to \mathbb{P}^1$. Since char$k \neq 2$, this morphism is separable. The Hurwitz-Riemann formula implies that

$$2 \cdot 2 - 2 = 2 \cdot (2 \cdot 0 - 2) + \deg R$$

where $R \in \mathrm{Div}(X)$ is the ramification divisor of f. Thus $\deg R = 6$. Suppose $P \in \mathrm{Supp}(R)$. Then, we have

$$1 < e_P \leq \sum_{P' \in f^{-1}(f(P))} e_{P'} = \deg f = 2.$$

Hence, each ramified point of f has ramification degree exactly 2, and moreover $f|_{\mathrm{Supp}(R)} : \mathrm{Supp}(R) \to \mathbb{P}^1$ is injective. Since $\deg R = 6$, there are exactly 6 ramified points of f, and moreover f maps them to an unordered subset of size 6 in $\mathbb{P}^1$. Since we can choose to order this subset in anyway we like, and since f is determined only up to an isomorphism of $\mathbb{P}^1$, it follows that to each genus 2 curve X we have assigned a unique element of $(k\backslash\{0,1\})^3/S_6$.

It thus remains to check that any such element $[\beta_1, \beta_2, \beta_3] \in (k\backslash\{0,1\})^3/S_6$ corresponds to a genus 2 curve X. Let $\alpha_i \in k$ $(1 \leq i \leq 6)$ be any 6 distinct points of k (where WLOG we shift the branch point ∞ to another point in k). Consider the degree 2 extension $K = k(t)[u]$ of $k(t) = K(\mathbb{P}^1)$ given by the quadratic equation $u^2 = \prod_{i=1}^{6}(t - \alpha_i)$. The inclusion $k(t) \hookrightarrow K$ yields a curve X with $K(X) = K$ and a degree 2 finite morphism $f : X \to \mathbb{P}^1$. It suffices to show that f is ramified at P iff $f(P) = \alpha_i \in \mathbb{P}^1$ for $1 \leq i \leq 6$, that $g(X) = 2$, and that f is precisely the map induced by $K_{\mathbb{P}^1}$.

- First consider $Q = \alpha_i \in \mathbb{P}^1$. Then, $t - \alpha_i$ is a local parameter at Q while $(t - \alpha_j)$ for $j \neq i$ are all units. Since f^* is just the inclusion map $k(t) \hookrightarrow k(t)[u]$, we have $f^*(t - \alpha_i) = \left(\prod_{j \neq i} \frac{1}{(t-\alpha_j)}\right) \cdot u^2$ in $\mathcal{O}_{X,P}$ for any $P \in f^{-1}(Q)$. Since $t - \alpha_i$ vanishes at Q, it follows that u^2 and thus u vanishes at all points in $f^{-1}(Q)$. In particular, u is a local parameter at each point in $f^{-1}(Q)$. Hence we see that Q is a branch point of f. As $\deg f = 2$, we also see that $f^{-1}(Q)$ is a singleton, so that we have at least 6 ramified points P_i of ramification index 2 with $f(P_i) = \alpha_i$.

  Now, one checks that the sub-ring $k[t,u]/\left\langle u^2 - p(t) \right\rangle$ is the integral closure of $k[t]$ in K, where $p(t) = \prod_{i=1}^{j}(t - \alpha_i)$. By definition, f^* is the injection $k(t) \hookrightarrow K$. Consider the DVR $\mathcal{O}_{\mathbb{P}^1,Q}$. This is embedded into K. If $Q \neq \alpha_i$ and $Q \neq \infty$, then $p \in \mathcal{O}^*_{X,Q}$, so that $u^2 \in f^*\mathcal{O}^*_{X,Q}$. It follows that $u \in \mathcal{O}^*_{X,P}$ for any $P \in f^{-1}(Q)$. Since the Galois group of K over $k(t)$ is $\mathbb{Z}_2$ generated by $u \mapsto -u$, any maximal ideal $\mathfrak{n}$ containing $f^*\mathfrak{m}_{\mathbb{P}^1,Q}$ does not contain u. Hence, under the Galois action $u \mapsto -u$, a maximal ideal $\mathfrak{n}$ gives another maximal ideal $\mathfrak{n}'$, and we see that $\mathfrak{m}_{\mathbb{P}^1,Q} = \mathfrak{n} \cap \mathfrak{n}'$. Since there is a 1-1 correspondence between DVRs of K and points on X, it follows that $f^{-1}(Q)$ has two points P_1 and P_2. Since $f^*Q = e_{P_1}P_1 + e_{P_2}P_2$ and $\deg f = 2$, we see that f is unramified at P_1 and P_2.

  Finally, for the point $Q = \infty$, we need to consider the integral closure of $k[\frac{1}{t}]$ in K and argue from there. The argument is exactly the same since $\alpha_i \neq \infty$ for all i

- The previous calculation shows that $\deg R = 6$. By the Riemann-Hurwitz formula, noting that $\deg f = 2$ and $g(\mathbb{P}^1) = 0$, it follows that $g(X) = 2$. It remains to show that this morphism f corresponds to the morphism induced by the canonical divisor. Let $h = f^*(t) \in K(X)$, so that $f : X \to \mathbb{P}^1$ is given by $P \mapsto h(P) \in \mathbb{P}^1$. We claim that $D := \mathrm{div}_0(h) \sim K_X$. Indeed, we have $f^*(1 \cdot 0) = f^*(\mathrm{div}(t)) = \mathrm{div}_0(h) = D$, so that $\deg D = 2$. Hence $\deg(K - D) = 0$. Next, we notice by Riemann-Roch that $\ell(D) - \ell(K - D) = 2 + 1 - 2 = 1$. Moreover, since both 1 and h^{-1} lie in $\Gamma(X, \mathcal{O}_X(D)) \subset K(X)$, it follows that $\ell(D) \geq 2$ so that $\ell(K - D) > 0$. Therefore $K \sim D$ as required.

□

1.4.6 Embedding Curves and Classifications

Suppose $\mathcal{L}$ is any line bundle on X generated by global sections $s_0, ..., s_n$. Since $\mathcal{L}$ is locally of rank 1, it follows that if $s_i(P) \notin \mathfrak{m}_{X,P}\mathcal{L}_P$ then $s_j(P)/s_i(P)$ gives a well-defined element of $\mathcal{O}_{X,P}/\mathfrak{m}_{X,P} \cong k$. Since the $s_i(P)$ generate the stalk $\mathcal{L}_P$ as an $\mathcal{O}_{X,P}$-module, there is at least one i such that $s_i(P) \notin \mathfrak{m}_{X,P}\mathcal{L}_P$. Thus, we get the map $\varphi : X \to \mathbb{P}^n_k$ given by

$$P \mapsto [s_0(P) \,(\mathrm{mod}\, \mathfrak{m}_{X,P}) : \cdots : s_n(P) \,(\mathrm{mod}\, \mathfrak{m}_{X,P})],$$

where by abuse of notation we write $s_i(P) \,(\mathrm{mod}\, \mathfrak{m}_{X,P}) \in k$ to be the image in $\mathcal{O}_{X,P}/\mathfrak{m}_{X,P}$ of the coefficient when $s_i(P)$ is written in terms of a fixed basis element of $\mathcal{L}_P$. Moreover, notice that in local affine coordinates this map φ induces a map $k[x_0, ..., x_n] \to \Gamma(X, \mathcal{L})$ $(x_i \mapsto s_i)$, and more generally, it induces the isomorphism $\mathcal{L} = \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$. This is one part of a more general important theorem.

Theorem 1.4.39. *Suppose X is a variety over k. Then, every invertible sheaf which is generated by global sections is of the form $\varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ for some morphism $\varphi : X \to \mathbb{P}^n$, where this morphism φ is described above.*

Moreover, if this morphism $\varphi : X \to \mathbb{P}^n$ is obtained from global sections $s_0, ..., s_n$ of $\mathcal{L}$, then the following three statements are equivalent:

1. *φ is a closed immersion (i.e. $\varphi(X)$ is closed in $\mathbb{P}^n$ and homeomorphic via φ to X, and such that $\varphi^\# : \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X$ is surjective);*
2. *each open set $X_i := \{P \in X : s_i(P) \notin \mathfrak{m}_{X,P}\mathcal{L}_P\}$ is affine and for each i the map $k[y_0, ..., y_n] \to \Gamma(X_i, \mathcal{O}_{X_i}), y_i \mapsto s_j/s_i$, is surjective;*
3. *if $V = \operatorname{span}_k(s_i) \subseteq \Gamma(X, \mathcal{L})$, then*
   - *elements of V* separate points, *i.e. for any distinct points $P, Q \in X$, there exists $s \in V$ such that $s \in \mathfrak{m}_{X,P}\mathcal{L}_P$ but $s \notin \mathfrak{m}_{X,Q}\mathcal{L}_Q$, and*
   - *elements of V* separate tangent vectors, *i.e. for each $P \in X$, the set $\{s \in V : s_P \in \mathfrak{m}_{X,P}\mathcal{L}_P\}$ spans the k-vector space $\mathfrak{m}_{X,P}\mathcal{L}_P/\mathfrak{m}^2_{X,P}\mathcal{L}_P$.*

Definition. $\mathcal{L}$ is *very ample* (over $\operatorname{Spec}(k)$) if there exists a closed immersion $i : X \hookrightarrow \mathbb{P}^n_k$ for some $n \in \mathbb{N}$ such that $\mathcal{L} \cong i^*(\mathcal{O}_{\mathbb{P}^n_k}(1))$. Thus $\mathcal{L}$ is very ample if it satisfies any of the three conditions above.

A line bundle $\mathcal{L}$ is said to be *ample* if there exists $m > 0$ such that $\mathcal{L}^m$ is very ample over $\operatorname{Spec}(k)$.

Note: Our definition of an ample line bundle is *not* the standard definition for an arbitrary line bundle over a scheme. However, over varieties it suffices.

Example 1.4.40. Let $X = \mathbb{P}^n_k$. For any $d > 0$, the line bundle $\mathcal{O}_X(d)$ is generated by the global sections of homogeneous monomials of degree d; indeed, any $P \in \mathbb{P}^n$ lies in $\{x_i \neq 0\}$ for some i, and for such an i we have $x_i^d \notin \mathfrak{m}_P\mathcal{O}_X(d)_P$. Moreover, notice that the corresponding morphism is simply the d-uple embedding.

Example 1.4.41. The line bundles $\mathcal{O}_{\mathbb{P}^n}(d)$ are very ample over $\operatorname{Spec}(k)$ for all $d > 1$, and in particular $\mathcal{O}_{\mathbb{P}^n}(d)$ are all ample. On the other hand, $\mathcal{O}_{\mathbb{P}^n}(d)$ for $d \leq 0$ are not ample.

Given an effective divisor D, we associate the line bundle $\mathcal{O}_X(D)$. Of course, very ample line bundles provide embeddings into projective space. We can thus ask for conditions on D so that $\mathcal{O}_X(D)$ is generated by global sections, is ample, etc.

Proposition 1.4.42. *Suppose D is a divisor on X, and let $\mathcal{L} = \mathcal{O}_X(D)$.*

1. *$\mathcal{L}$ is generated by global sections iff $|D|$ does not have any base points, where P is defined to be a* base point *of $|D|$ if for all $D' \in |D|$ we have $P \in \operatorname{Supp}(D')$.*
2. *$|D|$ is base-point free iff for all $P \in X$, we have $\ell(D - P) = \ell(D) - 1$.*
3. *$\mathcal{L}$ is very ample iff for every $P, Q \in X$ (not necessarily distinct), $\ell(D - P - Q) = \ell(D) - 2$.*
4. *$\mathcal{L}$ is ample iff $\deg D > 0$.*

Corollary 1.4.42.1. *Suppose D is a divisor on a curve X with genus g. If $\deg D \geq 2g$ then $|D|$ has no base points. If $\deg D \geq 2g + 1$, then $|D|$ is very ample.*

Proof. If $\deg D \geq 2g$, then $\deg D > \deg(D-P) \geq 2g-1 > \deg K$, and so by Riemann-Roch, $\ell(D) = \deg D+1-g$ and $\ell(D - P) = \deg D - 1 + 1 - g$, and thus $\ell(D) = \ell(D - P) + 1$. Similarly, if $\deg D \geq 2g + 1$, then $\deg D > \deg(D-P-Q) \geq 2g-1 > \deg K$, and by Riemann-Roch, $\ell(D) = \deg D+1-g$ and $\ell(D-P-Q) = \deg D-2+1-g$. Thus $\ell(D - P - Q) = \ell(D) - 2$. □

Corollary 1.4.42.2. *If X is an elliptic curve, then $|D|$ is very ample iff $\deg D \geq 3$. Moreover, if X is embedded into $\mathbb{P}^2$ via a very ample divisor of degree 3, then the image is a cubic curve in $\mathbb{P}^2$.*

Proof. If $\deg D \geq 3 = 2g(X) + 1$ then $|D|$ is very ample by Riemann-Roch. On the other hand, if $\deg D = 2$, then $|D|$ is base-point free, and D is non-special so that $\dim |D| = \ell(D) - 1 = 2 + 1 - 1 - 1 = 1$. However, it is not possible to embed an elliptic curve into $\mathbb{P}^1$: this is because a closed dimension 1 subset of $\mathbb{P}^1$ is $\mathbb{P}^1$ itself so that the embedding is in fact an isomorphism. This is impossible as $g(X) = 1$.

The second statement follows from the following proposition noting that $\deg D = 3$. □

Proposition 1.4.43. *If X is a curve and D a very ample divisor on X corresponding to the closed immersion $\varphi : X \to \mathbb{P}^n$, then the pull-back $\varphi*$ of any hyperplane divisor of $\varphi(X) \subset \mathbb{P}^n$ is linearly equivalent to D. In particular, the degree of the projective variety $\varphi(X)$ is $\deg D$.*

Example 1.4.44. Suppose X is an elliptic curve, and $P_0 \in X$ is fixed. Embed X into $\mathbb{P}^2$ via the very ample system $|3P_0|$. Then, the points P, Q, R are collinear in the image in $\mathbb{P}^2$ iff $P + Q + R \sim 3P_0$.

Let $\varphi : X \to \mathbb{P}^2$ be the embedding induced by $|3P_0|$. In other words, we can find linearly independent non-constant functions f, g with $\text{div}(f) + 3P_0 \geq 0$ and $\text{div}(g) + 3P_0 \geq 0$ (so that P_0 is the only pole of f and g), such that φ is the unique extension of the morphism

$$[1 : f : g] : X \backslash P_0 \to U_0 \subset \mathbb{P}^2$$

to X.

Now, $\varphi(P), \varphi(Q), \varphi(R)$ are collinear iff there exists a line L in $\mathbb{P}^2$ such that $L \cap \varphi(X) = \varphi\{P, Q, R\}$ iff there exists a line L such that $\varphi(X) \cdot L = \varphi(P) + \varphi(Q) + \varphi(R)$. Since this intersection divisor is linearly equivalent to any other hyperplane section divisor, we consider the line $L_0 = Z(x_0) \subset \mathbb{P}^2$. Since $1/f, 1/g \in \mathfrak{m}_{X,P_0}$, it follows that $\text{Supp}(\varphi(X) \cdot L_0) = \varphi(X) \cap L_0 = \{P_0\}$. However by Bezout's theorem, $\deg(X \cdot L_0) = 3$ since X is cubic. Hence $\varphi(X) \cdot L_0 = 3P_0$. Therefore P, Q, R are collinear in the image of X under φ iff $P + Q + R \sim 3P_0$.

Rational Points on Elliptic Curves

Let X be an elliptic curve over an algebraically closed field k. Suppose X can be embedded in $\mathbb{P}^2_k$ by the linear system $3P_0$ for some point $P_0 \in X$ in such a way so that the defining equation of the image of X is a polynomial in $k_0[x_0, x_1, x_2]$ where $k_0 \subset k$ is some sub-field, and such that the coordinates of P_0 under this embedding is also in $\mathbb{P}^2_{k_0} \subset \mathbb{P}^2_k$. Denote by $X(k_0)$ the set of points whose coordinates under this embedding are in $\mathbb{P}^2_{k_0}$. If $X(k_0) \neq \emptyset$, then we say that (X, P_0) is *defined over* k_0. It is easy to check that $X(k_0)$ is a subgroup of the abelian group X (recall that any elliptic curve with a fixed point has an induced group structure).

We say that a line bundle $\mathcal{L}$ on X is *defined over* k_0 if the corresponding divisor of $\mathcal{L}$ is linearly equivalent to a divisor in $\text{Div}(X(k_0))$.

Example 1.4.45. The elliptic curve $x^3 + y^3 = z^3$ in $\mathbb{P}^2_{\mathbb{C}}$ is defined over $\mathbb{Q}$.

Example 1.4.46 (Spring 2020 Day 2)**.** Suppose C is a non-singular irreducible curve of genus 1 defined over $\mathbb{Q}$, and suppose L and M are line bundles on C of degree 3 and 5 respectively, also defined over $\mathbb{Q}$. We show that $C(\mathbb{Q}) \neq \emptyset$.

Indeed, as L and M are defined over $\mathbb{Q}$, the line bundle $N = L^2 \otimes M^{-1}$ is also defined over $\mathbb{Q}$. Since $\deg L = 3$ and $\deg M = 5$, it follows that $\deg N = 1$. By Riemann-Roch applied to N, noting that N is non-special, it follows that $\ell(N) = 1 + 1 - 1 = 1$. Hence, there exists a non-zero global section $\sigma \in \Gamma(C, N)$. Since N is a degree 1 line bundle defined over $\mathbb{Q}$, the corresponding divisor $\text{div}_0 \sigma$ of N is a degree 1 divisor in $\text{Div}(C)$, i.e. $\text{div}_0 \sigma = P$ for some point $P \in C$. As $\ell(N) = 1$ so that σ spans the space of global sections of N, and since N is defined over $\mathbb{Q}$, it follows that $P \in C(\mathbb{Q})$. In particular, $C(\mathbb{Q}) \neq \emptyset$.

Chapter 2

Differential Geometry

Unless otherwise specified, we assume standard topology on $\mathbb{R}^n$ which is induced by the Euclidean norm $\|.\|$. Points $x \in \mathbb{R}^n$ are always written $x = (x_1, ..., x_n)$ without otherwise specifying. If we consider subsets S of some topological space X, then unless otherwise specified we assume S is endowed with subspace topology. Define $B^n(a,r) = \{p \in \mathbb{R}^n : \|p-a\| < r\}$, $\bar{B}^n(a,r) = \{p \in \mathbb{R}^n : \|p-a\| \leq r\}$, and $S^{n-1}(a,r) = \partial \bar{B}^n(a,r)$ (here ∂U stands for boundary of some subset U). The notation $B^n(r), \bar{B}^n(r), S^n(r)$ etc means a is taken to be the origin 0. The notation $B^n, \bar{B}^n, S^n$ means $a = 0$ and $r = 1$.

The notation $\hat{}$ always denotes omission.

For this chapter only, $\mathbb{Z}_n$ denotes $\mathbb{Z}/n\mathbb{Z}$.

2.1 Basic Notions

2.1.1 Basic Definitions

Manifolds

Recall that a map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is *smooth* if all possible partial derivatives exist. The *rank of F at p* is the rank of the $m \times n$ *Jacobian matrix* $(\frac{\partial F_i}{\partial x_j}(p))_{1 \leq i \leq m, 1 \leq j \leq n}$ where $F = (F_1, ..., F_m)$.

Definition. A *topological manifold of dimension n* is a Hausdorff second countable (i.e. has a countable basis) topological space M that is locally homeomorphic to $\mathbb{R}^n$, i.e. for any $p \in M$ there exists open neighbourhood $p \in U \subset M$ and a continuous map $\varphi : U \to \mathbb{R}^n$ that is a homeomorphism onto its image, which is open in $\mathbb{R}^n$. The pair (U, φ) is a *chart* on M. If $\varphi = (x_1, ..., x_n)$, then $x_1, ..., x_n$ are said to be coordinates on U (or around p). If φ sends p to the origin, then these coordinates are centred at p. An *atlas* is a collection $\{(U_i, \varphi_i)\}$ of charts such that $\{U_i\}$ is an open cover of M.

Two charts (U, φ) and (V, ψ) are *(smoothly) compatible* if the *transition maps*

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V) \quad \text{and} \quad \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

are smooth.

A topological manifold (equipped with a certain atlas) is said to be *smooth* or *differentiable* if every pair of charts in the given atlas is smoothly compatible.

From now on, *manifold* stands for smooth manifold equipped with some understood coordinate charts. If M is a manifold of dimension n, we sometimes write M^n to indicate that M has dimension n.

We list a bunch of important examples:

1. $\mathbb{R}^n$ is trivially a manifold.

2. Any finite-dimensional vector space V is a manifold, where charts are constructed by simply choosing a basis. The topology and smooth structure on V are independent of the choice of basis, since change of basis maps are linear and thus differentiable.

3. In particular, the space $M_{m \times n}(\mathbb{R})$ of all $m \times n$ matrices is a smooth manifold of dimension mn.

4. Open subsets U of smooth manifolds M^m are also manifolds of dimension m. Such U^m are called *open submanifolds*.

5. (*Spheres*) Consider the unit sphere S^n. There are two sets of charts that define the same atlas (thus equipping S^n with the same smooth structure):

   - (Hemispherical Coordinates) Let $U_i^+ = \{p \in S^n : p_i > 0\}$ and $U_i^- = \{p \in S : p_i < 0\}$. Define $\varphi_i^+ : U_i^+ \to B^n$ by $\varphi_i^+(p) = (p_1, ..., \hat{p}_i, ..., p_{n+1})$ for all $1 \le i \le n+1$, and similarly $\varphi_i^- : U_i^+ \to B^n$ by $\varphi_i^-(p) = (p_1, ..., \hat{p}_i, ..., p_{n+1})$ for all $1 \le i \le n+1$. Inverses are easily constructed by using $p_1^2 + \cdots + p_{n+1}^2 = 1$.
   - (Stereographic Projections from p) Fix a point $p \in S^n$. Define $V_N = S^n \backslash \{p\}$ and $V_S = S^n \backslash \{-p\}$. Let $\psi_N : V_N \to \mathbb{R}^n$ and $\psi_S : V_S \to \mathbb{R}^n$ be the stereographic projection from p and $-p$ respectively, where the stereographic projection of a point q from a point p in $\mathbb{R}^{n+1}$ is the unique intersection point of the line pq with the hyperplane of $\mathbb{R}^{n+1}$ perpendicular to p. If we choose $p = (0, ..., 0, 1)$, then
   $$\psi_N(x) = \frac{1}{1 - x_{n+1}}(x_1, ..., x_n) \quad \text{and } \psi_S(x) = \frac{1}{1 + x_{n+1}}(x_1, ..., x_n)$$
   with inverses
   $$\psi_N^{-1}(y) = \frac{1}{1 + \|y\|^2}\left(2y_1, ..., 2y_n, \|y\|^2 - 1\right) \quad \text{and } \psi_S(x) = \frac{1}{1 + \|y\|^2}\left(2y_1, ..., 2y_n, 1 - \|y\|^2\right).$$

6. (Projective Space) Consider $\mathbb{RP}^n := S^n/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on S^n by $x \mapsto -x$. Equivalently, we can consider $\mathbb{RP}^n := (\mathbb{R}^{n+1} - 0)/\mathbb{R}^*$. This is a smooth manifold with the quotient topology from S^n, i.e. if $\pi : S^n \to \mathbb{RP}^n$ is the projection, then $U \subset \mathbb{RP}^n$ is open iff $\pi^{-1}(U)$ is open. Since action of $\mathbb{Z}_2$ is discrete, π is an open map. The charts (U_i^+, φ_i^+) then induce the charts $(\tilde{U}_i = \pi(U_i^+), \tilde{\varphi}_i)$ where $\tilde{\varphi}_i([x]) \mapsto \varphi(x) = (x_1, ..., \hat{x}_i, ..., x_{n+1})$.

7. Products of manifolds are also manifolds with the product topology.

8. Since S^1 is a manifold, the n-fold product of S^1 is also a manifold, called the *n-dimensional torus T^n*. Thus $T^1 = S^1$, $T^2 = S^1 \times S^1$, etc.

9. (Grassmannians) Let $\text{Gr}(k, V)$ denote the space of all k-dimensional vector subspaces of the n-dimensional vector space V. We construct a topology and a smooth manifold structure on $\text{Gr}(k, V)$ by constructing charts (U_Q, φ_Q) where $Q \in \text{Gr}(n-k, V)$. Let $U_Q = \{P \in \text{Gr}(k, V) : P \cap Q = 0\}$. For any $P \in U_Q$, the map $\varphi_{P,Q} : U_Q \to L(P, Q) \cong \mathbb{R}^{k(n-k)}$ (where $L(P, Q)$ is the space of linear maps from P to Q) maps $R \in U_Q$ to the unique linear map $A \in L(P, Q)$ such that $R = (I_V + A)P$. By fixing bases and calculating from there, one checks that all transition maps are smooth.

Smooth Maps

Definition. A map $F : M^m \to N^n$ is *smooth at* $p \in M$ if for some (in fact, any) chart (U, φ) on M around p and for some (in fact, any) chart (V, ψ) on N around $F(p)$, where WLOG $U \subseteq F^{-1}(V)$, the map

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n$$

is a smooth map. A map $F : M \to N$ is *smooth* if it is smooth at all $p \in M$.

A map $F : M \to N$ is a *diffeomorphism* if F is bijective and both $F : M \to N$ and $F^{-1} : N \to M$ are smooth. In this case, M and N are said to be *diffeomorphic.*

A map $F : M \to N$ is a *local diffeomorphism* if for every $p \in M$, there exists open neighbourhood U of p in M and V of $F(p)$ in N such that $F|_U : U \to V$ is a diffeomorphism. It is clear that injective local diffeomorphisms are diffeomorphisms.

Dimension of a manifold is a diffeomorphic-invariant. Since smoothness is a local condition, various obvious glueing/restriction lemmas clearly hold. Smooth maps are clearly continuous, and composition of smooth maps is smooth.

Some examples of smooth maps:

1. Inclusion $I : S^n \to \mathbb{R}^{n+1}$ is smooth as can be checked in coordinate charts.

2. Projection map $\pi : S^n \to \mathbb{RP}^n$ is smooth.

Definition. If $F : M^m \to N^n$ is smooth, then the *rank of F at p* is the rank of the smooth map $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$, where φ is some chart around p and ψ is some chart around $F(p)$. The definition of rank is independent of choice of coordinate charts.

A smooth map F is said to have *constant rank k* if it has rank $k \le \min\{m, n\}$ at every point in M.

The following are three extremely important theorems in analysis/differential geometry. All three theorems are in fact equivalent to one another. Also, each theorem follows at once from the corresponding statement restricted to open submanifolds of Euclidean space.

Theorem 2.1.1 (Inverse Function Theorem)**.** *Suppose we have two manifolds M^n and N^n of the same dimension, and suppose $F : M \to N$ smooth. Suppose F has rank n at $p \in M$. Then there are connected open neighbourhoods U of p in M and V of $F(p)$ in N such that $F|_U : U \to V$ is a diffeomorphism.*

Theorem 2.1.2 (Constant Rank Theorem)**.** *Suppose $F : M^m \to N^n$ is a smooth map having constant rank k. Then, for any $p \in M$, there exist smooth coordinate charts (U, φ) on M centred at p and (V, ψ) on N centred at $F(p)$ with $F(U) \subset V$ such that for any $x \in \varphi(U)$,*

$$(\psi \circ F \circ \varphi^{-1})(x) = (x_1, ..., x_k, 0, ..., 0).$$

In other words, if F has constant rank k, then there exist smooth coordinates $(x_1, ..., x_m)$ centred at p and smooth coordinates $(y_1, ..., y_n)$ centred at $F(p)$ such that, in these coordinates,

$$F_i(x_1, ..., x_n) = (x_1, ..., x_k, 0, ..., 0).$$

Theorem 2.1.3 (Implicit Function Theorem)**.** *Suppose $F : U \subset \mathbb{R}^n \to \mathbb{R}^k$ is smooth. Fix $y_0 \in F(U)$, and consider the* level set *$M = F^{-1}(y_0) \subset U$. Suppose $x \in M$ is such that F has rank k (i.e. full rank) at x. By permuting variables, suppose the matrix $(\frac{\partial F_i}{\partial x_j})_{1\leq i \leq k, n-k+1 \leq j \leq n}$ is invertible. Then, there exists an open neighbourhood V of x in U, an open neighbourhood V' of $(x_1, ..., x_{n-k})$ in $\mathbb{R}^k$ and a smooth map $G = (g_1, ..., g_k) : V \to \mathbb{R}^k$ such that*

$$V \cap M = \{(x_1, ..., x_{n-k}, g_1(x_1, ..., x_{n-k}), ..., g_k(x_1, ..., x_{n-k})) : (x_1, ..., x_{n-k}) \in V'\}.$$

This latter set is the graph *of G.*

Definition. Suppose $F : M^m \to N^n$ is a smooth map. If $m \leq n$ and F has constant rank m on M, then F is said to be an *immersion*. If $m \geq n$ and F has constant rank n on M, then F is said to a *submersion*.

A smooth map $F : M^m \to N^n$ is an *embedding* if it is an injective immersion, and if $F : M \to F(M) \subset N$ is a homeomorphism with $F(M)$ endowed with the subspace topology on N.

Theorem 2.1.4. *Suppose $F : M \to N$ is a smooth map of constant rank.*

1. *If F is surjective, then it is a submersion.*
2. *If F is injective, then it is an immersion.*
3. *If F is bijective, then it is a diffeomorphism.*

Some properties/important facts on submersions:

1. If $F : M \to N$ is a submersion, then F is an open map (i.e. $F(U)$ open in N for all $U \subset M$ open), and is a quotient map on its image (i.e. $V \subset F(M) \overset{\text{open}}{\subset} N$ open iff $F^{-1}(V) \subset M$ open).
2. Suppose M, N, P are manifolds, and $\pi : M \to N$ a surjective submersion. Then, $F : N \to P$ is smooth iff $F \circ \pi : M \to P$ is smooth.
3. Suppose M, N, P are manifolds, and $\pi : M \to N$ a surjective submersion. Suppose $F : M \to P$ is a smooth map such that F is constant on all fibres of π. Then, there exists a unique smooth map $\tilde{F} : N \to P$ such that $F = \tilde{F} \circ \pi$.

   Note: the above two facts are extremely useful to study/construct maps from a manifold N to P, given a surjective submersion $M \to N$. In a very real sense, a surjective submersion $\pi : M \to N$ iff N is a quotient manifold of M. For instance, $S^n \to \mathbb{RP}^n$ is a surjective submersion, so that in order to construct smooth maps on projective space it suffices to construct (appropriate) smooth maps on S^n.
4. (Local Section Theorem) Suppose $\pi : M \to N$ smooth. Then, π is a smooth submersion iff for every $p \in M$, there exists an open subset $V \subset N$ and a smooth map $\sigma : V \to M$ such that $p \in \sigma(V)$ and $\pi \circ \sigma = \text{Id}_V$.
5. (Uniqueness of Smooth Quotients) Suppose M, N_1, N_2 are smooth manifolds and suppose $\pi_i : M \to N_i$ are surjective submersions that are constant on each others' fibres. Then, there exists a unique diffeomorphism $F : N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$.

Some properties/facts on immersions:

1. If $F : M \to N$ is an injective immersion and M is compact, then F is an embedding.
2. If $F : M \to N$ is an injective immersion that is either open or closed, then F is an embedding.
3. If $F : M \to N$ is an injective immersion and F is proper (i.e. F^{-1} takes compact sets to compact sets), then F is an embedding.
4. A smooth map is a local diffeomorphism iff it is both an immersion and a submersion.
5. If $\dim M = \dim N$ and $F : M \to N$ either a submersion or an immersion, then F is a local diffeomorphism.
6. Suppose $F : M \to N$ is smooth. Then F is an immersion iff for every $p \in M$, there exists an open neighbourhood $p \in U \subset M$ such that $F|_U : U \to N$ is an embedding.

Example 2.1.5. The map $\gamma : (-\pi, \pi) \to \mathbb{R}^2$, $\gamma(t) = (\sin 2t, \sin t)$ is an injective immersion, but not an embedding. The image of γ is a figure 8 in the plane.

Sub-Manifolds

Throughout, we fix a manifold M with dimension m. We have already encountered *open submanifolds*, which are open subsets of M.

Definition. An *immersed submanifold* is the image $F(M)$ of an injective immersion $F : M \to N$, where the topology and smooth structure on $F(M)$ are carried over from M by F. Note here that $F(M)$ *need not* have the subspace topology from N.

Definition. Suppose $N \subset M$, and suppose $n \leq m$. Then, N is a *regular (or embedded) submanifold of M of dimension n* if either one of the following two equivalent conditions hold:

1. N is the image of an embedding $F : N' \to M$ with $\dim N' = n$ (i.e. N is an immersed submanifold where the topology on N coincides with the subspace topology from M).

   Here, local charts on N are carried over from N' by F.

2. There exists a collection of smooth coordinate charts $\{(U_i, \varphi_i)\}$ on M such that $\{U_i \cap N\}$ covers N and such that each $(U_i, \varphi_i = (x_1^{(i)}, ..., x_m^{(i)}))$ is *adapted to* N, i.e.

$$U_i \cap N = \{y \in M : x_k^{(i)}(y) = 0 \ \ \forall \, n+1 \leq k \leq m\}.$$

   Here, local coordinates on N are given by $\{(U_i, (x_1^{(i)}, ..., x_n^{(i)}))\}$.

In either case, the topology on N is the subspace topology induced by M.

Definition. Suppose $F : M^m \to K^k$ is a smooth map with $m \geq k$. If $y \in K$ is such that F has rank k at every point $x \in F^{-1}(y)$, then y is said to be a *regular value of F*.

Basic properties of immersed and regular submanifolds $N^n \subset M^m$:

1. The inclusion $N \hookrightarrow M$ is an immersion if N is an immersed submanifold, and if N is a regular submanifold then the inclusion is an embedding.
2. If $F : M \to L$ is any smooth map, and if N is an immersed or regular submanifold of M, then $F|_N : N \to L$ is also smooth.
3. If $F : L \to M$ is any smooth map such that $F(L) \subset N$, and if N is embedded into M, then $F : L \to N$ is also smooth.
4. If $F : L \to M$ is any smooth map such that $F(L) \subset N$ where N is an immersed submanifold of M, and if $F : L \to N$ is continuous, then $F : L \to N$ is also smooth.
5. If $F : M^m \to K^k$ is smooth and y a regular value of F, then $N := F^{-1}(y)$ is a regular $(m-k)$-dimensional submanifold of M.

Example 2.1.6. The sphere S^n is an embedded submanifold of $\mathbb{R}^{n+1}$ (for instance, it is the fibre of the smooth map $\|.\|^2 : \mathbb{R}^{n+1} \to \mathbb{R}$ with 1 a regular value).

Open submanifolds may be regarded as embedded submanifolds as well.

Example 2.1.7. We can embed $\mathbb{RP}^2$ into $\mathbb{R}^4$ as follows. Consider the map $F : S^2 \to \mathbb{R}^4$ given by $F(x, y, z) = (x^2 - y^2, xy, yz, zx)$. One checks that F is a smooth map with $F(y) = F(x)$ iff $y = \pm x$. Since $\pi : S^2 \to \mathbb{RP}^2$ is a surjective submersion, it follows that the map $\tilde{F} : \mathbb{RP}^2 \to \mathbb{R}^4$ given by $F(x : y : z) = (x^2 - y^2, xy, yz, zx)$ is injective and smooth. One easily checks that F is of constant rank 2 on $\mathbb{RP}^2$, and that F is an embedding.

Example 2.1.8. If $F : M^m \to N^n$ is a smooth map, then the set $\Gamma(F) := \{(x, y) \in M \times N : x \in M, y = F(x)\}$ is a regular m-dimensional submanifold of the $m + n$-dimensional product manifold $M \times N$.

2.1.2 Smooth Functions and Partitions Of Unity

Suppose M a smooth manifold of dimension m.

Definition. A smooth map $f : M \to \mathbb{R}$ is called a *smooth function*. The $\mathbb{R}$-algebra of smooth functions on M is denoted by $C^\infty(M, \mathbb{R})$ or $\mathcal{C}^\infty(M, \mathbb{R})$.

Definition. Given $f \in C^\infty(M, \mathbb{R})$, the *support* $\operatorname{supp}(f)$ of f is the closure of the set $\{x \in M : f(x) \neq 0\}$.

A smooth function f is *compactly supported* if $\operatorname{supp}(f)$ is compact. A smooth function f is *supported in* $A \subset M$ if $\operatorname{supp}(f) \subset A$.

Definition. Given $U \subset M$ open and $A \subset U$ closed, a *bump function (on M) for A supported in U* is a smooth function f supported in U such that $0 \leq f \leq 1$ on M, and $f|_A \equiv 1$.

Bump functions allow us to extend local functions to global ones, i.e. suppose $f : C^\infty(U, \mathbb{R})$, and suppose $A \subset U$ is closed. If $\rho \in C^\infty(M, \mathbb{R})$ is supported in U, then the function $\tilde{f}(p) = \rho(p)f(p)$ for $p \in U$ and $\tilde{f}(p) = 0$ for $p \notin U$ is smooth. Moreover, if ρ is a bump function for A, then $\tilde{f}|_A = f|_A$.

Example 2.1.9. Fix $0 < r < R$. We construct a bump function for $\bar{B}^n(r)$ supported in $B^n(R)$.

First, set $\psi : \mathbb{R} \to \mathbb{R}$ to be the smooth function given by $\psi(t) = e^{-1/t^2}$ if $t > 0$, and $\psi(t) = 0$ for $t \leq 0$. We now define a bump function $\rho_{r,R} : \mathbb{R} \to \mathbb{R}$ for $(-\infty, r]$ whose support is $(-\infty, R]$, given by

$$\rho_1(t) = \frac{\psi(R - t)}{\psi(R - t) + \psi(t - r)}.$$

Finally, for any $n \in \mathbb{N}$, we can define the bump function $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ for $\bar{B}^n(r)$ supported in $B^n(R)$ by picking any $\epsilon \in (0, R - r)$, and setting

$$f(x) := \rho_{r^2, R^2 - \epsilon}(\|x\|^2).$$

Definition. Suppose $\mathcal{U}$ is an open cover on the manifold M. A *partition of unity subordinate to $\mathcal{U}$* is an assignment $\rho_U \in C^\infty(M, \mathbb{R})$ for each $U \in \mathcal{U}$ such that:

- $\{\operatorname{supp}(\rho_U) : U \in \mathcal{U}\}$ is *locally finite*, i.e. for each $p \in M$, there are only finitely many $U \in \mathcal{U}$ such that $p \in \operatorname{supp}(\rho_U)$;
- $0 \leq \rho_U \leq 1$, and ρ_U is supported in U for all $U \in \mathcal{U}$; and
- for each $p \in M$, we have $\sum_{U \in \mathcal{U}} \rho_U(p) = 1$.

Proposition 2.1.10. *For any manifold M and any open cover $\mathcal{U}$, there exists a partition of unity subordinate to $\mathcal{U}$.*

Partitions of unity allow us to stitch local functions to form global ones, to create bump functions for arbitrary closed subsets A, and so on. It also allows us to embed any compact manifold M into $\mathbb{R}^N$ for some $N \in \mathbb{N}$; this result is a weak version of *Whitney's Theorem*. The embedding is done by taking a finite cover of M by n local charts $(U_i, \varphi_i : U_i \to B^m(3))$, and then lifting a bump function from $B^m(3)$ to a bump function g_i on M. This allows us to define a function $F : M \to \mathbb{R}^{(m+1)n} = (\mathbb{R}^m)^n \times \mathbb{R}^n$ by

$$F(p) = (\varphi_1(p), ..., \varphi_n(p), g_1(p), ..., g_n(p)).$$

2.2 Differential Calculus on Manifolds

2.2.1 Tangent Spaces and Differentials

Definition. Suppose M^m is a manifold and $p \in M$. We give two different definitions of the *tangent space $T_p(M)$ at p.*

- Consider the $\mathbb{R}$-vector space $\mathcal{C}_p$ of curves at p, which are smooth maps $c : (-\epsilon, \epsilon) \to M$ for some $\epsilon > 0$, where $c(0) = p$. Define an equivalence relation on $\mathcal{C}_p$ where $c \sim c'$ iff for some (in fact, all) charts (U, ϕ) with $p \in U$, we have $(\phi \circ c)'(0) = (\phi \circ c')'(0)$ as vectors in $\mathbb{R}^m$ (where $\phi \circ c : (-\epsilon, \epsilon) \to \phi(U) \subset \mathbb{R}^m$). The tangent space $T_p(M)$ is simply the $\mathbb{R}$-vector space $\mathcal{C}_p/\sim$.
- Consider the $\mathbb{R}$-algebra $\mathcal{G}_p$ of *germs of functions at* p, i.e. the set of pairs $(U, f \in C^\infty(U, \mathbb{R}))$ with U open and $p \in U$, modulo the equivalence relation $(U, f) \sim (V, g)$ iff $f|_W = g|_W$ for some open $W \subset U \cap V$ with $p \in W$. We usually write $[f]$ for the equivalence class of germs of f (where we tacitly assume U). Let $\mathrm{Der}_p(\mathcal{G}_p, \mathbb{R})$ be the space of all $\mathbb{R}$-linear *local derivations* on $\mathcal{G}_p$, i.e. $\mathbb{R}$-linear maps $D : \mathcal{G}_p \to \mathbb{R}$ such that $D([fg]) = f(p)D([g]) + g(p)D([f])$. The tangent space $T_p(M)$ is simply $\mathrm{Der}_p(\mathcal{G}_p, \mathbb{R})$.

These two definitions of the tangent space are equivalent. More precisely, we have the isomorphism of vector spaces $\mathcal{C}_p/\sim \xrightarrow{\sim} \mathrm{Der}_p(\mathcal{G}_p, \mathbb{R})$ given by $[c] \mapsto D_{[c]}$, where $D_{[c]}$ is the *directional derivative in the direction of* c, defined by $D_{[c]}([f]) := (f \circ c)'(0) \in \mathbb{R}$ (this is independent of the choice of representative).

Clearly $T_p(\mathbb{R}^m) = \mathbb{R}^m$.

Definition. Suppose $F : M \to N$ is a smooth map. The *differential of* F *at* $p \in M$ is a linear map $d_pF : T_p(M) \to T_{F(p)}(N)$, given by the following.

- If we take $T_p(M) = \mathcal{C}_p/\sim$, then d_pF is the map $d_pF([c]) := [F \circ c]$.
- If we take $T_p(M) = \mathrm{Der}_p(\mathcal{G}_p, \mathbb{R})$, then d_pF takes the derivation $D : \mathcal{G}_p(M) \to \mathbb{R}$ to the derivation $d_pF(D) : \mathcal{G}_{F(p)}(N) \to \mathbb{R}$ given by $(d_pF(D))([f]) := D([f \circ F])$.

In both cases, one easily checks that the maps are well-defined, i.e. independent of choice of representative.

We list some important results about tangent spaces and differentials of maps that are immediate from the above definitions.

1. If $F : M \to N$ is such that $d_pF = 0$ for all $p \in M$, and if M is connected, then F is a constant map.
2. If $\varphi : U \to \mathbb{R}^m$ is a local chart at p, then $d_p\varphi : T_p(M) \to T_p(\mathbb{R}^m) \cong \mathbb{R}^m$ is an isomorphism. In particular, $\dim_{\mathbb{R}} T_p(M) = \dim M$.
3. (Chain Rule) If $F : M \to N$ and $G : N \to L$, then
$$d_p(G \circ F) = (d_{F(p)}G) \circ (d_pF) : T_p(M) \to T_{F(p)}(N) \to T_{G \circ F(p)}(L).$$
4. If $F : M \to N$ is a diffeomorphism, then d_pF is an isomorphism and $d_p(F^{-1}) = (d_pF)^{-1}$.
5. If M and N are manifolds, and if $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are the projections onto the first and second factor respectively, then for any $(p, q) \in M \times N$ the map
$$T_{(p,q)}(M \times N) \to T_pM \oplus T_qN, \quad v \mapsto d_p\pi_1(v) + d_p\pi_2(v)$$
is a vector space isomorphism.

Example 2.2.1. If $F = \mathrm{Id}_M : M \to M$, then $d_pF = \mathrm{Id}_{T_p(M)}$ for all $p \in M$.

We now study the differential of a smooth map in terms of local coordinates. Fix $p \in M^m$, and let (U, φ) be a local chart at p. Let $\varphi = (x_1, ..., x_m)$ be local coordinates.

- Suppose e_j is the j'th standard basis vector in $\mathbb{R}^m$. Then, we have the vector $d_p\varphi^{-1}(e_j) \in T_pM$, usually denoted by $\frac{\partial}{\partial x_j}|_p$ (*just notation!*).
    1. As an element of $\mathcal{C}_p(M)/\sim$, it is simply the (equivalence class of the) curve $t \mapsto \varphi^{-1}(\varphi(p) + te_j)$ (take domain small enough to fit in $\varphi(U)$ of course).
    2. As an element of $\mathrm{Der}_p(\mathcal{G}_p, \mathbb{R})$, it is the derivation $\frac{\partial}{\partial x_j}|_p([f]) := \frac{\partial f \circ \varphi^{-1}}{\partial x_j}(p)$.

  The latter point explains the notation, since $f \circ \varphi^{-1}$ is a function from $\varphi(U) \subset \mathbb{R}^m$ to $\mathbb{R}$.

  Clearly, $\{\frac{\partial}{\partial x_j}|_p\}$ is a basis for T_pM.

- Suppose $(U, \phi = (z_1, ..., z_m))$ are another set of local coordinates at p (WLOG, keep the open neighbourhood same by making smaller if necessary). The map $\phi \circ \varphi^{-1} : \varphi(U) \to \phi(U)$ allows us to think of the z_i as functions $z_i = z_i(x_1, ..., x_m)$ of the x_js. With this abuse of notation, we have the change of coordinates

$$\left(\frac{\partial}{\partial x_1}|_p, \cdots, \frac{\partial}{\partial x_m}|_p\right) = \left(\frac{\partial}{\partial z_1}|_p, \cdots, \frac{\partial}{\partial z_m}|_p\right) \cdot \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \cdots & \frac{\partial z_1}{\partial x_m} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \cdots & \frac{\partial z_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z_m}{\partial x_1} & \frac{\partial z_m}{\partial x_2} & \cdots & \frac{\partial z_m}{\partial x_m} \end{pmatrix} |_{\varphi(p)}.$$

The above $m \times m$ matrix is simply the Jacobian matrix (evaluated at $\varphi(p)$) of the diffeomorphism $\phi \circ \varphi^{-1} : \varphi(U) \to \phi(U)$. Writing the above system as

$$\frac{\partial}{\partial x_i}|_p = \sum_{j=1}^{m} \frac{\partial z_j}{\partial x_i}(\varphi(p)) \frac{\partial}{\partial z_j}|_p;$$

in this reformulation this is simply the change of variables formula.

- Suppose $F : M^m \to N^n$ is smooth, and suppose $p \in M$ with local coordinates $(U, \varphi = (x_1, ..., x_m))$. Let $(V, \psi = (y_1, ..., y_n))$ be local coordinates at $F(p)$. Define the smooth functions $f_j := \psi \circ F \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to \mathbb{R}$ for $1 \leq j \leq n$; these f_j are functions of $x_1, ..., x_m$. The chain rule yields

$$d_pF\left(\frac{\partial}{\partial x_i}|_p\right) = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i}|_{\varphi(p)} \cdot \frac{\partial}{\partial y_j}|_{F(p)},$$

or in matrix form as

$$d_pF\left(\frac{\partial}{\partial x_1}|_p, \cdots, \frac{\partial}{\partial x_m}|_p\right) = \left(\frac{\partial}{\partial y_1}|_{F(p)}, \cdots, \frac{\partial}{\partial y_n}|_{F(p)}\right) \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix} |_{\varphi(p)}.$$

In particular, the previous change of coordinates formula is simply the above formula applied to $F = \mathrm{Id}_U$.

From the above discussion in local coordinates, we see the following facts.

1. The rank of the smooth map $F : M \to N$ at p is simply the rank of the linear map $d_pF : T_pM \to T_{F(p)}N$.

2. In particular, F is an immersion iff $d_pF : T_pM \to T_{F(p)}N$ is injective for all $p \in M$; F is a submersion iff $d_pF : T_pM \to T_{F(p)}N$ is surjective for all $p \in M$; F is a local diffeomorphism iff $d_pF : T_pM \to T_{F(p)}N$ is a vector space isomorphism for all $p \in M$.

3. If $d_pF : T_pM \to T_{F(p)}N$ is injective at p, then there exists an open neighbourhood $U \subset M$ of p such that $F|_U : U \to N$ is an immersion. Similarly, if $d_pF : T_pM \to T_{F(p)}N$ is surjective at p, then there exists an open neighbourhood $U \subset M$ of p such that $F|_U : U \to N$ is a submersion.

   This follows because in local charts, the map taking $p \in M$ to the $n \times m$ Jacobian matrix of F (in fixed local coordinates) is a smooth map.

4. If $F : M \to N$ is a local diffeomorphism, then the rank of $G : N \to L$ at $q \in N$ is the same as the rank of $G \circ F$ at any $p \in F^{-1}(q)$. Similarly, the rank of $G : L \to M$ at $p \in L$ is the same as the rank of $F \circ G$ at p.

2.2.2 Tangent Bundle

Definition. Consider a manifold M^n. Set $TM = \bigsqcup_{p \in M} T_pM$. Let $\pi : TM \to M$ be the map sending vectors in T_pM to p. For any local coordinates $(U, (x_1, ..., x_n))$ on M, we have the injection $\pi^{-1}(U) \to \mathbb{R}^{2n}$ sending p to $(x_1(p), ..., x_n(p), \frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p)$. The collection of these local maps defines a topology and smooth structure on TM, making it a smooth manifold. This smooth manifold TM is called the *tangent bundle* of M. This smooth structure makes π a submersion.

For instance, if M is a manifold with a global coordinate chart, then $TM \cong M \times \mathbb{R}^n$.

Lemma-Definition. If $F : M \to N$ is a smooth map, then the differential maps $d_pF : T_pM \to T_{F(p)}N$ induce a smooth map $F_* : TM \to TN$ given by $F_*(p, v_p) = d_pF(v_p)$. Moreover, if F is an immersion then F_* is injective, thus giving a canonical isomorphism from T_pM to a vector subspace of $T_{F(p)}N$ for all $p \in M$ such that this subspace 'varies smoothly'.

Proposition 2.2.2. *Suppose N is an embedded submanifold of M. We have the following characterizations of T_pN as a subspace of T_pM.*

1. *If $F : M \to L$ is a map such that $N = F^{-1}(q)$ for some regular point $q \in L$, then $T_pN = \ker(d_pF : T_pM \to T_qL)$ for any $p \in N$.*

   More generally, if there exists an open subset U of M and a smooth map $F : U \to L$ such that $N \cap U = F^{-1}(q)$ for some regular point $q \in L$, then for any $p \in N \cap U$ we still have $T_pN = \ker d_pF$.

2. *We have $T_pN = \{v \in T_pM : vf = 0 \text{ whenever } f \in C^\infty(M) \text{ such that } f|_N = 0\}$.*

If N is only an immersed submanifold of M (not necessarily embedded), then $[c] \in T_pN$ iff there exists $c' \sim c$ such that $\mathrm{Im}(c') \subset N$.

If $\iota : N \to M$ is an immersion, then $T_pN = \mathrm{Im}(d_p\iota)$ for all $p \in N$.

Corollary 2.2.2.1. *If $\iota : N \to M$ is an immersion, then $d\iota : TN \to TM$ is an immersion. If ι is an embedding, then $d\iota$ is an embedding.*

Proof. Since ι is an immersion, $I := d\iota$ is injective. Fix coordinate patches $U, (x_1, ..., x_n)$ at $p \in N$ and $V, (y_1, ..., y_m)$ at $\iota(p) \in M$. Then TN and TM have local trivializations $U \times \mathbb{R}^n$ and $V \times \mathbb{R}^m$, where the local trivializations also induce local coordinates $(x_1, ..., x_n, v_1, ..., v_n)$ on TN and $(y_1, ..., y_m, w_1, ..., w_m)$ on TM (here, $v_i := \frac{\partial}{\partial x_i}$ and $w_i = \frac{\partial}{\partial y_i}$). The map $I : TN \to TM$ in these local trivializations is simply the map $U \times \mathbb{R}^n \to V \times \mathbb{R}^m$ given by $I(q, v_q) = (\iota(q), D\iota(q) \cdot v_q)$, where $D\iota(q)$ denotes the Jacobian matrix of ι at $q \in U$ in the local coordinates on U and V. In such a case, we see that

$$DI(q, v_q) = \begin{pmatrix} D\iota(q) & O \\ * & D\iota(q) \end{pmatrix}.$$

The bottom right block follows since $w_i \circ I$ is simply the i'th coordinate of $D\iota(q)v_q$. The top right block follows since $y_i \circ I$ does not actually depend on the v_j, but only on the x_j. Finally, the top left block follows since $y_i \circ I = y_i \circ \iota$ so that $\left(\frac{\partial y_i \circ I}{\partial x_j}\right)_{i,j} = \left(\frac{\partial y_i \circ \iota}{\partial x_j}\right)_{i,j} = D\iota$.

However, ι being an immersion implies that $D\iota$ has full rank, so that the matrix $DI(q, v_q)$ also has full rank. Therefore I is an immersion.

Now suppose ι is an embedding, so that we may identify N as a regular submanifold of M, it follows that we can pick the coordinates (x_i) and (y_j) such that $\iota(x_1, ..., x_n) = (y_1, ..., y_n, 0, ..., 0)$, so that $w_i = d\iota(v_i)$. Thus $D\iota(q) = \begin{pmatrix} I_n \\ O_{m-n,n} \end{pmatrix}$ for all $q \in U$, and thus

$$I(x_1, ..., x_n, v_1, ..., v_n) = (x_1, ..., x_n, \underbrace{0, ..., 0}_{m-n \text{ zeros}}, v_1, ..., v_n, \underbrace{0, ..., 0}_{m-n \text{ zeros}}).$$

It follows that $\mathrm{Im}(I)$ is a regular submanifold of TM, i.e. $I = d\iota$ is an embedding. $\square$

Example 2.2.3 (Spring 2020 Day 1)**.** Consider $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 - 1$. It is easy to see that f has constant rank 1 on $f^{-1}(0)$, so that as a consequence of the constant rank theorem $M = f^{-1}(0)$ is a two-dimensional embedded submanifold of $\mathbb{R}^3$.

For $a, b, c \in \mathbb{R}$, let $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$ ($\in \mathfrak{X}^1(\mathbb{R}^3)$). Let us find the values of a, b, c for which X tangent to M at $p = (1, 0, 1)$. Since M is defined globally as the fibre at a regular point of f, it follows that $T_pM = \ker d_pf$. Since $d_pf : T_p\mathbb{R}^3 = \mathbb{R}^3 \to T_p\mathbb{R} = \mathbb{R}$ takes $\frac{\partial}{\partial x}$ to $2x\frac{d}{dt}|_p = 2\frac{d}{dt}|_p$, $\frac{\partial}{\partial y}$ to $2y\frac{d}{dt}|_p = 0$, and $\frac{\partial}{\partial z}$ to 0, it follows that $\ker d_pf = \mathrm{span}\{\frac{\partial}{\partial y}|_p, \frac{\partial}{\partial z}|_p\}$. Hence, X_p is tangent to M iff $X_p \in \mathrm{span}\{\frac{\partial}{\partial y}|_p, \frac{\partial}{\partial z}|_p\}$ iff $a = 0$ (b, c are free variables).

2.2.3 Vector Fields

Basic Definitions

Definition. A *vector field* is any smooth map $X : M \to TM$ such that $X(p) \in T_p(M)$. Equivalently, if $\pi : TM \to M$ is the projection submersion, then a vector field is any smooth map $X : M \to TM$ such that $\pi \circ X = \mathrm{Id}_M$. In sheaf-theoretic language, it is simply a (smooth) section of the tangent bundle.

We usually denote $X(p)$ by X_p. The $\mathbb{R}$-vector space of vector fields on M is denoted by $\mathfrak{X}^1(M)$. More generally, we can define the space $\mathfrak{X}^1(U)$ of vector fields on U (smooth sections of $U \to \pi^{-1}(U)$) for any open set $U \subset M$. If $V \subset U$, then we have the obvious restriction maps $\mathfrak{X}^1(U) \to \mathfrak{X}^1(V)$.

If $U, (x_1, ..., x_m)$ are local coordinates, then the $\frac{\partial}{\partial x_i}$ are vector fields in $\mathfrak{X}^1(U)$. Moreover, these vector fields span $\mathfrak{X}^1(U)$ since at points they span the respective tangent spaces.

Suppose $\gamma : (a, b) \subset \mathbb{R} \to M$ is a smooth curve such that $0 \in (a, b)$ and $\gamma(0) = p$. Since $T_t\mathbb{R} = \mathbb{R}$ is spanned by the global vector field $\frac{d}{dt}$, we have the element

$$\gamma'(t_0) := d_{t_0}\gamma(\frac{d}{dt}) \in T_{\gamma(t_0)}M.$$

Definition. An *integral curve* of X at p is a smooth curve $\gamma : (a, b) \to M$ such that $0 \in (a, b)$, $\gamma(0) = p$, and $\gamma'(t) = X_{\gamma(t)}$.

The *maximal integral curve through* p is an integral curve $\gamma : (a, b) \to M$ that is maximal in the sense that for any other integral curve $\eta : (c, d) \to M$ at p with $0 \in (c, d)$ and $\eta(0) = p$, we have $(c, d) \subset (a, b)$ and $\eta = \gamma|_{(a,b)}$.

Proposition 2.2.4. *For any $X \in \mathfrak{X}^1(M)$ and any $p \in M$, a maximal integral curve through p exists and is unique. Moreover, if $\gamma : (a, b) \to M$ is the maximal integral curve through p, and if $q = \gamma(t_0) \in \gamma((a, b))$, then the maximal integral curve through q is the curve $\delta : (a-t_0, b-t_0) \to M$, $\delta(t) = \gamma(t+t_0)$ is the maximal integral curve through q.*

Finding (maximal) integral curves is simply attempting to solve systems of ODEs.

Definition. Let $p \in M$ and $X \in \mathfrak{X}^1(M)$. Let $\gamma_p : I_p \to M$ ($0 \in I_p$, $\gamma_p(0)$) be the maximal integral curve through p, where I_p is the domain of definition of γ_p. For each $t \in \mathbb{R}$, set $D^t_X := \{p \in M : t \in I_p\}$ be the set of points $p \in M$ such that the maximal integral curve through p is defined at t. For instance, $D^0_X = M$.

A vector field X such that $D^t_X = M$ for all $t \in \mathbb{R}$, i.e. such that the maximal integral curve through any point is defined on $\mathbb{R}$, is said to be *complete*.

The *time t-flow of X* is the map $\phi_X(t, \bullet) : D^t_X \to M$ given by $\phi_X(t, p) = \gamma_p(t)$. In other words, $\phi_X(t, p)$ is simply the position at time t on the integral curve of X through p. Clearly $\phi_X(0, \bullet) = \mathrm{Id}_M$. Also, $\phi_X(\bullet, p)$ is precisely the maximal integral curve through p.

Proposition 2.2.5. *For any $t_0 \in \mathbb{R}$ and any $p \in D^{t_0}_X$, there exists $\epsilon > 0$ and open neighbourhood U of p in M such that $U \subset D^t_X$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$. In particular, D^t_X is open for every $t \in \mathbb{R}$.*

Moreover, the map $(t_0 - \epsilon, t_0 + \epsilon) \times U \to M, (t, q) \mapsto \phi_X(t, q)$, is smooth.

Proposition 2.2.6. *Fix $X \in \mathfrak{X}^1(M)$. If $p \in M$ and $s_0, t_0 \in \mathbb{R}$ such that $p \in D^{t_0}_X$ and $\phi_X(t_0, p) \in D^{s_0}_X$, then $p \in D^{s_0+t_0}_X$ and $\phi_X(t_0 + s_0, p) = \phi_X(s_0, \phi_X(t_0, p))$.*

In particular, if X is a complete vector field, then for any $s, t \in \mathbb{R}$ we have $\phi_X(s+t, \bullet) = \phi_X(s, \bullet) \circ \phi_X(t, \bullet) : M \to M$, so that $\phi_X(t, \bullet) : M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$, with $\phi_X(t, \bullet)^{-1} = \phi_X(-t, \bullet)$. This gives us a one-parameter subgroup *of the group of diffeomorphisms on M. This also gives us an $\mathbb{R}$-action on M, in which setting we say that X is an* infinitesimal generator *of this action of $\mathbb{R}$ on M.*

Example 2.2.7. Consider $X = -x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \in \mathfrak{X}^1(\mathbb{R}^2)$. For the point $(a, b) \in \mathbb{R}^2$, if $\gamma = (x, y)$ is the (maximal) integral curve through (a, b), then $\gamma'(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y}$ so that $\gamma'(t) = X(\gamma(t))$ is equivalent to solving the system $x' = -x^2, y' = xy, x(0) = a, y(0) = b$. Solving this ODE gives us $x(t) = \frac{a}{at+1}$ and $y(t) = b(at + 1)$. We thus see that

$$I_{(a,b)} = \begin{cases} (-\frac{1}{a}, \infty) & a > 0, \\ \mathbb{R} & a = 0, \\ (-\infty, \frac{1}{|a|}) & a < 0, \end{cases}, \quad D^t_X = \begin{cases} \mathbb{R}^2 & t = 0, \\ (-\infty, -\frac{1}{t}) \times \mathbb{R} & t < 0, \\ (-\frac{1}{t}, \infty) \times \mathbb{R} & t > 0, \end{cases} \quad \text{and} \quad \phi_X(t, (x, y)) = \left(\frac{a}{at+1}, abt + b\right).$$

Example 2.2.8. Consider $X = xy\frac{\partial}{\partial x} - x^2\frac{\partial}{\partial y} \in \mathfrak{X}^1(\mathbb{R}^2)$. We check that $X \in \mathfrak{X}^1(S^1)$; indeed, for any $p = (x_0, y_0) \in S^1$, notice that

$$T_pS^1 = \ker d_p\left((x, y) \mapsto x^2 + y^2 - 1\right) = \ker\left(\frac{\partial}{\partial x}|_p \mapsto 2x_0\frac{d}{dt}|_0, \frac{\partial}{\partial y}|_p \mapsto 2y_0\frac{d}{dt}|_0\right) = \mathbb{R} \cdot \left(y_0\frac{\partial}{\partial x}|_p - x_0\frac{\partial}{\partial y}|_p\right)$$

so that $X \in \mathfrak{X}^1(S^1)$. Now consider the stereographic coordinate s on $S^1\backslash\{(0, 1)\}$ given by $s((x, y)) = \frac{x}{1-y}$. Set $N = (0, 1)$. The inclusion $\iota : S^1\backslash N \hookrightarrow \mathbb{R}^2$ induces the inclusion $T_pS^1 \to \mathbb{R}^2$ for all $p \in S^1\backslash N$. Since s is a coordinate on $S^1\backslash N$ and (x,y) is a coordinate on $\mathbb{R}^2$, we see that $x = \frac{2s}{s^2+1}, y = \frac{s^2-1}{s^2+1}$ (and $1-y = \frac{2}{s^2+1}$), and thus

$$d\iota|_{S^1\backslash N}\frac{d}{ds} = \frac{2(1-s^2)}{(s^2+1)^2}\frac{\partial}{\partial x} + \frac{4s}{(s^2+1)^2}\frac{\partial}{\partial y} = -y(1-y)\frac{\partial}{\partial x} + x(1-y)\frac{\partial}{\partial y} = -\frac{(1-y)}{x}X|_{S^1\backslash N}.$$

Since $X_p \in \mathrm{Im}(d_p\iota)$ and since ι is an immersion, we see that $X|_{S^1\backslash N} = -s\frac{d}{ds}$.

Now suppose $\gamma(t) : I \to S^1$ is an integral curve through a point in $S^1\backslash N$. From $\gamma'(t) = s'(t)\frac{\partial}{\partial ds}|_{s(t)} = -s(t)\frac{d}{ds}|_{s(t)} = X|_{\gamma(t)}$, we want to solve $s' = -s$ with $s(0) = s_0$. This has solution $s = s_0e^{-t}$. Thus,

$$\iota \circ \phi_X\left(t, \left(\frac{2s_0}{s_0^2+1}, \frac{s_0^2-1}{s_0^2+1}\right)\right) = \left(\frac{2s_0e^{-t}}{s_0^2e^{-2t}+1}, \frac{s_0^2e^{-2t}-1}{s_0^2e^{-2t}+1}\right).$$

From this we see that for any $p \in S^1\backslash N$, we have $I_p = \mathbb{R}$. However, notice also that $\phi_X(\bullet, p) \subset S^1\backslash N$ for all $p \in S^1\backslash N$.

We now try to calculate at N. For this, we pick the upper semi-circle coordinates $u((x,y)) = x$ which are centred at N. Then $x = u$ and $y = \sqrt{1-u^2}$ so that

$$d\iota\frac{d}{du} = \frac{\partial}{\partial x} - \frac{u}{\sqrt{1-u^2}}\frac{\partial}{\partial y}|_N = \frac{1}{u\sqrt{1-u^2}}X.$$

We thus want to solve $u' = X = u\sqrt{1-u^2}$ with $u' = 0$. Since $u \equiv 0$ is a solution, it is the unique solution, and hence $\phi_X(t, N) = N$. Therefore X is a complete vector field. Moreover, we have

$$\phi_X(t,(x_0,y_0)) = \left(\frac{2x_0(1-y_0)e^{-t}}{x_0^2e^{-2t}+(1-y_0)^2}, \frac{x_0^2e^{-2t}-(1-y_0)^2}{x_0^2e^{-2t}+(1-y_0)^2}\right) \quad \forall(x_0,y_0) \neq (0,1), \quad \text{and} \quad \phi_X(t,(0,1)) = (0,1).$$

We thus see that the integral curves of X tend towards the south pole $(0,-1)$, except for an 'unstable' equilibrium at $(0,1)$.

Proposition 2.2.9. *If M is compact, then every vector field of M is complete.*

$\mathfrak{X}^1(M)$ as a Lie Algebra

Definition. A real *Lie algebra* is a real vector space $\mathfrak{g}$ together with a map $[,] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ (the *Lie bracket*) such that $[x,y] = -[y,x]$, $[ax+by,z] = a[x,z]+b[y,z]$, and $[[x,y],z]+[[y,z],x]+[[z,x],y] = 0$ (Jacobian Identity). An *abelian Lie algebra* is simply a Lie algebra whose Lie bracket is identically zero.

We have the obvious definitions of Lie sub-algebras and of Lie algebra homomorphisms.

Definition. Any $X \in \mathfrak{X}^1(M)$ can be considered as a map $X : C^\infty(M) \to C^\infty(M)$ by $X(f)(p) := X_p([f]_p) = d_pf(X_p) \in T_{f(p)}\mathbb{R} = \mathbb{R}$. Then, $\mathfrak{X}^1(M)$ is the space of all derivations on $C^\infty(M)$, i.e. $X \in \mathfrak{X}^1(M)$ iff $X : C^\infty(M) \to C^\infty(M)$ is a linear map such that $X(fg) = fX(g)+gX(f)$.

We endow $\mathfrak{X}^1(M)$ with a Lie bracket defined by $[X,Y](f) := X(Yf)-Y(Xf)$. It turns out that $[X,fY] = X(f)Y + f[X,Y]$.

Proposition 2.2.10. *Suppose ϕ_X, ϕ_Y are the integral flows on X and Y. For any $p \in M$, define the smooth function $\gamma : [0,\epsilon) \to M$ by*

$$\gamma_p(t) = \phi_Y(-\sqrt{t},\bullet)\circ\phi_X(-\sqrt{t},\bullet)\circ\phi_Y(\sqrt{t},\bullet)\circ\phi_X(\sqrt{t},\bullet).$$

Then $[X,Y]_p = \frac{d}{dt}|_{t=0^+}\gamma_p(t) \in T_pM$.

We also have

$$[X,Y]_p = \lim_{t\to 0}\frac{1}{t}\left(\phi_X(-t,\bullet)_*Y_{\phi_X(t,p)} - Y_p\right) = \frac{d}{dt}|_{t=0}(\phi_X(-t,\bullet))_*Y_{\phi_X(t,p)}.$$

A more practical method of computation is to use local coordinates. If $U,(x_1,...,x_m)$ are local coordinates, then we write $X = \sum_{i=1}^m X_i\frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^m Y_i\frac{\partial}{\partial x_i}$ where $X_i, Y_j \in C^\infty(U)$. Since partial derivatives commute on $\mathbb{R}^m$, it follows that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ for all i,j. We can thus expand $[X,Y]$ using linearity and the fact that $[X,fY] = X(f)Y + f[X,Y]$, to get the following formula:

$$[X,Y] = \sum_{j=1}^m\left(\sum_{i=1}^m X_i\frac{\partial Y_j}{\partial x_i} - Y_i\frac{\partial X_j}{\partial y_i}\right)\frac{\partial}{\partial x_j}.$$

Definition. If $F : M \to N$ is smooth, $X \in \mathfrak{X}^1(M)$ and $Y \in \mathfrak{X}^1(N)$, say that X and Y are *F-related* if any of the following equivalent conditions hold: (i) for all $p \in M$, $d_pF(X_p) = Y_{F(p)}$, (ii) $\frac{d}{dt}|_{t=0}F(\phi_X(t,\bullet)) = Y_{F(p)}$ for all $p \in M$, or (iii) $X \circ F^* = F^* \circ Y$ where $F^* : C^\infty(N) \to C^\infty(M)$ is the linear map $f \mapsto f \circ F$.

The reason this definition is useful is due to the following result.

Proposition 2.2.11. *If X_i is F-related to Y_i for $i = 1, 2$, then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.*

Example 2.2.12. Suppose $X \in \mathfrak{X}^1(M)$, $p \in M$, $f \in C^\infty(M)$. If $\phi_X(t,p) : (-\epsilon, \epsilon) \to M$ is an integral curve of X through p, then $g(t) := f \circ \phi_X(t,p) : (-\epsilon, \epsilon) \to \mathbb{R}$ is a smooth function. The Taylor series of g at $t = 0$ is

$$f(p) + X(f)(p)t + \frac{1}{2!}X^2(f)(p)t^2 + \frac{1}{3!}X^3(f)(p)t^3 + \cdots = \sum_{i\geq 0} \frac{1}{i!}X^i(f)(p) \cdot t^i,$$

where $X^i(f)$ is the function obtained by X acting on f i-times.

2.2.4 Lie Groups

Definition. A *Lie group* is a smooth manifold G that is also a group with multiplication $m : G \times G \to G$ and inverse $i : G \to G$ such that both m and i are smooth maps. We write the group operation as usual.

A *Lie subgroup* of G is a subgroup H such that H is a Lie group in its own right, and the inclusion $H \hookrightarrow G$ is an immersion. If the inclusion is an embedding, then we say that H is an *embedded Lie subgroup.*

For each $g \in G$, we have well-defined diffeomorphisms $\ell_g, r_g, C_g : G \to G$ given by $\ell_g(h) = gh = m(g,h)$, $r_g(h) = m(h,g)$, and $c_g(h) = ghg^{-1}$.

A *Lie group homomorphism* $\varphi : G \to H$ is simply a group homomorphism φ that is also smooth.

Definition. We say that $X \in \mathfrak{X}^1(M)$ is *left invariant* if X is ℓ_g-related to itself for all $g \in G$, i.e. for any $p \in G$, we have $d_p\ell_g(X_p) = X_{gp}$ for all $g \in G$.

There is a 1-1 correspondence between left-invariant vector fields and $\mathfrak{g} := T_eG$ where $e \in G$ is the identity: given any $v \in T_eG$, we can define the left-invariant $v^\ell \in \mathfrak{X}^1$ by $v^\ell_p := d_e\ell_g(v)$. The space $\mathfrak{g}$ becomes a Lie subalgebra of $\mathfrak{X}^1(M)$ by the above map $v \mapsto v^\ell$, i.e. we have the Lie bracket $[,]$ on $\mathfrak{g}$ defined by $[x,y] := [x^\ell, y^\ell]_e$.

This space $\mathfrak{g}$ with the above Lie bracket is called *the Lie algebra of the Lie group G.*

It is known that all left-invariant vector fields are complete.

Definition. Suppose G is a Lie group with Lie algebra $\mathfrak{g}$. For $x \in \mathfrak{g}$, let ϕ^t_x ($t \in \mathbb{R}$) be the integral flow of the left-invariant vector field x^ℓ. Define the *exponential map* $\exp : \mathfrak{g} \to G$ by $\exp(x) := \phi_x(1, e) \in G$. More generally, we set $\exp(tx) := \phi_x(t, e)$; one checks that $\phi_{tx}(1, e) = \phi_x(t, e)$ so that this is well-defined.

The map $\mathbb{R} \to G$, $t \mapsto \exp(tx)$, is called *the one-parameter subgroup of G defined by x.*

Important facts:

1. We have $d_{(a,b)}m(v,w) = d_b\ell_a(v) + d_ar_b(w) : T_aG \oplus T_bG \to T_{ab}G$ for all $a, b \in G$, and all $v \in T_aG, w \in T_bG$.
2. Clearly $\exp(0)$ is the identity of G.
3. The map $\mathbb{R} \times \mathfrak{g} \to G$, $(t,x) \mapsto \exp(tx)$ is smooth. In particular, $\exp : \mathfrak{g} \to G$ is smooth, and moreover $d_0 \exp : T_0\mathfrak{g} \cong \mathfrak{g} \to T_eG = \mathfrak{g}$ is the identity map.
4. The integral curve for x^ℓ is $\phi_x(t,g) = r_{\exp(tx)}(g) = g\exp(tx)$.
5. $\exp(t+s)X = \exp(tX)\exp(sX)$ for all $t, s \in \mathbb{R}$. In particular, $\exp(-X) = \exp(X)^{-1}$.
6. We have $d_e\iota : \mathfrak{g} \to \mathfrak{g}$ is given by $X \mapsto -X$; more generally we have $d_g\iota = -d_g(R_{g^{-1}} \circ L_{g^{-1}})$.
7. $(\exp X)^n = \exp(nX)$ for all $n \in \mathbb{Z}$.
8. The exponential map restricts to a diffeomorphism from some open neighbourhood of $0 \in \mathfrak{g}$ to an open neighbourhood of $e \in G$.
9. Suppose $x, y \in \mathfrak{g}$. There exists a smooth curve $z : (-\epsilon, \epsilon) \to \mathfrak{g}$ with $z(0) = 0$ such that $\exp(tx)\exp(ty) = \exp\big(t(x+y) + tz(t)\big)$.
10. For any $x, y \in \mathfrak{g}$, $\lim_{n\to\infty} \big(\exp(\frac{t}{n}x)\exp(\frac{t}{n}y)\big)^n = \exp(tx + ty)$.

11. Suppose $\Phi : G \to H$ is a Lie group homomorphism. Then the map $d_e\Phi : T_eG = \mathfrak{g} \to T_eH = \mathfrak{h}$ is a Lie algebra homomorphism. Moreover
$$\Phi \circ \exp_G = \exp_H \circ d_e\Phi;$$
in fact, for any $x \in \mathfrak{g}$, we have $\Phi\big(\exp_G(tx)\big) = \exp_H\big(td_e\Phi(x)\big)$.

Proof. That $d_e\Phi$ is a Lie algebra homomorphism follows since the left-invariant vector field associated to $x \in \mathfrak{g}$ is Φ-related to the left-invariant vector field associated to $d_e\Phi(x) \in \mathfrak{h}$.

Now fix $x \in \mathfrak{g}$. By definition, the map $t \mapsto \exp_G(tx)$ is the integral flow of x^ℓ through e, and $\exp_H\big(td_e\Phi(x)\big)$ is the integral flow of $(d_e\Phi(x))^\ell$ through $\Phi(e) = e$. By uniqueness of integral flow, it suffices to show that $\gamma : \mathbb{R} \to H, t \mapsto \Phi\big(\exp_G(tx)\big)$ is the integral flow of $(d_e\Phi(x))^\ell$ through e. To see this, note that $\gamma(0) = e$ and, by the chain rule,
$$\gamma'(t_0) = d_{\exp_G(t_0x)}\Phi\big(d_{t_0}(\exp_G tx)(\tfrac{d}{dt}|_{t_0})\big) = d_{\exp_G(t_0x)}\Phi\big(x^\ell|_{\exp_G(t_0x)}\big) = d_{\exp_G(t_0x)}\Phi\big(d_e\ell_{\exp_G(t_0x)}(x)\big),$$
where the second last equality follows since $t \mapsto \exp_G(tx)$ is the integral flow of x^ℓ through e, and the last equality follows by definition of x^ℓ. By the chain rule once again, the right-most expression is simply $d_e(\Phi \circ \ell_{\exp_G(t_0x)})(x)$. However, as Φ is a Lie group homomorphism, we see that
$$\Phi \circ \ell_{\exp_G(t_0x)}(g) = \Phi(\exp_G(t_0x)g) = \ell_{\Phi(\exp_G t_0x)} \circ \Phi(g) = \ell_{\gamma(t_0)} \circ \Phi(g).$$
Thus
$$\gamma'(t_0) = d_e\big(\ell_{\gamma(t_0)} \circ \Phi(g)\big)(x) = d_e(\ell_{\gamma(t_0)}) \circ d_e\Phi(x) = \big(d_e\Phi(x)\big)^\ell|_{\gamma(t_0)}.$$
Hence γ is indeed the integral flow of $(d_e\Phi(x))^\ell$ through e, as required. □

12. We have $\phi_{x^\ell}(t, \bullet) = r_{\exp(tx)}$, i.e. the integral curve of x^ℓ through g is simply $t \mapsto g\exp(tx)$.

Definition. The *adjoint action* of G on $\mathfrak{g}$ is the Lie group homomorphism $Ad : G \to GL(\mathfrak{g})$ which sends $g \in G$ to the linear operator $d_ec_g : \mathfrak{g} \to \mathfrak{g}$. It is known that $Ad_* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \cong End(\mathfrak{g})$ takes $x \in \mathfrak{g}$ to the linear operator $ad_x := [x, \bullet] \in End(\mathfrak{g})$ (i.e. $ad_x(y) = [x, y]$).

The *Killing Form* of a Lie algebra is a symmetric bilinear map $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $\langle x, y\rangle := \mathrm{Trace}(ad_xad_y)$. This bilinear map is adjoint invariant, i.e. $\langle ad_xy, z\rangle = -\langle y, ad_xz\rangle$.

It is known that the Killing form is invariant under any automorphism of $\mathfrak{g}$, i.e. if $\rho : \mathfrak{g} \to \mathfrak{g}$ satisfies $\rho([x, y]) = [\rho x, \rho y]$ for all $x, y \in \mathfrak{g}$, then $\langle \rho x, \rho y\rangle = \langle x, y\rangle$ (this is a direct calculation). In particular, if $\varphi : G \to G$ is a diffeomorphism taking e to e, then $\langle d_e\varphi x, d_e\varphi y\rangle = \langle x, y\rangle$ for all $x, y \in \mathfrak{g}$, and so in some basis for $\mathfrak{g}$ the matrix of $d_e\varphi$ is a real symmetric matrix, and thus diagonalizable.

Theorem 2.2.13 (Closed Subgroup Theorem)**.** *Suppose G a Lie group and H a subgroup. If H is a closed subset of G, then H is an embedded Lie subgroup of G. Moreover, the Lie algebra of H can be characterized either by*
$$\mathfrak{h} = \{x \in \mathfrak{g} : \exp(tx) \in H \forall t \in \mathbb{R}\}$$
or by
$$\mathfrak{h} = \{x \in \mathfrak{g} : x \in T_eH\} = d_e\iota(T_eH) \subset T_eG = \mathfrak{g}$$
where $\iota : H \to G$ is the inclusion map, which clearly induces the map $d_e\iota : T_eH = \mathfrak{h} \to T_eG = \mathfrak{g}$.

The exponential map $\exp_H : \mathfrak{h} \to H$ of H is simply the restriction to $\mathfrak{h}$ of the exponential map $\exp$ of G.

Here are some examples of Lie groups, their Lie algebras, and the exponential map.

1. Any finite dimensional vector space V is an abelian Lie group. The Lie algebra is V itself, and the exponential map is the identity map.

2. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a 1-dimensional Lie group. The Lie algebra is the abelian Lie algebra $\mathbb{R}$, with the exponential map $\exp : \mathbb{R} \to S^1, t \mapsto e^{2\pi it}$.

3. Product of Lie groups is again a Lie group; the corresponding Lie algebra is the direct sum of the two Lie algebras. If G and H are Lie groups, then the exponential map $\mathfrak{g} \oplus \mathfrak{h} \to G \times H$ is $\exp(x + y) = (\exp(x), \exp(y))$ for any $x + y \in \mathfrak{g} \oplus \mathfrak{h}$ $(x \in \mathfrak{g}, y \in \mathfrak{h})$.

4. In particular, the n-dimensional torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ is a Lie group with Lie algebra canonically isomorphic to $\mathbb{R}^n$. The exponential map $\exp : \mathbb{R}^n \to T^n$ is the map
$$\exp(x_1, ..., x_n) = \big(e^{2\pi ix_1}, ..., e^{2\pi ix_n}\big)$$

5. The *general linear group* $GL(n,\mathbb{R})$, the group of all invertible $n\times n$ real matrices, is an open submanifold of $M_{n\times n}(\mathbb{R})$, with group operation multiplication of matrices. The Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ is $M_{n\times n}(\mathbb{R})$, and the exponential map $\exp : \mathrm{gl}(n,\mathbb{R}) \to GL(n,\mathbb{R})$ sends A to $\sum_{k=0}^{\infty} \frac{1}{k!}A^k$. We have $\exp(A)\exp(B) = \exp(A+B)$, and $\exp(O_n) = I_n$. One also checks that
$$\det(\exp(A)) = e^{\mathrm{Tr}(A)}$$
by diagonalizing A (over $\mathbb{C}$).

6. The map $\det : GL(n,\mathbb{R}) \to \mathbb{R}^\times$ is a Lie group homomorphism (where the smooth structure on $\mathbb{R}^\times$ is the open submanifold structure of $\mathbb{R}$). This along with the closed subgroup theorem gives us for free the following Lie groups as embedded Lie subgroups of $GL(n,\mathbb{R})$:
   (a) The *special linear subgroup* $SL(n,\mathbb{R}) := \det^{-1}(1)$ is the n^2-1-dimensional embedded Lie subgroup of $GL(n,\mathbb{R})$ of invertible matrices whose determinant is 1. Since $\det\circ\exp = e^{\mathrm{Tr}}$, it follows that $A \in \mathfrak{sl}(n,\mathbb{R})$ iff $\exp(tA) \in SL(n,\mathbb{R})$ for all $t\in\mathbb{R}$ iff $e^{t\mathrm{Tr}(A)} = \det\exp(tA) = 1$ iff $\mathrm{Tr}(A) = 0$. Thus
   $$\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl} : \mathrm{Trace}(A) = 0\}.$$
   (b) The *orthogonal group* $O(n)$ is the subgroup of $GL(n,\mathbb{R})$ of matrices X such that $XX^T = I_n$. Note that $A\in\mathfrak{o}(n)$ iff $\exp(tA)\in O(n)$ iff $\exp(tA)\exp(tA)^T = I_n$ iff $\exp(tA+tA^T) = I_n$ iff $A + A^T = O$. Thus
   $$\mathfrak{o}(n) = \{A\in\mathfrak{gl}(n,\mathbb{R}) : A^T = -A\},$$
   is the space of all skew-symmetric matrices. In particular, $\dim O(n) = \binom{n}{2}$.
   (c) The *special orthogonal group* $SO(n)$ is the subgroup of $O(n)$ with determinant 1. The constant rank theorem (noting $SO(n) = \det|_{O(n)}^{-1}(1)$) implies that $SO(n)$ has dimension $\binom{n}{2}-1$. Clearly,
   $$\mathfrak{so}(n) = \{A\in\mathfrak{gl}(n,\mathbb{R}) : A^T = -A, \mathrm{Trace}(A) = 0\}.$$
   (d) The *symplectic group* $Sp(2n,\mathbb{R})$ is the subgroup of $GL(2n,\mathbb{R})$ of matrices X such that $X^TJ_{2n}X = J_{2n}$, where $J_{2n} = \left(\begin{smallmatrix} O_n & I_n \\ -I_n & O\end{smallmatrix}\right)$. One checks similarly to above that
   $$\mathfrak{sp}(2n,\mathbb{R}) = \{A\in\mathfrak{gl}(2n,\mathbb{R}) : A^TJ_{2n} = -J_{2n}A\}.$$

7. Similarly, $GL(n,\mathbb{C})$ (the $n\times n$ invertible matrices over $\mathbb{C}$) is a Lie group of dimension $2n^2$ (since $\mathbb{C}^m\cong\mathbb{R}^{2m}$ as smooth manifolds). The Lie algebras $\mathrm{gl}(n,\mathbb{C})$ and $\mathrm{sl}(n,\mathbb{C})$ are obvious. The exponential map is again $X\mapsto e^X = \sum_{k=0}^{\infty}\frac{1}{k!}X^k$.
   We also have the *unitary group* $U(n)$ and the *special unitary group* $SU(n)$, which are subgroups of $GL(n,\mathbb{C})$ defined by $U(n) = \{g\in GL(n,\mathbb{C}) : gg^* = I_n\}$ and $SU(n) = \det|_{U(n)}^{-1}(1)$ (here, g^* is the Hermitian transpose $g^* = \overline{g^T}$, where we take complex conjugates). We see easily that $\dim U(n) = n(n-1)$, $\dim SU(n) = n^2-n-2 = (n+1)(n-2)$, and that
   $$\mathfrak{u}(n) = \{X\in\mathfrak{gl}(n,\mathbb{C}) : X^* = -X\} \quad \text{and } \mathfrak{su}(n) = \{X\in\mathfrak{gl}(n,\mathbb{C}) : X^* = -X, \mathrm{Trace}(X) = 0\}.$$

Example 2.2.14 (Fall 2020 Day 2)**.** Suppose G is a compact Lie group with Lie algebra $\mathfrak{g}$, and fix $g\in G$. Let $\mathfrak{c}$ be the subalgebra of $\mathfrak{g}$ given by $\mathfrak{c} = \{x\in\mathfrak{g} : Ad_g(x) = x\}$. Show that there exists an $\epsilon > 0$ such that for any $x\in\mathfrak{g}$ with $|x| < \epsilon$, there exists $y\in\mathfrak{c}$ such that $g\exp(y)$ is conjugate to $g\exp(x)$ in G.

Indeed, since $Ad_g = d_e(c_g)$ where $c_g : G\to G$ is the conjugation by g diffeomorphism, it follows by a property of exponential maps that $\exp\circ Ad_g = c_g\circ\exp$, i.e. for any $x\in\mathfrak{g}$, we have $\exp(Ad_g(x)) = c_g(\exp(x)) = g\exp(x)g^{-1}$. It follows that $Ad_g(x) = x$ iff $g\exp(x) = \exp(x)g$, i.e. iff $\exp(x)\in C(g)$, the centralizer of g in G. Thus $y\in\mathfrak{c}$ iff $\exp(y)\in C(g)$. Since c_g is a diffeomorphism, and $C(g) = \{h\in G : c_g(h) = h\}$, it follows that $C(g)$ is a closed subgroup. By the closed subgroup theorem, it follows that $\mathfrak{c}$ is the Lie algebra of $C(g)$.

Now, consider the Lie group $M := G\times\mathfrak{c}$, and let $\psi : M\to G$ be the map $\psi(h,y) = hg\exp(y)h^{-1}$. The map ψ is clearly smooth. At the identity $(e,0)\in M$, notice that $T_{(e,0)}M = T_eG\oplus\mathfrak{c} = \mathfrak{g}\oplus\mathfrak{c}$. Let us evaluate the map $\psi_* = d_{(e,0)}\psi : \mathfrak{g}\oplus\mathfrak{c}\to\mathfrak{g}$; we have $\psi(h,y) = m(m(h,g\exp(y)),i(h))$ and $\psi(e,0) = g$, and by using $d_{(a,b)}m(v,w) = d_ar_b(v) + d_b\ell_a(w)$ and $d_ei = -1_{\mathfrak{g}}$, we have
$$\psi_*(x,y) = d_gr_e\big(d_er_g(x) + d_g\ell_e(d_e\ell_g(y))\big) + d_e\ell_g(-x) = d_er_g(x) + d_e\ell_g(y) + d_e\ell_g(-x) = d_e\ell_g\big(Ad_g(x) + y - x\big).$$

Since $\psi_* : \mathfrak{g}\oplus\mathfrak{c}\to T_eG \underset{d_e\ell_g}{\cong} \mathfrak{g}$, we see that $\mathrm{rank}\psi_*(x,y)$ is the dimension of the subspace
$$\{(x,y)\in\mathfrak{g}\oplus\mathfrak{c} : Ad_g(x) + y - x\} = \mathfrak{c} + \mathrm{Im}(Ad_g - I)$$

in $\mathfrak{g}$. Now, as c_g is a Lie group automorphism of G, $Ad_g = (c_g)_*$ preserves the Killing form of $\mathfrak{g}$, and so in particular is diagonalizable. This implies that for any $v \in \mathfrak{g}$ if $(Ad_g - I)^2 v = 0$, then $(Ad_g - I)v = 0$. In other words, $\ker(Ad_g - I) \cap \mathrm{Im}(Ad_g - I) = 0$. The rank nullity theorem then implies that $\mathfrak{g} = \mathfrak{c} \oplus \mathrm{Im}(Ad_g - I)$, and thus ψ_* has full rank. Since exp is a local diffeomorphism from 0 to e, it follows that the smooth map $\Psi : G \times \mathfrak{c} \to G \times C(g)$ given by $\Psi(h, y) = (\psi(h, y), \exp y)$ has full rank at $(e, 0)$, and so by the Inverse Function Theorem is a local diffeomorphism at $(e, 0)$. Hence, there exists a small open neighbourhood U of (g, e) in $G \times C(g)$ that is mapped diffeomorphically via Ψ^{-1} to a small open neighbourhood V of $(e, 0)$ in $G \times \mathfrak{c}$.

Now, for all $X \in \mathfrak{g}$ with small enough norm we have that $(g \exp X, a)$ is in U for some $a \in C(g)$. If $(h, Y) \in G \times \mathfrak{c}$ is the pre-image under Ψ of $(\exp X, g)$, then $hg \exp(Y)h^{-1} = g \exp X$ and $\exp Y = a$. Hence, we have found $Y \in \mathfrak{c}$ such that $g \exp(Y)$ is conjugate (via $h \in G$) to $g \exp X$.

We briefly discuss Lie group actions.

Definition. A *smooth (left) action* or a *(left) Lie group action* is a group action G on M where G is a Lie group, M is a manifold, and the group action map $G \times M \to M$ is smooth. In such a case we also say that M is a *left smooth G-space.*

Proposition 2.2.15. *Define $\sigma : \mathfrak{g} \to \mathfrak{X}^1(M)$ by $\sigma(x)_p := \frac{d}{dt}|_{t=0} \exp(-tx) \cdot p \in T_pM$. This is a Lie algebra homomorphism*

We give some examples of smooth left actions.

1. $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ on the left.
2. $GL(n, \mathbb{R})$ has a left smooth action on $M_{n,m}(\mathbb{R})$ by multiplying on the left. Similarly, $GL(m, \mathbb{R})$ has a right smooth action on $M_{n,m}(\mathbb{R})$ by multiplying on the right.
3. In general, $\mathfrak{g}$ acts on a Lie group G via the exponential map.
4. If G a Lie group and H a closed Lie subgroup, then there is a unique smooth structure on the set of cosets G/H with underlying topology the quotient topology. Moreover, the action $G \times G/H \to G/H$, $(g_1, g_2H) \mapsto (g_1g_2)H$, is a left Lie group action. Moreover, G/H is a *homogeneous space* of G, i.e. the Lie group action is transitive.
5. In fact, any homogeneous space M of G is diffeomorphic to G/H for some closed subspace H of G, where for a fixed $p \in M$ we can suppose $H = \{g \in G : gp = p\}$ is the stabilizer subgroup of the G-action on M.
6. The Grassmannian $\mathrm{Gr}(k, n)$ is a homogeneous space of $GL(n, \mathbb{R})$ via the action $(g, V) \mapsto gV$. If we fix $V = \mathrm{span}_{\mathbb{R}}\{e_1, ..., e_k\}$, then the stabilizer subgroup of the action at V is
$$H = \left\{ \begin{pmatrix} A & B \\ O & C \end{pmatrix} : A \in GL(k, \mathbb{R}), B \in M_{k,n-k}(\mathbb{R}), C \in GL(n-k, \mathbb{R}) \right\}$$
so that $\mathrm{Gr}(k, n) \cong GL(n, \mathbb{R})/H$.

Definition. If G is a Lie group, if M is a manifold, then the manifold P is *a principal G-bundle over M* if there exists a smooth map $\pi : P \to M$ and a smooth right group action $P \times G \to P$ such that

1. we have $\pi(pg) = \pi(p) \in M$ for all $p \in P$ and all $g \in G$;
2. for any $x \in M$, there exists an open neighbourhood U of x in M such that there exists a diffeomorphism $\varphi_U : \pi^{-1}(U) \subset P \to U \times G$ of the form $\varphi_U(p) = (\pi(p), \phi(p))$ where the smooth map $\phi : \pi^{-1}(U) \to G$ satisfies $\phi(pg) = \phi(p)g$;
3. For any $x \in M$, the action of G restricted to the fibre $P_x := \pi^{-1}(x) \subset P$ is free and transitive;
4. For each $x \in G/H$ and for each $g \in \pi^{-1}(x)$, the map $H \to \pi^{-1}(x), h \mapsto gh$ is a diffeomorphism.

Examples:

1. The product space $P = M \times G$ is the *trivial* principal G-bundle over M.
2. If $H \subset G$ is a closed (thus embedded) subgroup, then G is a principal H-bundle over the homogeneous space G/H.

3. (*Frame bundle of smooth manifold*) Let M^n be a manifold, and for $p \in M$ let FM_p be the set of all ordered bases of T_pM. Let $FM := \bigsqcup_{p\in M} FM_p$ and endow FM with the structure of a smooth manifold by using coordinate charts and local frames on M. Then $GL(n,\mathbb{R})$ acts on FM_p in the obvious way, so that $FM \to M$ is a principal $GL(n,\mathbb{R})$-bundle over M.

Properties:

1. The base space M is diffeomorphic to P/G, the space of all G-orbits in P.
2. The action of G on P is *proper*, i.e. for fixed $g \in G$ the map $g : P \to P$ is a proper map (pre-images of compact sets are compact).
3. In fact, if P is any manifold and G any Lie group such that there is a smooth, free and proper right action of G on P, then P/G is a smooth manifold, the projection $\pi : P \to G$ is a submersion, and P is a principal G-bundle over P/G.

2.3 Differential Forms and Integration

2.3.1 Multi-linear Algebra

Throughout, we fix a vector space V over a field k (for concreteness, either $k = \mathbb{R}$ or $k = \mathbb{C}$). Recall that the *dual vector space* V^* is the n-dimensional k-vector space of linear functionals on V.

1. The k-vector space of all multi-linear maps from $V^k = V \times V \times \cdots \times V \to \mathbb{R}$ is denoted by $\bigotimes^k V^*$, called the *k-th tensor product of* V^*. Also set $\bigotimes^0 V^* = \mathbb{R}$.
2. An element $\phi \in \bigotimes^k V^*$ is *skew-symmetric* if
$$\phi(v_1, ..., v_{i-1}, v_j, v_{i+1}, ..., v_{j-1}, v_i, v_{j+1}, ..., v_n) = -\phi(v_1, ..., v_n)$$
for all $v_1, ..., v_n \in V$. The subspace of skew-symmetric elements of $\bigotimes^k V^*$ is denoted by $\bigwedge^k V^*$. This space is called the *k'th exterior product of* V^*
3. An element $\phi \in \bigotimes^k V^*$ is *symmetric* if
$$\phi(v_1, ..., v_{i-1}, v_j, v_{i+1}, ..., v_{j-1}, v_i, v_{j+1}, ..., v_n) = \phi(v_1, ..., v_n)$$
for all $v_1, ..., v_n \in V$. The subspace of skew-symmetric elements of $\bigotimes^k V^*$ is denoted by S^kV^*.
4. We have the bilinear map $\otimes : \bigotimes^k V^* \times \bigotimes^\ell V^* \to \bigotimes^{k+\ell} V^*$ which sends (ϕ, ψ) to
$$(\phi \otimes \psi)(x_1, ..., x_{k+\ell}) := \phi(x_1, ..., x_k)\phi(x_{k+1}, ..., x_{k+\ell}).$$
We also have the bilinear map $\wedge : \bigwedge^k V^* \times \bigwedge^\ell V^* \to \bigwedge^{k+\ell} V^*$ which sends (ϕ, ψ) to
$$(\phi \wedge \psi)(x_1, ..., x_{k+\ell}) := \frac{1}{k!\ell!} \sum_{\sigma\in S_{k+\ell}} \mathrm{sign}(\sigma)\phi(x_{\sigma(1)}, ..., x_{\sigma(k)})\phi(x_{\sigma(k+1)}, ..., x_{\sigma(k+\ell)}).$$
5. The direct sums $\bigoplus_{m=0}^\infty \bigotimes^m V^*$ and $\bigoplus_{m=0}^\infty \bigwedge^m V^*$ are graded k-algebras, called the *tensor algebra* and the *exterior algebra* of V^*.
6. If $\xi_1, ..., \xi_m \in V^*$, then we have
$$(\xi_1 \otimes \cdots \otimes \xi_m)(x_1, ..., x_m) = \prod_{i=1}^m \xi_i(x_i) \quad \text{and} \quad (\xi_1 \wedge \cdots \wedge \xi_m)(x_1, ..., x_m) = \det(\xi_i(x_j)).$$
If $j > \dim_k V$, then $\bigwedge^j V^* = 0$, and if $1 \le j \le \dim_k V$ then $\dim_k \bigwedge^j V^* = \binom{\dim_k V}{j}$. If $\{\xi_1, ..., \xi_n\}$ is a basis for V^*, then $\bigwedge^j V^*$ has the basis
$$\{\xi_{i_1} \wedge \cdots \wedge \xi_{i_j} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}.$$
We also have that $\dim_k S^jV^* = \binom{\dim_k V + j - 1}{j}$.
7. Any linear map $T : V \to W$ induces the linear *pull-back map* $T^* : W^* \to V^*$ ($T^*f = f \circ T$), which then induces the *pull-back map*
$$T^* : \bigotimes^m W^* \to \bigotimes^m V^*, \quad (T^*\omega)(x_1, ..., x_m) = \omega\big(T(x_1), ..., T(x_m)\big).$$

2.3.2 Vector Bundles

Definition. A *(real) vector bundle over a manifold* M^m *of rank* r is a smooth manifold E^{r+m} equipped with a surjective submersion $\pi : E \to M$ such that:

- for each $p \in M$, the *fibre* $E_p := \pi^{-1}(p) \subset E$ is a $\mathbb{R}$-vector space of dimension r; and
- for any $p \in M$, there exists an open neighbourhood U of p in M such that the open submanifold $E_U := \pi^{-1}(U)$ of E is diffeomorphic to $U \times \mathbb{R}^r$ via a diffeomorphism $\varphi_U : E_U \to U \times \mathbb{R}^r$ such that
  - $\pi|_U \circ \varphi_U^{-1} : U \times \mathbb{R}^r \to U$ is simply the projection onto the first factor,
  - for each $q \in U$, the map $\varphi_U|_{E_q} : E_q \to \{q\} \times \mathbb{R}^r \cong \mathbb{R}^r$ is a vector space isomorphism.

  Such a U for which $E_U \cong U \times \mathbb{R}^r$ is called a *local trivialization.*

A rank r vector bundle over M that is diffeomorphic to $M \times \mathbb{R}^r$ with the projection onto M simply the projection onto the first factor is called a *trivial vector bundle.*

A *complex vector bundle of rank* r is defined similarly, except now we need the fibres E_p to be complex vector spaces, and local trivializations look like $E_U \cong \mathbb{C}^r \times U$. Clearly, a complex vector bundle of rank r is a real vector bundle of rank $2r$ (though the converse need not be the case).

In a vector bundle $\pi : E \to M$, the space E is the *total space*, M is the *base (space)*, and π the *projection (onto the base).*

Definition. A *(smooth) global section* of a vector bundle $\pi : E \to M$ is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$. More generally, a *(smooth) local section* on the open subset $U \subset M$ is a smooth map $\sigma : U \to \pi^{-1}(U)$ such that $\pi \circ \sigma = \mathrm{Id}_U$. The space of smooth sections on $U \subseteq M$ is sometimes denoted by $\Gamma(U, E)$. The space $\Gamma(U, E)$ is a $C^\infty(U)$-module under the action $(fs)_p := f(p)s_p$, where $f(p)$ is a scalar acting on the vector space E_p.

The $C^\infty(U)$-modules $\Gamma(U, E)$ for all open subsets $U \subset M$ forms a sheaf of modules over the sheaf of smooth functions C^∞ of M.

We can define various operations on vector bundles, by carrying out operations of vector spaces on each individual fibre. Throughout, suppose $E \to M$ and $F \to M$ are vector bundles over M of ranks k and ℓ respectively.

1. The vector $E^* := \bigcup_{p \in M} E_p^*$ is the *dual vector bundle of* E, with rank k once again.
2. The vector bundle $E \oplus F := \bigcup_{p \in M} E_p \oplus F_p$ is the *direct product bundle of* E *and* F, with rank $k + \ell$.
3. The vector bundle $E \otimes F := \bigcup_{p \in M} E_p \otimes F_p$ is the *tensor product bundle of* E *and* F, with rank $k\ell$.
4. For $0 \leq a \leq k$, the vector bundle $\bigwedge^a E := \bigcup_{p \in M} \bigwedge^a E_p$ is the *a-th exterior product bundle of* E, with rank $\binom{k}{a}$.
5. For $n \geq 1$, the vector bundle $S^n E := \bigcup_{p \in M} S^n E_p$ is the *n-th symmetric power bundle of* E, with rank $\binom{k+a-1}{a}$.

Definition. A *bundle map* $\Phi : E \to F$ of vector bundles $\pi : E \to M$ and $\rho : F \to M$ over M is a smooth map such that $\rho \circ \Phi = \pi$ (i.e. Φ maps the fibre E_p to the fibre F_p) and such that $\Phi_p := \Phi|_{E_p} : E_p \to F_p$ is a linear map for all $p \in M$.

2.3.3 Cotangent Bundles and Tensor Fields

Recall the tangent bundle TM. It is obvious that $TM \to M$ is a vector bundle over M with the obvious projection map, and that vector fields are smooth sections of TM. Local coordinates on M induce local trivializations of TM.

Definition. Let T_p^*M be the dual space of T_pM, called the *cotangent space of* M *at* p. The dual vector bundle to TM is the *cotangent bundle* T^*M. Clearly, the fibres of the cotangent bundle are the cotangent spaces.

We give local trivializations of T^*M. Let $U, (x_1, ..., x_m)$ be a local coordinate patch. Then, we have the local trivialization $TM_U =: TU$ of TM given by $(x_1, ..., x_m, \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m})$. Here, the local vector fields $\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_m}|_p$

span T_pM for all $p \in U$. We also have the smooth local function $x_i : U \to \mathbb{R}$, which induces the differential $d_px_i : T_pM \to T_{x_i(p)}\mathbb{R} = \mathbb{R}$ for all $p \in U$. Notice that

$$d_px_i\left(\frac{\partial}{\partial x_j}\right) := \frac{\partial}{\partial x_j}(x_i) = \delta_{ij}$$

so that the d_px_is span T_p^*M, and moreover that $\{d_px_i\}$ is the dual basis of $T_p * M$ to the basis $\{\frac{\partial}{\partial x_i}|_p\}$ of T_pM. Since the local differential maps $dx_i : TU \to \mathbb{R}$ are all smooth maps, it follows that T^*M has the local coordinates $(x_1, ..., x_m, dx_1, ..., dx_m)$, which also induce a local trivialization on T^*M.

Let us write down the formula for change of coordinates. Suppose $(y_1, ..., y_m)$ and $(x_1, ..., x_m)$ are two local coordinates on U. Then, we have the change of basis $(\frac{\partial}{\partial x_j}) = (\frac{\partial}{\partial y_i})(\frac{\partial y_i}{\partial x_j})$. By dualizing, it follows that

$$(dy_1, \cdots, dy_m) = (dx_1, \cdots, dx_m) \cdot \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_m} & \frac{\partial y_2}{\partial x_m} & \cdots & \frac{\partial y_m}{\partial x_m} \end{pmatrix},$$

or equivalently

$$dy_j = \sum_{i=1}^{m} \frac{\partial y_j}{\partial x_i} dx_i.$$

Definition. Any smooth section of T^*M is called a 1-*form*. The space of smooth 1-forms is denoted by $\Omega^1(M)$ (more generally, the space of smooth local sections $\alpha : U \to T^*M|_U$ is $\Omega^1(U)$).

Notice that for any $f \in C^\infty(M)$, we have the 1-form $df : M \to T^*M$, and so we have the induced map $C^\infty(U) \to \Omega^1(U)$ of $C^\infty(U)$-modules.

Locally, if α is a 1-form and if $(x_1, ..., x_m)$ are local coordinates on U, then

$$\alpha|_U = f_1dx_1 + \cdots + f_mx_m$$

for some smooth functions $f_i \in C^\infty(U)$.

If $X \in \mathfrak{X}^1(M)$ and $\alpha \in \Omega^1(M)$, then $\alpha(X) \in C^\infty(M)$ is given by $\alpha(X)(p) := \alpha_p(X_p)$. In local coordinates $U, (x_1, ..., x_m)$, if we express $X = \sum_{i=1}^m f_i \frac{\partial}{\partial x_i}$ and $\alpha = \sum_{j=1}^m g_j dx_j$, then

$$\alpha(X) = f_1g_1 + f_2g_2 + \cdots + f_mg_m \in C^\infty(U).$$

In particular, if $f \in C^\infty(M)$, then $df(X)$ is the function $df(X)(p) = d_pf(X_p)$. In local coordinates, we have

$$df = \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_m}dx_m$$

where $\frac{\partial f}{\partial x_i}$ is the smooth function given by the vector field $\frac{\partial}{\partial x_i}$ acting on f, or equivalently, the smooth function

$$\frac{\partial f}{\partial x_i}(p) = d_pf\left(\frac{\partial}{\partial x_i}|_p\right) = \frac{\partial f \circ \phi^{-1}}{\partial x_i}|_{\phi(p)}$$

where $\phi = (x_1, ..., x_m)$ is the coordinate chart.

Now, on the vector bundles TM and T^*M, we can carry out the various vector bundle operations to get the vector bundles $\bigwedge^k TM$, $\bigwedge^k T^*M$, $\bigotimes^k TM$, and $\bigotimes^k T^*M$.

Definition.

1. Smooth sections of $\bigwedge^k TM$ are called k-*vector fields*. This space is also denoted by $\mathfrak{X}^k(M)$.
2. Smooth sections of $\bigwedge^k T^*M$ are called *differential* k-*forms*. This space is also denoted by $\Omega^k(M)$. We denote the exterior algebra $\bigoplus_{i\geq 0} \Omega^i(M)$ by $\Omega(M)$. We have $\Omega^0(M) = C^\infty(M)$.
3. More generally, smooth sections of vector bundles of the form $\bigwedge^k TM \otimes S^\ell TM$, $\bigwedge^k TM \otimes S^\ell T^*M$, $\bigwedge^k T^*M \otimes S^\ell TM$, or $\bigwedge^k T^*M \otimes S^\ell T^*M$ are called *tensor fields.*

Here is one way to think of differential forms: a differential k-form ω is a skew-symmetric $C^\infty(M)$-linear map

$$\omega : \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \times \cdots \times \mathfrak{X}^1(M) \to C^\infty(M).$$

The converse also holds; any such map is induced by a differential k-form.

In local coordinates $(x_1, ..., x_m)$ on U, any differential k-form ω is written as

$$\omega = \sum_{I \subset [1,n], |I|=k} \omega_I dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

where $I = (i_1 < i_2 < \cdots < i_k)$, and the $\omega_I \in C^\infty(U)$.

For ease of notation, if α is a differential k-form then we write $|\alpha| = k$.

2.3.4 Operations on Differential Forms

Throughout this chapter, we adopt the following notation: small Greek letters α, β, ω, etc for differential forms, small Latin letters f, g, h for smooth functions, capital Greek letters Φ, Ψ for smooth maps between manifolds, and capital letters X, Y, Z for vector fields.

Throughout, we denote the multi-index I with $I \subset [1, n], |I| = k$ by $I = (i_1 < \cdots < i_k)$.

Exterior Product

For any $k, \ell \geq 0$, we have the exterior product $\wedge : \Omega^k(M) \times \Omega^\ell(M) \to \Omega^{k+\ell}(M)$ which acts point-wise. Thus, for instance, if $\alpha_i \in \Omega^1(M)$ and $X_i \in \mathfrak{X}^1(M)$ then

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(X_1, ..., X_k) = \det\big(\alpha_i(X_j)\big) \in C^\infty(M).$$

Properties:

1. $(f\alpha + g\beta) \wedge \gamma = f\alpha \wedge \gamma + g\beta \wedge \gamma$.
2. $\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\beta \wedge \alpha$.
3. $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.
4. In particular, $\Omega(M)$ equipped with the exterior product is a graded commutative $C^\infty(M)$-algebra.

Pull-back by Smooth Maps

Suppose $\Phi : M \to N$ is smooth. Then, we have an induced homomorphism $\Phi^* : \Omega(N) \to \Omega(M)$ of graded commutative $C^\infty(M)$-algebras which sends the k-form $\omega \in \Omega^k(N)$ to the k-form $\Phi^*\omega \in \Omega^k(M)$ defined by

$$\Phi^*\omega|_p(v_1, ..., v_k) := \omega|_{\Phi(p)}\big(d_p\Phi(v_1), ..., d_p\Phi(v_k)\big) \quad \forall p \in M \forall v_1, ..., v_k \in T_pM.$$

In local coordinates $(x_1, ..., x_m)$ on M and $(y_1, ..., y_n)$ on N, and writing $y_i \circ \Phi \circ (x_1, ..., x_m)^{-1}$ as a function $\phi_i(x_1, ..., x_m)$ of the x_j, we see that Φ^* sends the differential form $\alpha = \sum_{I \subset [1,n], |I|=k} f_I dy_{i_1} \wedge \cdots \wedge dy_{i_k}$ to

$$\Phi^*\alpha = \sum_{I \subset [1,n], |I|=k} (f_I \circ \Phi) d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}.$$

In particular, if α is a *top-degree form* (i.e. $|\alpha| = \dim M$), and if $F : M^m \to N^m$ is smooth, then

$$F^*(f dy_1 \wedge \cdots \wedge dy_m) = (f \circ F) \det\left(\frac{\partial y_i \circ F}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx^n.$$

Clearly, we have $\Phi^*(f) = f \circ \Phi$ on 0-forms, $\Phi^*(df) = d(f \circ \Phi)$ on 1-forms, and more generally

$$\Phi^*(\alpha + \beta) = \Phi^*(\alpha) + \Phi^*(\beta) \quad \text{and } \Phi^*(\alpha \wedge \beta) = (\Phi^*\alpha) \wedge (\Phi^*\beta).$$

We also have $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$

Definition. Suppose G a Lie group. A differential form $\omega \in \Omega(G)$ is *left-invariant* if for any $g \in G$, we have $\ell_g^*\omega = \omega$. In other words, for any $h \in G$ and any $v_1, ..., v_k \in T_hG$ ($k = |\omega|$), we require

$$\omega_{gh}(d_h\ell_g v_1, ..., d_h\ell_g v_k) = \omega_h(v_1, ..., v_k).$$

Suppose now that $\iota : N^n \hookrightarrow M^m$ is an embedding of manifolds. We can then identify $\Omega^k(N)$ as a quotient space of $\Omega^k(M)$; indeed, any $\omega \in \Omega^k(M)$ can be pulled back by ι to get a k-form $\iota^*\omega \in \Omega^k(N)$, so that the $\mathbb{R}$-linear map ι^* induces the $\mathbb{R}$-vector space isomorphism $\Omega^k(N) \cong \Omega^k(M)/\ker\iota^*$. Clearly if $n < k \leq m$, then $\ker\iota^* = \Omega^k(M)$.

Contraction by Vector Fields

For any $X \in \mathfrak{X}^1(M)$, we have the induced map $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ defined by

$$(i_X\omega)(X_1, ..., X_{n-1}) := \omega(X, X_1, ..., X_{n-1}) \in C^\infty(M).$$

We define $i_X(f) = 0$ for all $f \in C^\infty(M) = \Omega^0(M)$. Properties:

1. If $\alpha \in \Omega^1(M)$, then $i_X(\alpha) = \alpha(X) \in C^\infty(M)$.
2. $i_{fX+gY}\omega = fi_X\omega + gi_Y\omega$.
3. $i_X(f\alpha + g\beta) = fi_X\alpha + gi_X\beta$.
4. $i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge i_X\beta$.
5. If $\Phi : M \to N$ is a smooth map and $X \in \mathfrak{X}^1(M), Y \in \mathfrak{X}^1(N)$ are Φ-related, then $\Phi^* \circ i_Y = i_X \circ \Phi^*$ as maps on $\Omega(N)$.

Example 2.3.1 (Fall 2019 Day 2)**.** Let $S^2 \subset \mathbb{R}^3$ be the unit 2-sphere with the usual orientation. Let X be the vector field generating the flow given by

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let ω be the volume form on S^2 induced by the embedding in $\mathbb{R}^3$ so that the total surface area of S^2 with respect to ω is 4π. Find a function $f : S^2 \to \mathbb{R}$ satisfying $df = \iota_X\omega$ where ι_X is contraction.

Let $j : S^2 \to \mathbb{R}^3$ denote the embedding, and $F : \mathbb{R}^3 \to \mathbb{R}$ the map $F(x, y, z) = x^2+y^2+z^2$ so that S^2 is a level set of F. Throughout, we embed TS^2 into $T\mathbb{R}^3$ via j_*, or equivalently, by identifying TS^2 with $\ker F_*$. Then, the vector field $E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ satisfies $TS^2 \oplus (S^2 \times \mathbb{R} \cdot E) \cong T\mathbb{R}^3$ since $E \notin \ker F_*$. The identification $j_* : TS^2 \hookrightarrow T\mathbb{R}^3$ induces an identification of T^*S^2 with $T^*\mathbb{R}^3$ modulo the kernel of j^*; this kernel is the space of 1-forms that vanish identically on $TS^2 \hookrightarrow T\mathbb{R}^3$, which is the image of $T^*\mathbb{R}$ via F^*.

Now, a volume form on S^2 is simply $\iota_E(dx\wedge dy\wedge dz) = xdy\wedge dz - ydx\wedge dz + zdx\wedge dy$ (of course, modulo the vector bundle $\bigwedge^2 \ker j^*$). A quick calculation in any dense coordinate patch shows that $\int_{S^2} j^*\iota_E(dx\wedge dy\wedge dz) = 4\pi$, and so $\omega = j^*\iota_E(dx \wedge dy \wedge dz)$. We can calculate X by noting that the flow $\phi_X(t, p)$ of X satisfies $d_0\phi_X(\bullet, p)(\frac{d}{dt}) = X_p$ by definition, and thus

$$X = (-x\sin t - y\cos t)|_{t=0}\frac{\partial}{\partial x} + (x\cos t - y\sin t)|_{t=0}\frac{\partial}{\partial y} + 0 \cdot \frac{\partial}{\partial z} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Hence, modulo $\ker j^*$, we have

$$\iota_X\omega = x(xdz) - y(-ydz) + z(-ydy - xdx) = (x^2 + y^2)dz - xzdx - yzdy = dz - z(xdx + ydy + zdz)$$

where we use the fact that $x^2 + y^2 + z^2 = 1$. However, notice that $F^*(dt) = 2xdx + 2ydy + 2zdz$ and so, modulo $\ker j^*$, we have $\iota_X\omega = dz$. It follows that the function $f : S^2 \to \mathbb{R}$ given by $f(x, y, z) = z$ is a smooth function satisfying $df = \iota_X\omega$. More precisely, the function $f : S^2 \to \mathbb{R}$ given by the composition $S^2 \overset{j}{\hookrightarrow} \mathbb{R}^3 \overset{z}{\to} \mathbb{R}$ satisfies $df = j^*(\iota_X\omega)$.

Exterior Differential

We define a map $d : \Omega^k(M) \to \Omega^{k+1}(M)$, in any of the following three equivalent ways:

1. (Global Definition) for any $\alpha \in \Omega^k(M)$, we have

$$(d\alpha)(X_1, ..., X_{k+1}) := \sum_{i=1}^{k+1}(-1)^{i-1}X_i\big(\alpha(X_1, ..., \hat{X}_i, ..., X_{k+1})\big) + \sum_{1\le i<j\le k+1}(-1)^{i+j}\alpha\big([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{k+1}\big).$$

2. (Local Definition) for the k-form $fdx_{i_1} \wedge \cdots \wedge dx_{i_k}$ (where $(x_1, ..., x_m)$ are local coordinates), d is defined by

$$d\big(fdx_{i_1} \wedge \cdots \wedge dx_{i_k}\big) := df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_{j\notin I}\frac{\partial f}{\partial x_j}dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and then extended linearly.

3. (Characterizing Property) The map $d : \Omega(M) \to \Omega(M)$, where $d : \Omega^k(M) \to \Omega^{k+1}(M)$, is the unique $\mathbb{R}$-linear map satisfying: (i) $(df)(X) = X(f)\ \forall X \in \mathfrak{X}^1(M)$, (ii) $d^2 = 0$, (iii) $d(\alpha\wedge\beta) = (d\alpha)\wedge\beta+(-1)^{|\alpha|}\alpha\wedge d\beta$.

We now list some important properties:

1. For $|\alpha| = 0, 1, 2$, we have $(d\alpha)(X) = X(\alpha)$, $d\alpha(X,Y) = X\big(\alpha(Y)\big) - Y\big(\alpha(X)\big) - \alpha([X,Y])$, and

$$(d\alpha)(X,Y,Z) = X\big(\alpha(Y,Z)\big) - Y\big(\alpha(X,Z)\big) + Z\big(\alpha(X,Y)\big) - \alpha([X,Y],Z) + \alpha([X,Z],Y) - \alpha([Y,Z],X)$$

respectively. We can rewrite the second condition in a neater way:

$$(d\alpha)(X,Y,Z) = X\big(\alpha(Y,Z)\big) + Y\big(\alpha(Z,X)\big) + Z\big(\alpha(X,Y)\big) - \alpha([X,Y],Z) - \alpha([Z,X],Y) - \alpha([Y,Z],X);$$

here we see some cycling of inputs.

2. If $\Phi : M \to N$ and $\alpha \in \Omega(N)$, then $d(\Phi^*\alpha) = \Phi^*(d\alpha)$.

Definition. $\alpha \in \Omega(M)$ is said to be *closed* if $d\alpha = 0$; it is said to be *exact* if there exists $\beta \in \Omega^{|\alpha|-1}(M)$ such that $\alpha = d\beta$. Since $d^2 = 0$, all exact forms are closed.

The k'th *de Rham Cohomology group* $H^k(M,\mathbb{R})$ is the quotient vector space

$$H^k(M,\mathbb{R}) = H^k_{dR}(M,\mathbb{R}) := \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))},$$

i.e. it is the space of all closed k-forms modulo the space of all exact k-forms. Obviously, $H^k(M,\mathbb{R}) = 0$ for $k < 0$ or for $k > \dim M$. It is known that $H^k(M,\mathbb{R})$ is a homotopy invariant, and in particular, is invariant under homeomorphisms.

Any smooth map $F : M \to N$ induces a pull-back map $F^* : H^k(N,\mathbb{R}) \to H^k(M,\mathbb{R})$ by $F^*([\omega]) = [F^*\omega]$. This pull-back map respects composition, and the pull-back of the identity map is the identity.

Example 2.3.2. $H^0(M,\mathbb{R}) = \mathbb{R}^d$, where d is the number of connected components of M. In particular, if M is connected, then $H^0(M,\mathbb{R}) = \mathbb{R}$ (the space of constant functions on M).

More generally, if $M_1, ..., M_d$ are the connected components of M, then $H^k(M,\mathbb{R}) = \prod_{i=1}^d H^k(M_i,\mathbb{R})$ for all $k \in \mathbb{Z}$.

Example 2.3.3. $H^1(S^1,\mathbb{R}) = \mathbb{R}$, where the basis is the cohomology class of the closed form $\omega = xdy - ydx$.

Example 2.3.4. Suppose G is a Lie group, with $\iota : G \to G$ the inversion map, $L_g : G \to G$ the left multiplication, and $R_g : G \to G$ the right multiplication. A differential form ω is *bi-invariant* if $L_g^*\omega = \omega = R_g^*\omega$. We show that $\iota^*\omega = (-1)^k\omega$ for all bi-invariant k-forms ω, and that all bi-invariant forms are closed.

We know that $d_e\iota = -1$. Using the identity $\iota \circ L_{g^{-1}} = R_g \circ \iota$ as well as the bi-invariance of ω, we get

$$L_{g^{-1}}^*(\iota^*\omega) = (R_g \circ \iota)^*\omega = \iota^* \circ R_g\omega = \iota^*\omega.$$

Evaluating at e, we get

$$\big(L_{g^{-1}}^*(\iota^*\omega)\big) = \omega_e(d_e\iota(\bullet), ..., d_e\iota(\bullet)) = (-1)^k\omega_e.$$

Composing both sides by L_g^*, we then get $(\iota^*\omega)_g = (-1)^k L_g^*\omega_e = (-1)^k\omega_g$. Thus $\iota^*\omega = (-1)^k\omega$. Finally,

$$(-1)^{k+1}d\omega = \iota^*(d\omega) = d(\iota^*\omega) = (-1)^k d\omega.$$

Therefore $d\omega = 0$, and hence all bi-invariant forms are closed.

A subset A of $\mathbb{R}^n$ is said to be *star-shaped* if there exists a point $p \in A$ such that for every point $q \in A$, the line segment $\{tp + (1-t)q : t \in [0,1]\} \subset A$ lies in A. For example, convex sets are star-shaped.

Theorem 2.3.5 (Poincare's Lemma). *If M is homeomorphic to a star-shaped open subset of $\mathbb{R}^n$, then $H^p_{dR}(M) = 0$ for all $p \geq 1$.*

In particular, every closed form α is locally exact*, i.e. for any $p \in M$, there exists an open neighbourhood U of p in M such that $\alpha|_U$ is exact.*

Example 2.3.6 (Fall 2021 Day 2)**.** For each of $S^2, \mathbb{RP}^2, S^1 \times S^1$, check whether all closed forms are exact, and if not, give an example of a closed non-exact form.

1. Let $N = (0,0,1)$ and $S = (0,0,-1)$ be points on S^2. Then the stereographic projection implies that $S^2\backslash\{N\}$ and $S^2\backslash\{S\}$ are both diffeomorphic to $\mathbb{R}^2$. Hence, if ω is a closed 1-form on S^2, then $\omega|_{S^2\backslash\{p\}} = df_p$ for some $f_p \in C^\infty(S^2\backslash\{p\})$ for $p \in \{N,S\}$. It follows that $d(f_N - f_S) = 0$ on $S^2\backslash\{N,S\}$. Since $S^2\backslash\{N,S\}$ is connected, it follows that $f_N - f_S = c$ on $S^2 \setminus \{N,S\}$ for some constant $c \in \mathbb{R}$. By glueing f_N and $f_S + c$ and noting that $d(f_S + c) = df_S$, we have found a smooth function $F : S^2 \to \mathbb{R}$ such that $\omega = dF$. Hence $H^1_{dR}(S^2, \mathbb{R}) = 0$.
2. Let $\pi : S^2 \to \mathbb{RP}^2$ be the projection map. Then π is a surjective local diffeomorphism. We have an induced map of graded spaces $\pi^* : \Omega(\mathbb{RP}^2) \to \Omega(S^2)$ which maps k-forms to k-forms. Suppose there is a k-form $\eta \in \ker \pi^*$; then $\pi^*(\eta)$ is the zero form on S^2. At any point $p \in \mathbb{RP}^2$, we can pick $q \in S^2$ such that $\pi(q) = p$. Then, $\pi_* : T_qS^2 \to T_p\mathbb{RP}^2$ is an isomorphism of vector spaces, and so it follows that $(\pi^*\eta)_p$ acts as the zero map on $\bigwedge^k T_qS^2 \overset{\pi_*}{\cong} \bigwedge^k T_p\mathbb{RP}^2$. Therefore $\eta_p = 0$ for all p, and so η is the zero form on $\mathbb{RP}^2$. Thus π^* is injective. Since π^* commutes with the exterior differential, it follows that π^* maps closed (resp. exact) forms on $\mathbb{RP}^2$ to closed (resp. exact) forms on S^2. It follows that π^* induces a map $\Pi : H^1(\mathbb{RP}^2, \mathbb{R}) \to H^1(S^2, \mathbb{R})$, and since π^* is injective, it follows that Π is injective. As $H^1(S^2, \mathbb{R}) = 0$ by the previous part, it follows that $H^1(\mathbb{RP}^2, \mathbb{R}) = 0$.
3. Let $\iota : S^1 \hookrightarrow \mathbb{R}^2$ be the usual inclusion map. Then, $\eta = \iota^*(xdy - ydx)$ is a global 1-form on S^1. Now, consider the diffeomorphism $S^1 \times S^1 \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$; this yields a local diffeomorphism $(\theta_1, \theta_2) : S^1 \times S^1 \to \mathbb{R}^2$, which induces local coordinates at each point. A quick calculation shows that $d\theta_1 = \eta$ locally, and so η is closed. Consider now the closed 1-form $\omega = (\eta, 0)$ on $S^1 \times S^1$. If ω is exact, then η must be exact, and so $\eta = df$ for some $f : S^1 \to \mathbb{R}$. Then, the volume form $d\theta_1 \wedge d\theta_2 = \eta \wedge d\theta_2 = d(fd\theta_2)$ is also exact, and so by Stokes Theorem $\int_{S^1\times S^1} d\theta_1 \wedge d\theta_2 = 0$, contradicting the simple calculation that
$$\int_{S^1\times S^1} d\theta_1 \wedge d\theta_2 = \int_0^{2\pi}\int_0^{2\pi} d\theta_1 d\theta_2 = 4\pi^2.$$
Hence the closed 1-form ω cannot be exact.

Lie Derivatives

Definition. If $X \in \mathfrak{X}^1(M)$ and $\alpha \in \Omega^k(M)$, then the *Lie derivative* is the k-form $L_X\alpha$ given by any of the following two (equivalent) definitions:

- (Local Definition) $(L_X\alpha)|_p := \lim_{t\to 0} \frac{1}{t}\left(\left(\phi_X(t, \bullet)\right)^* \alpha|_{\phi_X(t,p)} - \alpha|_p \right)$.
- (Global Definition) $(L_X\alpha)(X_1, ..., X_k) := X\big(\alpha(X_1, ..., X_k)\big) - \sum_{i=1}^n \alpha\big(X_1, ..., X_{i-1}, [X, X_i], X_{i+1}, ..., X_k\big)$

Example 2.3.7. Suppose $X = xy\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ and $\alpha = ydx \wedge dy + e^z dy \wedge dz$ on $\mathbb{R}^3$. We compute $L_X\alpha$ using the local definition (for sake of example). We first need to calculate $\phi_X(t,p)$, i.e. we need to solve $x' = 0$, $y' = xy$, and $z' = 1$; we get
$$\phi_X(t,(x,y,z)) = (x, ye^{tx}, z+t).$$
Then,
$$\phi_X(t,\bullet)^*\alpha|_p = ye^{xt}dx \wedge d(ye^{tx}) + e^{z+t}d(ye^{tx}) \wedge d(z+t) = ye^{2tx}dx \wedge dy + e^{t+z+tx}dy \wedge dz + tye^{t+z+tx}dx \wedge dz,$$
so that
$$L_X\alpha = \lim_{t\to 0}\left(y\frac{e^{2tx}-1}{t}dx \wedge dy + e^z\frac{e^{t+tx}-1}{t}dy \wedge dz + ye^{t+z+tx}dx \wedge dz \right) = 2xydx\wedge dy + e^z(1+x)dy\wedge dz + ye^z dx\wedge dz.$$

Properties:

1. $L_X f = X(f)$ for all $f \in C^\infty(M)$.
2. $L_X(f\alpha) = X(f)\alpha + fL_X\alpha$.
3. $L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge L_X\beta$.
4. (*Cartan's Magic Formula*) $L_X = i_X \circ d + d \circ i_X$.
5. $L_X \circ d = d \circ L$.
6. $L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X$.
7. If $\Phi : M \to N$ is a smooth map and $X \in \mathfrak{X}^1(M), Y \in \mathfrak{X}^1(N)$ are Φ-related, then $\Phi^* \circ L_Y = L_X \circ \Phi^*$ as maps on $\Omega(N)$.

2.3.5 Integration

Volume Forms and Orientations

Definition. A *volume form* on a manifold M^m is a differential m-form ω such that $\omega|_p \neq 0$ for all $p \in T_pM$.
A manifold is *orientable* if it admits a volume form.

Since $\Omega^m(M)$ is a rank 1 $C^\infty(M)$-module, and if $\mu \in \Omega^m(M)$ is a volume form, then every m-form is of the form $f\mu$ for some $f \in C^\infty(M)$, and $f\mu$ is a volume form iff f is no-where zero.

Some examples:

1. Every Lie group is orientable, and moreover we can find a left-invariant volume form. Indeed, for the Lie group G^n, if $\omega_e \in \bigwedge^n T_e^*(G)$ is non-zero, then we can define $\omega \in \Omega^n(G)$ by $\omega|_g := \ell_{g^{-1}}^*\omega_e$ for all $g \in G$. It is easy to see that, up to non-zero scalar multiples, a Lie group has a *unique* left-invariant volume form.

2. Suppose $\iota : N^n \hookrightarrow M^m$ is an embedding. Suppose there exists an n-form $\omega \in \Omega^n(M)$ such that at every point $p \in N$, the form $\omega|_p$ is non-zero on T_pN where we identify $T_pN \subset T_pM$ via $d_p\iota$. In such a case, $\mu := \iota^*\omega$ is a volume form on N.

3. We use the above example to explicitly describe a volume form on S^n. Consider the volume form $\mu' = dx_1 \wedge \cdots \wedge dx_{n+1}$ on $\mathbb{R}^{n+1}$ and the vector field $E = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$. Define

$$\omega = i_E\mu' = \sum_{i=1}^{n+1}(-1)^{i+1}x_i dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_{n+1} \in \Omega^n(\mathbb{R}^{n+1}).$$

If $\iota : S^n \to \mathbb{R}^{n+1}$ is the embedding, then the *standard volume form on* S^n is $\mu = \iota^*\omega \in \Omega^n(S^n)$.

We claim that μ is a volume form. Indeed, for any $p \in S^n$, recall that $S^n = \ker(x \mapsto \|x\|^2 - 1)$ so that T_pS^n is the kernel of the map $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}|_p \mapsto \sum_{i=1}^n 2p_i a_i$. Hence, identifying $T_pS^n \subset \mathbb{R}^{n+1}$, we see that T_pS^n is simply all vectors orthogonal to E under the standard inner product on $\mathbb{R}^{n+1}$. In particular, E_p is linearly independent of all vectors in T_pS^n, so that $\omega_p(v_1, ..., v_n) \neq 0$ for any basis $v_1, ..., v_n \in T_pS^n$. Hence $\omega_p \neq 0$ for all $p \in S^n$. Hence μ is a volume form on S^n. More generally, this argument shows that any $\alpha = \sum_{i=1}^{n+1}(-1)^{i+1}\alpha_i(x)dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_{n+1} \in \Omega^n(\mathbb{R}^{n+1})$, the n-form $\iota^*\alpha$ is a volume form on S^n iff $\sum_{i=1}^n x_i\alpha_i \equiv 0$.

4. We check that the standard volume form $\mu = \iota^*(xdy - ydx)$ on S^1 is left-invariant, so that it is also the volume form induced by the Lie group structure defined above. Consider any $p_1 = e^{2\pi i\theta_1}$ and $p_2 = e^{2\pi i\theta_2}$, so that $p_1p_2 = e^{2\pi i(\theta_1+\theta_2)} =: e^{2\pi i\theta_3}$. At p_i $(i = 2, 3)$, we have the coordinate chart (U_i, ϕ_i) centred at p_i where $\phi_i(U) = (-1, 1)$ and $\phi_i^{-1}(t) = e^{2\pi i(t+\theta_i)}$. Notice that $\iota \circ \phi_i^{-1}(t) = (\cos 2\pi(t + \theta_i), \sin 2\pi(t + \theta_i))$ so that

$$(\phi_i^{-1})^*\mu|_{U_i} = (\iota \circ \phi_i^{-1})^*(xdy - ydx) = 2\pi \cos^2 2\pi(t + \theta_i)dt + 2\pi \sin^2 2\pi(t + \theta_i)dt = 2\pi dt,$$

i.e. $\mu|_{U_i} = 2\pi d\phi_i$. Now,

$$\phi_3 \circ \ell_{p_1} \circ \phi_2^{-1}(t) = \phi_3 \circ \ell_{p_1}(e^{2\pi i(t+\theta_2)}) = \phi_3(e^{2\pi i(t+\theta_1+\theta_2)}) = t$$

and thus $d_{p_2}\ell_{p_1} : T_{p_2}S^1 \to T_{p_1p_2}S^1$ takes $\frac{\partial}{\partial\phi_2}$ to $\frac{\partial}{\partial\phi_3}$. Hence, $\ell_{p_1}^*\mu|_{U_3} = \mu|_{U_2}$. It follows that $\ell_{p_1}^*\mu = \mu$, i.e. μ is left-invariant.

5. We claim that $\mathbb{RP}^n$ is orientable iff n is odd. Let $\pi : S^n \to \mathbb{RP}^n$ be the quotient map. Since $\pi^{-1}([x]) = \{\pm x\}$, it follows that $\alpha \in \pi^*(\Omega^k(\mathbb{RP}^n))$ only if α is invariant under the action of $A : S^n \to S^n, x \mapsto -x$. On the other hand, if α is invariant under A (so that $A^*\alpha = \alpha$), then we have $\alpha|_{-x} = (d_{-x}A)^*\alpha|_x$, which along with $\pi \circ A = \pi$ implies that the n-form $\beta \in \Omega^k\mathbb{RP}^n$ given by $\beta|_{[x]} = \left((d_x\pi)^{-1}\right)^*\alpha|_x$ is well-defined. Here, $d_x\pi$ is an invertible map as π is a submersion between manifolds of the same dimension, so that π is a local diffeomorphism.

Now, if μ is the volume form on S^n defined above, we see that $A^*\mu = (-1)^{n+1}\mu$; this follows since the map $\bar{A} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, x \mapsto -x$ satisfies $\bar{A} \circ \bar{\iota} = \bar{\iota} \circ A$ where $\bar{\iota} : S^n \hookrightarrow \mathbb{R}^{n+1}$. If n is odd, then $\mu \in \pi^*(\Omega^n(\mathbb{RP}^n))$ and μ descends onto a volume form on $\mathbb{RP}^n$.

Now suppose n is even, and suppose $\omega \in \pi^*(\Omega^n(\mathbb{RP}^n))$ is arbitrary. Then, $\omega = f\mu$ for some $f \in C^\infty(S^n)$. Taking A^* on both sides, we get that $f\mu = \omega = A^*\omega = (f \circ A)A^*\mu = -(f \circ A)\mu$, i.e. $f(x) = -f(-x)$ for all $x \in S^n$. Since the map $\gamma : S^1 \to S^n$, $\gamma(x, y) = (x, y, 0, 0, ..., 0)$ is smooth, $f \circ \gamma$ induces a smooth map $f' : S^1 \to \mathbb{R}$ such that $f'(x, y) = -f'(-x, -y)$. By continuity of f', one sees that f' is zero somewhere on S^1, so that f is zero somewhere on S^n. Hence ω cannot be a volume form. Therefore $\mathbb{RP}^n$ is non-orientable if n even.

6. The product of orientable manifolds is orientable. More precisely, if M and N are orientable manifolds with volume forms μ_M and μ_N respectively, and if $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the projections onto the first and second factors respectively, then

$$\mu := \pi_M^*(\mu_M) + \pi_N^*(\mu_N)$$

is a volume form on $M \times N$.

In particular, the standard volume form $\iota^*(xdy - ydx)$ on S^1 induces a volume form μ on the compact torus T^n

Definition. Two volume forms μ_1 and μ_2 are said to *define the same orientation* if there exists $f \in C^\infty(M)$ such that $f > 0$ on M and $\mu_2 = f\mu_1$. An *orientation* on M is simply a choice of equivalence classes of volume forms that define the same orientation. An *oriented manifold* is simply a manifold with a specified orientation.

If M and N are oriented manifolds with orientations $[\mu]$ and $[\nu]$, then a smooth map $F : M \to N$ is *orientation preserving* if $[F^*\nu] = [\mu]$. On $\mathbb{R}^n$, one checks that a diffeomorphism is orientation preserving iff the determinant of its Jacobian matrix is positive.

An atlas on a manifold M is positive if the transition maps from open subsets of $\mathbb{R}^n$ to other open subsets of $\mathbb{R}^n$ preserve the *standard orientation* of $\mathbb{R}^n$ induced by the volume form $dx_1 \wedge \cdots \wedge dx_n$.

A positive atlas is said to *define the orientation* $[\mu]$ or *be positive with respect to the orientation* $[\mu]$ if for any chart (U, ϕ) in the atlas, we have $[\phi^*(dx_1 \wedge \cdots \wedge dx_n)] = [\mu|_U]$.

Clearly there are only two possible orientations that can be defined on a connected manifold M. A manifold has a positive atlas iff it is orientable, in which case there is a bijection between positive atlases and orientations. This bijection takes positive atlases to the unique orientation with respect to which the atlas is positive.

Integration

Definition. For any $\omega \in \Omega^n(M^n)$, we define the *support* $\mathrm{supp}(\omega)$ to be the closure of the set $\{p \in M : \omega|_p \neq 0\}$.

A top-degree form is *compactly supported* if its support is compact. The space of compactly supported top-degree forms is denoted by $\Omega_c^n(M)$; it is easy to see that it is a $C^\infty(M)$-submodule of $\Omega^n(M)$.

If M is compact, then clearly $\Omega_c^n(M) = \Omega^n(M)$.

Definition. Suppose M^n is an oriented manifold. We define the *integral* $\omega \in \Omega_c^n(M)$ by the following:

- If $\mathrm{supp}(\omega) \subset U$ for some positive coordinate chart (U, ϕ), then $\int_M \omega := \int_{\phi(U)} (\phi^{-1})^*(\omega)$.
- Otherwise, if $\{(U_i, \phi_i)\}$ is a (finite) cover of positive charts covering $\mathrm{supp}(\omega)$, then for any partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$, we set

$$\int_M \omega := \sum_i \int_M \rho_i \omega.$$

This integral is well-defined, i.e. independent of the choice of positive coordinate charts or partitions of unity.

Basic properties:

1. $\int_M (a\omega_1 + b\omega_2) = a \int_M \omega_1 + b \int_M \omega_2$.
2. Suppose $M = M_1 \cup M_2$ where M_1 and M_2 are open submanifolds of M. Then,

$$\int_M \omega = \int_{M_1} \omega + \int_{M_2} \omega - \int_{M_1 \cap M_2} \omega.$$

3. (*Change of Variables*) If $\Phi : M \to N$ is an orientation preserving diffeomorphism, then $\int_M \Phi^*(\omega) = \int_N \omega$. Otherwise, if Φ is not orientation preserving and N is connected, then

$$\int_M \Phi^*(\omega) = -\int_N \omega.$$

4. If C is a subset of M of *measure zero* (for instance, if C is a subset of a submanifold of dimension strictly less than $\dim M$), then $\int_C \omega = 0$ and $\int_M \omega = \int_{M \setminus C} \omega$ for any $\omega \in \Omega^n(M)$.
5. (*Stokes Theorem*) If M^n is a manifold without boundary, then $\int_M d\omega = 0$ for any $\omega \in \Omega^{n-1}(M)$. More generally, if M^n is a manifold *with boundary* ∂M (it is known that ∂M is a smooth manifold without boundary of dimension $n - 1$), then

$$\int_M d\omega = \int_{\partial M} \omega|_{\partial M}$$

for any $\omega \in \Omega^n(M)$.

2.3.6 Distributions and Foliations

Throughout we fix a smooth manifold M of dimension m.

Definition. A *distribution of rank k* is a rank k vector subbundle of TM. A *smooth distribution* is a smooth vector subbundle. We assume from now on that all distributions are smooth.

A k-form $\omega \in \Omega^k(M)$ is said to *annihilate* a distribution D if for any local sections $X_1, ..., X_k \in \Gamma(U, D)$ ($U \subset M$ open), we have $\omega(X_1, ..., X_k) = 0$.

Definition. Fix a distribution D. A non-empty immersed submanifold $N \subseteq M$ is an *integral manifold of D* if $T_pN = D_p$ (where we identify the tangent bundle of N as a subbundle of TM). A smooth distribution D is *integral* if every point of M is contained in an integral manifold of D.

A rank k distribution D is *completely integrable* if M is covered by local coordinates $(U, (x_1, .., x_m))$ such that $D|_U$ is spanned by $\frac{\partial}{\partial x_i}$, $1 \le i \le k$, and such that the image of U under these coordinates is a cube in $\mathbb{R}^m$. Clearly completely integrable distributions are integrable.

For instance, any nowhere vanishing smooth vector field X spans a rank 1 distribution D via $D_p = \mathbb{R}X_p$. Integral manifolds for D are given by the integral curves of X. Thus, such rank 1 distributions are always integral.

Definition. A distribution D is *involutive* if the Lie bracket of any two local sections of D is also a local section of D. It is clear that integrable distributions are involutive.

Proposition 2.3.8. *The following are equivalent for any distribution D:*

1. *D is involutive;*
2. *$\Gamma(M, D)$ is a Lie subalgebra of $\Gamma(X) = \mathfrak{X}^1(M)$;*
3. *for all 1-forms $\eta \in \Omega^1(M)$ annihilating D on some open subset U, the 2-form $d\eta$ also annihilates D on U;*

Theorem 2.3.9 (Local Frobenius Theorem). *Every involutive distribution is completely integrable. In particular, involutive iff integrable.*

Example 2.3.10 (Fall 2021 Day 3). Let a_{ij}, $1 \le i \le n-1, 1 \le j \le n$ be real constants. For $1 \le i \le n-1$ set

$$X_i = \frac{\partial}{\partial x_i} + \left(\sum_{j=1}^{n} a_{ij} x_j \right) \frac{\partial}{\partial x_n} \in \mathfrak{X}^1(\mathbb{R}^n).$$

Let Π be the rank $n-1$ distribution spanned by $X_i, 1 \le i \le n-1$. Determine necessary and sufficient conditions on a_{ij} so that Π is integrable.

By Frobenius' Theorem, it suffices to check when $[X_i, X_j] \in \Pi$. We have

$$\left[\frac{\partial}{\partial x_i} + \left(\sum_{k=1}^{n} a_{ik} x_k \right) \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_j} + \left(\sum_{h=1}^{n} a_{jh} x_h \right) \frac{\partial}{\partial x_n} \right] = \left(a_{ji} - a_{ij} + \sum_{k=1}^{n} (a_{ik} a_{jn} - a_{jk} a_{in}) x_k \right) \frac{\partial}{\partial x_n}.$$

Since all of the coefficients of $\frac{\partial}{\partial x_i}$ are 0, it follows that Π is integrable iff $[X_i, X_j] = 0$ for all i, j. Comparing coefficients, it follows that $a_{ij} = a_{ji}$ and $a_{ik}a_{jn} = a_{jk}a_{in}$ for all $1 \le i, j \le n-1$ and $1 \le k \le n$. We then see immediately that a necessary and sufficient condition for the integrability of Π is that $(a_{ij})_{1 \le i,j \le n-1}$ is a symmetric matrix and the vectors $\mathbf{a}_i = (a_{ik})_{1 \le k \le n}$ are all pair-wise linearly dependent.

Definition. Let $\mathcal{F}$ be a collection of k-dimensional submanifolds of M. A chart $(U, \varphi = (x_1, ..., x_m))$ is *flat for* $\mathcal{F}$ if $\varphi(U)$ is a cube in $\mathbb{R}^m$, and for all $N \in \mathcal{F}$ either $N \cap U = \emptyset$ or $N \cap U$ is a countable union of k-dimensional slices $\{x_i = c_i : k+1 \le i \le n\}$ for some $c_i \in \mathbb{R}$.

A *foliation* of dimension k on M is a collection $\mathcal{F}$ of disjoint connected non-empty immersed k-dimensional submanifolds of M whose union is M, and such that each point in M is covered by a flat chart for $\mathcal{F}$. The elements of $\mathcal{F}$ are called *the leaves of the foliation.*

Examples:

- Collection of all k-dimensional affine subspaces of $\mathbb{R}^n$ parallel to $\mathbb{R}^k \times \{0\}$ is a k-dimensional foliation. More generally, if M and N are connected smooth manifolds, then the collection of $M \times \{q\}$, $q \in N$, is a $\dim M$ foliation for $M \times N$ whose leaves are all diffeomorphic to M.

- Collection of open rays $\{\lambda x : \lambda > 0\}$ is a 1-dimensional foliation for $\mathbb{R}^n \setminus \{0\}$.
- Collection of all spheres centered at 0 is an $n-1$ dimensional foliation for $\mathbb{R}^n \setminus \{0\}$.

Proposition 2.3.11. *If $\mathcal{F}$ is a foliation on M, then the collection of tangent spaces to the leaves of $\mathcal{F}$ forms an involutive distribution on M.*

Theorem 2.3.12 (Global Frobenius Theorem)**.** *Let D be any involutive distribution on a smooth manifold M. The collection of all maximal connected integral manifolds of D forms a foliation of M.*

2.4 Riemannian Geometry

2.4.1 Metrics On Vector Bundles

Definition. A *Riemannian metric on a vector bundle* $\pi : E \to M$ is an assignment g that to each $p \in M$ gives an inner product $g_p(\bullet, \bullet)$ on each of the vector spaces E_p such that, for any two smooth sections $s, t : M \to E$, the function $g(s,t) : M \to \mathbb{R}, p \mapsto g_p(s_p, t_p)$ is smooth. A vector bundle endowed with a Riemannian metric is called a *Riemannian bundle*. Notice that g can be considered as a smooth section of S^2E^* that is positive definite; from this point of view, g is often called the *first fundamental form of the Riemannian bundle*. A Riemannian metric is also often denoted by $\langle , \rangle_g$ or $\langle , \rangle$ if the metric is fixed (i.e. g understood). The norm associated to the inner product $\langle , \rangle_g |_p$ is denoted by $\|.\|_g|_p$ or simply $\|.\|_p$.

If $\Phi : E \to F$ is an injective bundle map over M, and if F is a Riemannian bundle with Riemannian metric g, then we can define a pull-back tensor $\Phi^* g$ on E given by $(\Phi^* g)|_p(s_p, t_p) = g|_p(\Phi(s_p), \Phi(t_p))$ for all $s_p, t_p \in E_p$.

Using partitions of unity subordinate to local trivializations, it is possible to construct Riemannian metrics on any vector bundle.

Example 2.4.1. On the trivial bundle $M \times \mathbb{R}^r$, the inner product $\langle , \rangle$ on $\mathbb{R}^r$ induces a Riemannian metric on $M \times \mathbb{R}^r$ in the obvious way.

Definition. A *Riemannian metric on a manifold M* is a Riemannian metric on the tangent bundle TM.

Suppose $U \subset M$ open gives a local trivialization of $E \to M$. Let $E|_U \cong U \times \mathbb{R}^r$ be spanned by the sections $e_1, ..., e_r$, and let $e^1, ..., e^r$ be the dual basis to $e_1, ..., e_r$ (these are sections of $E^*|_U$). For each $\alpha, \beta \in [1, r]$, let $g_{\alpha\beta} \in C^\infty(U)$ such that $g_{\alpha\beta}(p) = \langle e_\alpha|_p, e_\beta|_p \rangle |_p$. Then, we see that

$$g|_U = \sum_{\alpha,\beta} g_{\alpha\beta} e^\alpha \otimes e^\beta,$$

where the matrix $(g_{\alpha\beta})$ is a positive definite symmetric matrix. By the *Gram-Schmidt process*, we can find an orthonormal frame of sections on U, i.e. we can choose $e_1, ..., e_r$ such that $g_{\alpha\beta} \equiv \delta_{\alpha\beta}$.

In particular, if $(x_1, ..., x_m)$ are local coordinates on a Riemannian manifold M, then the Riemannian metric g on M can be written as

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$$

with $g_{ij} \in C^\infty(U)$ and (g_{ij}) a positive definite matrix. If we choose local coordinates for which $\{\frac{\partial}{\partial x_i}\}$ is an orthonormal frame, then we write

$$g = (dx^1)^2 + \cdots + (dx^m)^2.$$

Definition. Suppose M and $\tilde{M}$ are Riemannian manifolds with Riemannian metrics g and $\tilde{g}$. Then, an *isometry* from M to $\tilde{M}$ is a diffeomorphism $\Phi : M \to \tilde{M}$ such that $\Phi^* \tilde{g} = g$. In other words, Φ is an isometry if it is a smooth bijection from M to $\tilde{M}$ such that each differential map $d_p\Phi : T_pM \to T_{\Phi(p)}\tilde{M}$ is a linear isometry, i.e. $\tilde{g}_{\Phi(p)}(d_p\Phi(v), d_p\Phi(w)) = g_p(v, w)$ for all $v, w \in T_pM$.

A *local isometry* is a smooth open map $\varphi : M \to \tilde{M}$ such that for each $p \in M$, there exists a neighbourhood U of p in M such that $\varphi|_U : U \to \varphi(U) \subset \tilde{M}$ is an isometry.

A *conformal diffeomorphism* is a diffeomorphism $\Phi : M \to \tilde{M}$ such that $\Phi^* \tilde{g} = fg$ where $f \in C^\infty(M)$ is everywhere positive.

Definition. For a Riemannian manifold (M^m, g), the *unit tangent bundle UTM* is the sub-bundle of TM consisting of $v_p \in T_pM$ such that $|v_p|_g = 1$. Since $\|.\|_g : TM \to M \times \mathbb{R}$ is a smooth map and UTM is the pre-image of $M \times \{1\}$, it follows that UTM is a $2m-1$ dimensional embedded submanifold of TM.

Proposition 2.4.2. *UTM is connected iff M is connected. UTM is compact iff M is compact.*

Proposition 2.4.3. *Suppose $F : M \to \tilde{M}$ is a smooth map, and $(\tilde{M}, \tilde{g})$ is a Riemannian manifold. Let $g = F^*\tilde{g}$ be the smooth two tensor field. Then, g is a Riemannian metric iff F is an immersion.*

In particular, any immersed/embedded submanifold of a Riemannian manifold has an induced metric*. With this metric, the submanifold is called a* Riemannian submanifold*.*

Definition. Suppose $F : (M, g) \to (\tilde{M}, \tilde{g})$ is a smooth map of Riemannian manifolds. If F is an immersion (resp. embedding) such that $F^*\tilde{g} = g$, then F is said to be an isometric immersion (resp. isometric embedding).

Definition. Suppose $(\tilde{M}, \tilde{g})$ are Riemannian manifolds, and suppose M is an immersed submanifold. Identify TM as a submanifold of $T\tilde{M}$. Since $\langle , \rangle_{\tilde{g}}$ gives an inner product on $T_p\tilde{M}$ for $p \in M$, we can talk about the orthogonal complement $T_pM^\perp \subset T_p\tilde{M}$. The *normal bundle* $NM = \bigsqcup_{p\in M}(T_pM)^\perp$ is the smooth rank $(n-m)$-vector subbundle of $T\tilde{M}$.

We now construct examples of Riemannian manifolds.

1. $\mathbb{R}^n$ is a Riemannian manifold, with the metric the usual Euclidean inner product on $T_p\mathbb{R}^n = \mathbb{R}^n$.

2. Consider the n-sphere S^n. The inclusion embedding $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ pulls back the standard Riemannian metric to obtain a Riemannian metric g° on S^n. This metric is called the *round metric* or *standard metric* on S^n. The stereographic projection from $S^n \backslash p$ to $\mathbb{R}^n$ is a conformal diffeomorphism with S^n equipped with the round metric.

   We see that NS^n is the vector sub-bundle of $T\mathbb{R}^{n+1}$ spanned by the global vector field $\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$.

3. (Hyperbolic space) Equip the standard unit ball B^n with the metric

$$g_{B^n} = \frac{4}{(1 - x_1^2 - \cdots - x_n^2)^2}\left((dx^1)^2 + \cdots + (dx^n)^2\right)$$

   where $(x_1, ..., x_n)$ is the standard coordinates on $B^n \subset \mathbb{R}^n$. This is diffeomorphic to the open upper-half space $U^n = \{(x_1, ..., x_{n-1}, y) : y > 0\} \subset \mathbb{R}^n$ endowed with the metric

$$g_{U^n} = \frac{1}{y^2}\left((dx^1)^2 + \cdots + (dx^{n-1})^2 + (dy)^2\right).$$

   Either (B^n, g_{B^n}) or (U^n, g_{U^n}) is *hyperbolic space.*

4. Suppose G is a Lie group. A Riemannian metric g on G is *left-invariant* if $\ell_a^* g = g$ for all $a \in G$. A Riemannian metric g on G is left-invariant iff for any $x, y \in \mathfrak{g}$, the function $g(x^\ell, y^\ell) \in C^\infty(G)$ is a constant function. Thus, the restriction map $g \mapsto g_e \in S^2(T_e^*G)$ along with the identification $T_eG = \mathfrak{g}$ yields a bijection between left-invariant Riemannian metrics on G and inner products on $\mathfrak{g}$.

2.4.2 Connections

Definition. Suppose $\pi : E \to M$ is a vector bundle. A *connection on E* is a map

$$\nabla : \mathfrak{X}^1(M) \times \Gamma(M, E) \to \Gamma(M, E),$$

written $(X, s) \mapsto \nabla_X s$, satisfying:

- $\nabla_X s$ is $C^\infty(M)$-linear in X for a fixed s,
- $\nabla_X s$ is $\mathbb{R}$-linear in s for a fixed X,
- For any $f \in C^\infty(M)$, we have $\nabla_X(fs) = f\nabla_X s + (Xf)s$.

For fixed $X \in \mathfrak{X}^1(M)$, the operator $\nabla_X : \Gamma(M, E) \to \Gamma(M, E)$ is called the *covariant derivative with respect to X*.

A connection on TM is called an *affine connection* on M.

Lemma 2.4.4. *Suppose $X, X' \in \mathfrak{X}^1(M)$ and $s, s' \in \Gamma(M, E)$, and let $p \in M$. If $X_p = X'_p$ and if there exists an open neighbourhood U of p in M such that $s|_U = s'|_U$, then $(\nabla_X s)|_p = (\nabla_{X'} s')|_p$.*

Suppose we have a local trivialization U of E with basis $(e_1, ..., e_r)$ such that (WLOG) U is also a coordinate patch on M with coordinates $(x_1, ..., x_n)$. Write $X = \sum_i X^i \frac{\partial}{\partial x_i}$ and $s = \sum_{\alpha=1}^r s^\alpha e_\alpha$ for $X^i, s^\alpha \in C^\infty(M)$. Then, we have

$$\nabla_X s = \sum_{i,\alpha} X^i \left(\frac{\partial s^\alpha}{\partial x_i} e_\alpha + s^\alpha \nabla_{\partial x_i} e_\alpha \right).$$

Here, we use the short-hand notation $\partial x_i = \frac{\partial}{\partial x_i}$. Hence, it suffices to know $\nabla_{\partial x_i} e_\alpha$ for a fixed local trivialization and fixed local coordinates.

Definition. In a fixed coordinate patch with local coordinates $U, (x_1, ..., x_n)$ such that $E|_U$ is trivial with local basis $e_1, ..., e_r$, the *connection coefficients* of the connection ∇ on E_U are the smooth functions $\Gamma^k_{i\alpha}$ given by

$$\nabla_{\partial/\partial x_i} e_\alpha = \sum_\beta \Gamma^\beta_{i\alpha} e_\beta.$$

Lemma 2.4.5. *If $X \in \mathfrak{X}^1(M)$ and $s \in \Gamma(E)$ with $X = \sum_i X^i \frac{\partial}{\partial x_i}$ and $s = \sum_\alpha s^\alpha e_\alpha$ in the local coordinates/trivialization $(x_1, ..., x_n)$, then*

$$\nabla_X s = \sum_\beta \left(X(s^\beta) + \sum_{i,\alpha} X^i s^\alpha \Gamma^\beta_{i\alpha} \right) e_\beta.$$

Suppose we fix $X \in \mathfrak{X}^1(M)$ and suppose ∇ is an affine connection on M. Then ∇_X is a covariant derivative on sections of TM. We can extend this to define the covariant derivative (with respect to X) on sections of $\bigoplus^k TM \oplus \bigoplus^\ell T^*M$ for arbitrary k, ℓ.

Proposition 2.4.6. *For a fixed affine connection ∇ on M, there is a uniquely define a connection ∇ on all of the tensor bundles $\bigoplus^k TM \oplus \bigoplus^\ell T^*M$ extending the affine connection, characterized by the following properties:*

1. *$\nabla_X f = X(f)$ for any $f \in C^\infty(M)$.*
2. *$(\nabla_X \omega)(Y) := X(\omega(Y)) - \omega(\nabla_X Y)$ for any $\omega \in \Omega^1(M)$.*
3. *$\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$*

In local coordinates $(x_1, ..., x_k)$, we have

$$\nabla_{\frac{\partial}{\partial x_i}} dx^j = -\sum_k \Gamma^j_{ik} dx^k.$$

In particular, we see that ∇ itself defines a map from $\bigoplus^k TM \oplus \bigoplus^\ell T^*M$ to $\bigoplus^k TM \oplus \bigoplus^{\ell+1} T^*M$, where for a given $F \in \bigoplus^k TM \oplus \bigoplus^\ell T^*M$ the affine connection sends F to the map $X \mapsto \nabla_X F \in \bigoplus^k TM \oplus \bigoplus^\ell T^*M$ for any $X \in \mathfrak{X}^1$.

2.4.3 Geodesics and Parallel Transport

Definition. Suppose M is a manifold and $\gamma : I \to M$ a fixed curve. A *(smooth) vector field along* γ is a smooth map $V : I \to TM$ such that $V_t \in T_{\gamma(t)}M$. The $\mathcal{C}^\infty(M)$-module of vector fields along γ is sometimes denoted by $\mathfrak{X}^1(\gamma)$.

A smooth vector field V along γ is *extendible* if there exists a vector field $\tilde{V}$ on a neighbourhood of $\mathrm{Im}(\gamma)$ such that $V = \tilde{V} \circ \gamma : I \to TM$.

Theorem 2.4.7. *Suppose ∇ is an affine connection on M. For each smooth curve $\gamma : I \to M$, the connection determines a unique linear operator $D_{\gamma'} : \mathfrak{X}^1(\gamma) \to \mathfrak{X}^1(\gamma)$, the* covariant derivative along γ, *such that $D_{\gamma'}(fV) = f'V + fD_{\gamma'}V$ for $f \in C^\infty(I)$ and such that if V is extendible, then for any extension $\tilde{V}$ of V we have $(D_{\gamma'}V)_t = \nabla_{\gamma'(t)}\tilde{V}$.*

Definition. Suppose ∇ is an affine connection on M. A *geodesic (with respect to ∇)* is a smooth curve $\gamma : I \to M$ such that $D_{\gamma'}\gamma' = 0$, where notice that $\gamma' \in \mathfrak{X}^1(\gamma)$.

Definition. The *arc-length* function of a smooth curve $\gamma : [a, b] \to M$ is the function $s = s_\gamma : [a, b] \to \mathbb{R}$ given by $s(t) = \int_a^t |\gamma'(t)|_g dt$. It is a smooth function such that $s'(t) = |\gamma'(t)|_g$.

A smooth curve is said to be *parametrized by arc-length* if its corresponding arc-length function is simply $s : [0, b] \to \mathbb{R}, s(t) = t$.

Lemma 2.4.8. *For any smooth curve $\gamma : [a,b] \to M$ such that $|\gamma'|_g \neq 0$ everywhere, there exists a unique* arc-length forward reparametrization*, i.e. there exists a unique diffeomorphism $\varphi : [0, b-a] \to [a,b]$ such that $\varphi(a) = 0$, $\varphi(b) = b-a$, and such that $\gamma \circ \varphi$ is parametrized by arc-length.*

Theorem 2.4.9. *For any $p \in M$ and any $v \in T_pM$, there is a unique maximal geodesic $\gamma : I \to M$ (I open interval containing 0) such that $\gamma(0) = p$ and $\gamma'(0) = v$.*

Moreover, if we remove our restriction on v, then γ can be chosen to be parametrized by arc-length.

Remark 2.4.10. Note that two geodesics that have the same image in M may still be distinct curves, since their velocity vectors may be very different (and may not even be linearly dependent).

Thus, if we want to solve for a geodesic through a point and if we are only concerned with the image in M of such a geodesic, then we can assume WLOG that our geodesic in parametrized by arc-length. This gives us a first order ODE $g(\gamma', \gamma') = 1$. This makes solving for γ easier, since the equation $\nabla_{\gamma'}\gamma' \equiv 0$ is a second order system of ODEs. Of course, any of these ODEs may be non-linear.

Definition. Suppose ∇ is an affine connection on M. A smooth vector field V along a smooth curve γ is *parallel along γ (with respect to ∇)* if $D_{\gamma'}V \equiv 0$. In particular, a geodesic is a curve whose velocity vector field is parallel along γ.

In local coordinates, $V = \sum_i V^i \frac{\partial}{\partial x_i}$ is parallel along $\gamma = (\gamma^1, ..., \gamma^m)$ ($\gamma^i = x_i \circ \gamma$) iff

$$(V^k)'(t) = -\sum_{i,j} V^j(t)(\gamma^i)'(t)\Gamma^k_{ij}(\gamma(t))$$

for all $t \in I$.

Lemma 2.4.11. *If X and Y are parallel vector fields along a curve γ, then $g(X,Y)$ is a constant function of t.*

Theorem 2.4.12. *Suppose ∇ an affine connection on M. Suppose given a smooth curve $\gamma : I \to M$ (with $0 \in I$) and a vector $v \in T_{\gamma(t_0)}M$. Then, there exists a unique vector field V along γ, called the* parallel transport *of v along γ, such that $V|_0 = v$.*

2.4.4 Levi-Cevita Connection

Definition. Suppose that (E, g) is a Riemannian bundle over M, and suppose ∇ is a connection on E. We say that ∇ is *compatible with the metric*, or a *metric connection*, if

$$X\big(g(s,t)\big) = g(\nabla_X s, t) + g(s, \nabla_X t).$$

Suppose ∇ is now an affine connection on a Riemannian manifold (M, g). Then ∇ is a *metric connection on M* if it is a metric connection on TM. Since a Riemannian metric g can be considered as a symmetric 2-tensor, and since connections can be extended to arbitrary tensor fields, the tensor field ∇g is well-defined. In this case, ∇ is a metric connection iff $\nabla g \equiv 0$.

Definition. Given an affine connection ∇ on M, the *torsion tensor (with respect to ∇)* $T = T_\nabla : \mathfrak{X}^1(M) \times \mathfrak{X}^1(M)$ is the map

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

The affine connection ∇ is *symmetric* or *torsion-free* if $T_\nabla \equiv 0$, i.e. $\nabla_X Y - \nabla_Y X = [X,Y]$ for all $X, Y \in \mathfrak{X}^1(M)$. In terms of local coordinates, an affine connection is symmetric iff its connection coefficients satisfy $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Theorem 2.4.13 (Fundamental Theorem of Riemannian Geometry)**.** *On any Riemannian manifold, there exists a unique torsion-free metric affine connection called the* Levi-Cevita Connection *of g. Moreover, we have the following formulae:*

1. *(Koszul's Formula)* $2g(\nabla_X Y, Z) = X\big(g(Y,Z)\big) + Y\big(g(Z,X)\big) - Z\big(g(X,Y)\big) - g(Y,[X,Z]) - g(Z,[Y,X]) + g(X,[Z,Y])$.

2. *Consider local coordinates $U, (x_1, ..., x_n)$, where we write $g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$ with $g_{ij} \in C^\infty(U)$ and $X = \sum_i X^i \frac{\partial}{\partial x_i}, Y = \sum_j Y^j \frac{\partial}{\partial x_j}$. Let (g^{ij}) be the inverse matrix of (g_{ij}). The connection coefficients of the Levi-Cevita connection, the* Riemann-Christoffel symbols*, are given by*

$$\Gamma^k_{ij} = \frac{1}{2}\sum_\ell g^{k\ell}\left(\frac{\partial g_{j\ell}}{\partial x_i} + \frac{\partial g_{i\ell}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_\ell}\right).$$

Remark 2.4.14. Rather than computing the Christoffel symbols directly, we can instead use the *Euler-Lagrange equation*. Here, set $L := \sum_{i,j} g_{ij} x_i' x_j'$ (where $t \mapsto (x_1(t), ..., x_n(t))$ is a local geodesic). Then, the Euler Lagrange equation states that

$$\frac{\partial L}{\partial x_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial x_k'}\right)$$

for all k (where for the purposes of differentiating with respect to x_k or x_k' in the Euler-Lagrange equation, we assume x_k and x_k' are independent variables). After simplifying, the Euler-Lagrange equation reduces to simply

$$x_k'' + \sum_{i,j} \Gamma_{i,j}^k x_i' x_k' = 0.$$

The Christoffel symbols can now be simply read off from this equation. This also gives us the geodesic equation, i.e. if $(x_1(t), ..., x_n(t))$ is a geodesic on M then it must satisfy

$$x_k'' + \sum_{i,j} \Gamma_{i,j}^k x_i' x_k' = 0.$$

As an example, if $g = e^x dx^2 + dy^2$, then $L = e^x (x')^2 + (y')^2$, and so

$$e^x (x')^2 = \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial x'}\right) = \frac{d}{dt} 2e^x x' = 2e^x x'' + 2e^x (x'^2).$$

Thus we get $x'' + \frac{1}{2}(x')^2 = 0$, and so we immediately see that $\Gamma_{xx}^x = \frac{1}{2}$, and $\Gamma_{xy}^x = \Gamma_{yx}^x = 0 = \Gamma_{yy}^x$.

Definition. The Levi-Cevita connection induces a metric space structure on a connected manifold M, with metric

$$d_g(p,q) := \inf\left\{\int_a^b |\gamma'(t)|_g dt : \gamma : [a,b] \to M \text{ piece-wise continuously differentiable}, \gamma(a) = p, \gamma(b) = q\right\}.$$

Definition. A Riemannian metric is said to be *(geodesically) complete* if the maximal geodesic at a point p has domain $\mathbb{R}$.

Theorem 2.4.15 (Hopf-Rinow)**.** *Suppose (M, g) is a Riemannian manifold with the above metric. The following are equivalent:*

1. *(M, d_g) is a complete metric space;*
2. *M is geodesically complete;*
3. *every closed and bounded subset of M is compact*

Moreover, for any $p, q \in M$ there exists a geodesic $\gamma : [a,b] \to M$ such that $\gamma(a) = p, \gamma(b) = q$, and for any $s, t \in [a,b]$ we have $d_g(\gamma(s), \gamma(t)) = |s - t|$.

Example 2.4.16 (Spring 2020 Day 2)**.** Consider the metric $g = (1 + x^2)dx^2 + dy^2$ on $M = \mathbb{R}^2$. We have $(g_{ij}) = \begin{pmatrix} 1+x^2 & 0 \\ 0 & 1 \end{pmatrix}$ and $(g^{ij}) = \begin{pmatrix} \frac{1}{1+x^2} & \\ & 1 \end{pmatrix}$. Thus the only non-zero Riemann-Christoffel symbol is

$$\Gamma_{xx}^x = \frac{1}{2}\frac{1}{1+x^2}(2x + 2x - 2x) = \frac{x}{x^2+1}.$$

In other words, $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \frac{x}{1+x^2}\frac{\partial}{\partial x}$ while $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0 = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}$.

We now calculate the parallel transport V of $(a,b) \in T_{(0,0)}\mathbb{R}^2 = \mathbb{R}^2$ along $\gamma(t) = (t,t)$. Notice that $\gamma'(t) = \frac{\partial}{\partial x}|_{(t,t)} + \frac{\partial}{\partial y}|_{(t,t)}$. Let $V(t) = a(t)\frac{\partial}{\partial x}|_{(t,t)} + b(t)\frac{\partial}{\partial y}|_{(t,t)}$ where a, b smooth and $a(0) = a, b(0) = b$. Then,

$$\begin{aligned}\nabla_{\gamma'} V &= \nabla_{\gamma'} a(t)\frac{\partial}{\partial x} + \nabla_{\gamma'} b(t)\frac{\partial}{\partial y} \\ &= a'(t)\frac{\partial}{\partial x} + a(t) \cdot \frac{t}{1+t^2}\frac{\partial}{\partial x} + b'(t)\frac{\partial}{\partial y}.\end{aligned}$$

Since $\nabla_{\gamma'} V = 0$, we have $(1 + t^2)a'(t) + ta(t) = 0$ and $b'(t) = 0$. Thus $b(t) = b$, and $a(t) = \frac{a}{\sqrt{1+t^2}}$, and hence $V(t) = a(1 + t^2)^{-1/2}\frac{\partial}{\partial x}|_{(t,t)} + b\frac{\partial}{\partial y}|_{(t,t)}$.

To check whether γ is a geodesic, we need to compute $\nabla_{\gamma'}\gamma'$, where $\gamma'(t) = \frac{\partial}{\partial x}|_{(t,t)} + \frac{\partial}{\partial y}|_{(t,t)}$. Since $\nabla_{\gamma'}\gamma' = \frac{t}{1+t^2}\frac{\partial}{\partial x}|_{(t,t)}$ is non-zero, it follows that γ is not a geodesic.

Finally, we claim that there are no parallel vector fields X, Y to the curve γ such that $g(X(t), Y(t)) = 2t$. Indeed, if such a pair exists, then taking $\nabla_{\gamma'}$ on both sides and noticing that $\nabla_{\gamma'}X = 0 = \nabla_{\gamma'}Y$, we have

$$2 = \nabla_{\gamma'}2t = \nabla_{\gamma'}g(X,Y) = g\big(\nabla_{\gamma'}X, Y\big) + g\big(X, \nabla_{\gamma'}Y\big) = 0,$$

an obvious contradiction.

Example 2.4.17 (Fall 2020 Day 1)**.** Consider the embedding $\Phi : S = S^1 \times (0,\infty) \to \mathbb{R}^3$ by $\Phi(e^{iu}, v) = (v\cos u, v\sin u, f(v))$. Throughout we use the local coordinates (u,v), where u is the local angular coordinate on S^1 and v the global coordinate on $(0,\infty)$. We have

$$\Phi^*(dx) = -v\sin u du + \cos u dv, \quad \Phi^*(dy) = v\cos u du + \sin u dv, \quad \text{and} \quad \Phi^*(dz) = f'(v)dv.$$

Hence, the pull-back metric g on S from $g_{Euc} = dx^2 + dy^2 + dz^2$ is

$$\begin{aligned} g &= v^2\sin^2 u du^2 - v\sin u\cos u dv du - v\sin u\cos u du dv + \cos^2 u dv^2 + v^2\cos^2 du^2 + \sin^2 u dv^2 \\ &\qquad + v\cos u\sin u du dv + v\cos u\sin u dv du + f'(v)^2 dv^2 \\ &= v^2 du^2 + (1 + f'(v)^2)dv^2. \end{aligned}$$

Thus $g_{uu} = v^2, g_{uv} = 0 = g_{vu}$, and $g_{vv} = 1 + f'(v)^2$. We now calculate the Levi-Cevita connection ∇ on S. To calculate the Riemann-Christoffel symbols, we use the hint. We have $E(u,v) = v^2, F \equiv 0$, and $G(u,v) = 1 + f'(v)^2$. We have $(EG - F^2) = v^2(1 + f'(v)^2)$, $E_u = 0 = F_u = F_v = G_u$, $E_v = 2v$, and $G_v = 2f'(v)f''(v)$. Thus

$$\begin{aligned} \Gamma^u_{uu} &= \frac{GE_u - 2F_u + FE_v}{2(EG - F^2)} = 0 & \Gamma^v_{uu} &= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)} = -\frac{v}{1 + f'(v)^2} \\ \Gamma^u_{uv} &= \frac{GE_v - FG_u}{2(EG - F^2)} = \frac{1}{v} & \Gamma^v_{uv} &= \frac{EG_u - FE_v}{2(EG - F^2)} = 0 \\ \Gamma^u_{vv} &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)} = 0 & \Gamma^v_{vv} &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)} = \frac{f'(v)f''(v)}{1 + f'(v)^2}. \end{aligned}$$

Consider the curve $\gamma(t) = (u(t), v(t))$. We have $\gamma' = u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}$, and so γ is a geodesic iff

$$0 = \nabla_{\gamma'}\gamma' = u''\frac{\partial}{\partial u} - \frac{(u')^2 v}{1 + f'(v)^2}\frac{\partial}{\partial v} + \frac{u'v'}{v}\frac{\partial}{\partial u} + v''\frac{\partial}{\partial v} + \frac{v'u'}{v}\frac{\partial}{\partial u} + \frac{(v')^2 f'(v)f''(v)}{1 + f'(v)^2}\frac{\partial}{\partial v},$$

i.e. iff u, v satisfy the differential equations

$$u'' + 2\frac{u'v'}{v} = 0 \quad \text{and} \quad v'' - \frac{(u')^2 v}{1 + f'(v)^2} + \frac{(v')^2 f'(v)f''(v)}{1 + f'(v)^2} = 0.$$

First consider the curve $u \equiv \alpha$ for some $\alpha \in [0, 2\pi]$, i.e. $\gamma(t) = (\alpha, v(t))$ for some smooth function v that takes on all values in $(0,\infty)$. Then $u' \equiv 0$ and the first equation is trivially satisfied. On the other hand, the second equation becomes

$$(1 + f'(v)^2)v'' + f'(v)f''(v)(v')^2 = 0.$$

Multiplying by $2v'$, we see that $(1 + f'(v)^2)\cdot 2v'v'' + 2f'(v)f''(v)v'\cdot(v')^2 = 0$, and so $(1 + f'(v)^2)(v')^2 = C$ where C is a constant of our choice. By the fundamental theorem of ODEs, noting that f is smooth, such a smooth v always exists. By picking $C \neq 0$, we force $v' \neq 0$ so that v is non-constant, regardless of the value of α. Hence $u = \alpha$ is always a geodesic for all $\alpha \in [0, 2\pi]$.

Next consider the curve $v \equiv \beta$ for some $\beta > 0$, i.e. $\gamma(t) = (u(t), \beta)$ for some smooth function u that takes on all values in $[0, 2\pi]$. We have $v' \equiv 0$, which implies $u'' = 0$ and $\beta(u')^2 = 0$. The only solution to this is $u \equiv \alpha$ for some $\alpha \in [0, 2\pi]$. Thus, the curve is a constant curve and so does not take on all values on $[0, 2\pi] \times \{\beta\}$. Hence $v = \beta$ is not a geodesic for all $\beta > 0$.

2.4.5 Curvature

Definition. Suppose (M, g) is a Riemannian manifold. The $C^\infty(M)$-multilinear map $R : \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \to \mathfrak{X}^1(M)$ given by

$$R(X,Y)(Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

where ∇ is the Levi-Cevita connection, is called the $(1,3)$-*curvature tensor*. The map $R(X,Y) : \mathfrak{X}^1(M) \to \mathfrak{X}^1(M)$ is a smooth endomorphism of TM, called the *curvature endomorphism determined by X and Y*.

Definition. The *(0,4)-Riemann curvature tensor* Riem or $R^\flat$ is defined by

$$\mathrm{Riem}(X,Y,Z,W) = g\big(R(X,Y)Z,W\big).$$

Definition. The *Ricci Curvature Ric*(X,Y) is the trace of the linear map $Z \mapsto R(Z,X)Y$. It is symmetric in X and Y.

Definition. The *Scalar Curvature* $S \in C^\infty(M)$ is the trace of the Ricci curvature, in the sense that the Ricci curvature is the linear map $\mathfrak{X}^1(M) \to \mathfrak{X}^1(M)$ that sends X to the unique $Y \in \mathfrak{X}^1(M)$ such that $\mathrm{Ric}(X,Z) = g(Y,Z)$ for all $Z \in \mathfrak{X}^1(M)$.

Theorem 2.4.18. *A Riemannian manifold M is* flat*, i.e. locally isometric to Euclidean space, iff its curvature tensor (either $(0,4)$ or $(1,3)$) is identically zero.*

Properties:

1. $Riem(W,X,Y,Z) = -Riem(X,W,Y,Z)$.
2. $Riem(W,X,Y,Z) = -Riem(W,X,Z,Y)$ (follows since Levi-Cevita connection is a metric connection).
3. (*First Bianchi Identity*, or the *Algebraic Bianchi Identity*)

$$Riem(W,X,Y,Z) + Riem(X,Y,W,Z) + Riem(Y,W,X,Z) = 0$$

(follows from torsion-freeness).

4. $Riem(W,X,Y,Z) = Riem(Y,Z,W,X)$.
5. (*Second Bianchi Identity* or the *Differential Bianchi Identity*)

$$\nabla_W\big(Riem(X,Y,Z,V)\big) + \nabla_Z\big(Riem(X,Y,V,W)\big) + \nabla_V\big(Riem(X,Y,W,Z)\big) = 0.$$

Suppose $(U,(x_1,...,x_n))$ are local coordinates. Let $g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$, and suppose Γ_{ij}^k are the Riemann-Christoffel symbols on U. We describe the curvatures above in terms of Γ_{ij}^k and g.

1. (*(1,3)-curvature tensor*) Write $R = \sum_{i,j,k,\ell} R_{ijk}^\ell dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x_\ell}$. Clearly $\sum_\ell R_{ijk}^\ell \frac{\partial}{\partial x_\ell} = R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k}$. We then have

$$\begin{aligned} R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} &= \nabla_{\partial_i}\left(\sum_\ell \Gamma_{jk}^\ell \frac{\partial}{\partial x_\ell}\right) - \nabla_{\partial_j}\left(\sum_m \Gamma_{ik}^m \frac{\partial}{\partial x_m}\right) \\ &= \sum_\ell \left(\frac{\partial \Gamma_{jk}^\ell}{\partial x_i}\frac{\partial}{\partial x_\ell} + \Gamma_{jk}^\ell \sum_m \Gamma_{i\ell}^m \frac{\partial}{\partial x_m}\right) - \sum_m \left(\frac{\partial \Gamma_{ik}^m}{\partial x_j}\frac{\partial}{\partial x_m} + \Gamma_{ik}^m \sum_\ell \Gamma_{jm}^\ell \frac{\partial}{\partial x_\ell}\right) \\ &= \sum_\ell \left(\frac{\partial \Gamma_{jk}^\ell}{\partial x_i} - \frac{\partial \Gamma_{ik}^\ell}{\partial x_j} + \sum_m (\Gamma_{jk}^m \Gamma_{im}^\ell - \Gamma_{ik}^m \Gamma_{jm}^\ell)\right)\frac{\partial}{\partial x_\ell}. \end{aligned}$$

Therefore

$$R_{ijk}^\ell = \frac{\partial \Gamma_{jk}^\ell}{\partial x_i} - \frac{\partial \Gamma_{ik}^\ell}{\partial x_j} + \sum_m (\Gamma_{jk}^m \Gamma_{im}^\ell - \Gamma_{ik}^m \Gamma_{jm}^\ell).$$

2. (*(0,4)-Riemann curvature tensor*) Write $\mathrm{Riem} = \sum_{i,j,k,\ell} R_{ijk\ell} dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell$. It is clear that

$$R_{ijk\ell} = \sum_m g_{\ell m} R_{ijk}^m.$$

The above symmetries of the Riemann curvature tensor then state that $R_{ijk\ell} = -R_{jik\ell}$, $R_{ijk\ell} = -R_{ij\ell k}$, (first Bianchi identity) $R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0$, and $R_{ijk\ell} = R_{k\ell ij}$. If we define $R_{ijk\ell;m} := \frac{\partial R_{ijk\ell}}{\partial x_m}$, then the second Bianchi identity states that $R_{ijk\ell;m} + R_{ij\ell m;k} + R_{ijmk;\ell} = 0$.

3. (*Ricci Curvature*) Write $Ric = \sum_{i,j} R_{ij} dx^i \otimes dx^j$. Then $R_{ij} = \sum_k R_{kij}^k = \sum_{k\ell} g^{k\ell} R_{kij\ell}$ where (g^{ij}) is the inverse matrix to g_{ij}.

4. (*Scalar Curvature*) $S = \sum_{i,j} g^{ij} R_{ij}$.

2.4.6 (Hyper)-Surfaces

In this subsection we assume throughout that S is 2-dimensional with local coordinates x, y, so that the first fundamental form is of the form $g = Edx^2 + 2Fdx\cdot dy + Gdy^2$ (here $dx\cdot dy$ is the symmetric product). Throughout, we let D denote the Lev-Cevita connection of S. Let us first calculate expressions for the Riemann-Christoffel symbols.

We see that

$$E_x = D_{\partial_x} g(\partial_x, \partial_x) = 2g(\Gamma^x_{xx}\partial_x + \Gamma^y_{xx}\partial_y, \partial x) = 2E\Gamma^x_{xx} + 2F\Gamma^y_{xx}.$$

Similar computations yield

$$\begin{aligned} & & E_y &= 2E\Gamma^x_{yx} + 2F\Gamma^y_{yx} \\ F_x &= F\Gamma^x_{xx} + G\Gamma^y_{xx} + E\Gamma^x_{xy} + F\Gamma^y_{xy} & F_y &= F\Gamma^x_{yx} + G\Gamma^y_{yx} + E\Gamma^x_{yy} + F\Gamma^y_{yy} \\ G_x &= 2F\Gamma^x_{xy} + 2G\Gamma^y_{xy} & G_y &= 2G\Gamma^y_{yy} + 2F\Gamma^x_{yy}. \end{aligned}$$

By solving for the various Riemann-Christoffel symbols, and using the tensor-free property, we see that

$$\begin{aligned} \Gamma^x_{xx} &= \frac{FE_y + GE_x - 2FF_x}{2(EG - F^2)} & \Gamma^x_{xy} = \Gamma^x_{yx} &= \frac{GE_y - FG_x}{2(EG - F^2)} & \Gamma^x_{yy} &= \frac{2GF_y - GG_x - FG_y}{2(EG - F^2)} \\ \Gamma^y_{xx} &= \frac{2EF_x - EE_y - FE_x}{2(EG - F^2)} & \Gamma^y_{xy} = \Gamma^y_{yx} &= \frac{EG_x - FE_y}{2(EG - F^2)} & \Gamma^y_{yy} &= \frac{FG_x + EG_y - 2FF_y}{2(EG - F^2)}. \end{aligned}$$

Now, on hyper surfaces there is another notion of curvature. Suppose M is an n-dimensional Riemannian manifold with an embedding $\iota : M \hookrightarrow \mathbb{R}^{n+1}$ such that the Riemannian metric g on M is given by $g = \iota^* g_{Euc}$. Then $d_p\iota : T_pM \hookrightarrow T_p\mathbb{R}^{n+1}$. The orthogonal complement of $d_p\iota(T_pM)$ in $T_p\mathbb{R}^{n+1}$ is a 1-dimensional space. A unit vector $\mathbf{n}_p$ in the orthogonal complement is called the *unit normal vector to* M. Suppose we fix a choice of smooth vector field $\mathbf{n} : M \to NM \subset T\mathbb{R}^{n+1}$ (recall NM is the normal bundle to M), i.e. we pick unit normals at each $p \in M$ such that our choices vary smoothly (if M is connected, then there are only two such choices, namely $\mathbf{n}$ and $-\mathbf{n}$). Let $\tilde{\nabla}$ be the Levi-Cevita connection of g_{Euc}.

Definition. The *second fundamental form* $\mathbf{II} : \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \to NM$ is $\mathbf{II}(X, Y) := g_{Euc}(\mathbf{n}, \tilde{\nabla}_X Y)\mathbf{n}$. We can write $\mathbf{II} = h \cdot \mathbf{n}$ where $h \to \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \to \mathbb{R}$; this map h is the *scalar second fundamental form.* It is known that the second fundamental form is symmetric. The second fundamental form depends on the embedding.

The *shape operator* $s : \mathfrak{X}^1(M) \to \mathfrak{X}^1(M)$ is defined by the map $g(sX, Y)\mathbf{n} = \mathbf{II}(X, Y)$. The induced map $s : TM \to TM$ is self-adjoint endomorphism. By identifying $T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$, the map $\mathbf{n} : S \to NS$ induces a smooth map $N : S \to S^n$ (here, S^n is the n-sphere) called the Gauss map. The shape operator is given by $s(p) = -d_pN$.

The *Gaussian curvature* K of M is the determinant of the shape operator. The *mean curvature* H of M is $\frac{1}{n}$ times the trace of the shape operator. The *principal curvatures* are the eigenvalues of the shape operator.

If $S \hookrightarrow \mathbb{R}^3$ is a surface, then in local coordinates x, y, the unit normal vector $\mathbf{n}_p$ may be calculated by normalizing the vector $\pm\big(d_p\iota(\partial_x) \times d_p\iota(\partial_y)\big)$ (cross product in $\mathbb{R}^3$).

Proposition 2.4.19. *If $g = Edx^2 + 2Fdxdy + Gdy^2$ and $\mathbf{II} = Ldx^2 + 2Mdxdy + Ndy^2$ for some coordinates x, y on S, then the matrix of the shape operator in this basis is given by*

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Example 2.4.20. Suppose S is a surface embedded in $\mathbb{R}^3$ via $f : U \to \mathbb{R}^3$ ($U \subset \mathbb{R}^2$ open). Then, some easy calculations show that

$$g = (f_x \cdot f_x)dx^2 + 2(f_x \cdot f_y)dxdy + (f_y \cdot f_y)dy^2, \quad \text{and} \quad \mathbf{II} = (f_{xx} \cdot \mathbf{n})dx^2 + 2(f_{xy} \cdot \mathbf{n})dxdy + (f_{yy} \cdot \mathbf{n})dy^2.$$

Example 2.4.21 (Fall 2021 Day 1)**.** Let S be the surface obtained by rotating $x = c\cosh\frac{z}{c}$ around the z-axis. Using the coordinates (θ, z) on S (where θ is the polar angular), find the first two fundamental forms, the mean curvature, and the Gaussian curvature.

We have $x = c\cos\theta\cosh\frac{z}{c}$ and $y = c\sin\theta\cosh\frac{z}{c}$. Thus the first fundamental form is

$$\begin{aligned} g &= (\sinh\frac{z}{c}\cos\theta dz - c\cosh\frac{z}{c}\sin\theta d\theta)^2 + (\sinh\frac{z}{c}\sin\theta dz + c\cosh\frac{z}{c}\cos\theta d\theta)^2 + dz^2 \\ &= \cosh^2\frac{z}{c}dz^2 + c^2\cosh^2\frac{z}{c}d\theta^2. \end{aligned}$$

Denoting by ι the embedding $S \hookrightarrow \mathbb{R}^3$, we have

$$\iota_* \frac{\partial}{\partial \theta} = -c\cosh\frac{z}{c}\sin\theta\frac{\partial}{\partial x} + c\cosh\frac{z}{c}\cos\theta\frac{\partial}{\partial y}, \quad \text{and} \quad \iota_*\frac{\partial}{\partial z} = \sinh\frac{z}{c}\cos\theta\frac{\partial}{\partial x} + \sinh\frac{z}{c}\sin\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Calculating the cross product and dividing by the norm, we get

$$\mathbf{n} = \frac{\cos\theta}{\cosh\frac{z}{c}}\frac{\partial}{\partial x} + \frac{\sin\theta}{\cosh\frac{z}{c}}\frac{\partial}{\partial y} - \frac{\sinh\frac{z}{c}}{\cosh\frac{z}{c}}\frac{\partial}{\partial z}.$$

Denoting by $\tilde{\nabla}$ the Levi-Cevita connection on Euclidean space, notice that

$$\begin{aligned}
\tilde{\nabla}_{\iota_*\partial_\theta}\iota_*\partial_\theta &= -c\cosh\frac{z}{c}\cos\theta\frac{\partial}{\partial x} - \cosh\frac{z}{c}\sin\theta\frac{\partial}{\partial y}, \\
\tilde{\nabla}_{\iota_*\partial_z}\iota_*\partial_\theta &= -\sinh\frac{z}{c}\sin\theta\frac{\partial}{\partial x} + \sinh\frac{z}{c}\cos\theta\frac{\partial}{\partial y}, \\
\tilde{\nabla}_{\iota_*\partial_\theta}\iota_*\partial_\theta &= \tfrac{1}{c}\cosh\frac{z}{c}\cos\theta\frac{\partial}{\partial x} + \tfrac{1}{c}\cosh\frac{z}{c}\sin\theta\frac{\partial}{\partial y}.
\end{aligned}$$

From $\mathbf{II}(X,Y) = g_{Euc}(\mathbf{n}, \tilde{\nabla}_X Y)$, it then follows that $\mathbf{II}(\partial_\theta, \partial_\theta) = -c, \mathbf{II}(\partial_\theta, \partial_z) = 0, \mathbf{II}(\partial_z, \partial_z) = \frac{1}{c}$, and hence

$$\mathbf{II} = -cd\theta^2 + \tfrac{1}{c}dz^2.$$

Now, the definition of the shape operator is $g(sX,Y) = \mathbf{II}(X,Y)$. If the matrix of s in the basis $\partial_\theta, \partial_z$ is $[s]$, then

$$\cosh^2\frac{z}{c}\begin{pmatrix} c^2 & 0 \\ 0 & 1\end{pmatrix}[s] = \begin{pmatrix} -c & 0 \\ 0 & \frac{1}{c}\end{pmatrix}, \quad \text{and so} \quad [s] = \frac{1}{\cosh^2\frac{z}{c}}\begin{pmatrix} -\frac{1}{c} & 0 \\ 0 & \frac{1}{c}\end{pmatrix}.$$

Taking the trace and the determinant respectively, it follows that th mean curvature H is identically 0, while the Gaussian curvature K is

$$K = -\frac{1}{c^2\cosh^4\frac{z}{c}}.$$

Proposition 2.4.22. *Let notation be as above for a hypersurface M embedded in $\mathbb{R}^{n+1}$. We also assume coordinates $(x_1, ..., x_n)$ on M.*

1. *(*Gauss Equation*) Writing $h = \sum_{i,j} h_{ij}dx^i dx^j$, we have $R_{ijk\ell} = h_{i\ell}h_{jk} - h_{ik}h_{jl}$.*
2. *(*Gauss' Formula*) $\tilde{\nabla}_X Y = \nabla_X Y + \mathbf{II}(X,Y)$.*
3. *(*Weingarten Equation*) $\tilde{\nabla}_X \mathbf{n} = -s(X)$.*
4. *(*Gauss' Theorema Egregium*) If $M = S$ is a 2-dimensional surface embedded in $\mathbb{R}^3$, then the Gaussian curvature K of S is an intrinsic quantity of S.*

We have explicit formulae for the Gaussian curvature of a surface:

- An explicit formula is: $K = \dfrac{1}{(EG-F^2)^2}\left(\begin{vmatrix} F_{xy} - \frac{1}{2}E_{yy} - \frac{1}{2}G_{xx} & \frac{1}{2}E_x & F_x - \frac{1}{2}E_y \\ F_y - \frac{1}{2}G_x & E & F \\ \frac{1}{2}G_y & F & G\end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_y & \frac{1}{2}G_x \\ \frac{1}{2}E_y & E & F \\ \frac{1}{2}G_x & F & G\end{vmatrix}\right).$
- We have the following symmetry properties of the $(0,4)$-Riemann curvature tensor: $R_{ijk\ell} = -R_{jik\ell}$ and $R_{ijk\ell} = -R_{ij\ell k}$. These two symmetries imply that $R_{ijk\ell} = 0$ whenever $i = j$ or $k = \ell$. Thus, there are only four non-zero terms namely $R_{xyxy}, R_{xyyx}, R_{yxxy}, R_{yxyx}$. Also, we can swap i and j as well as k and ℓ, so that the only number that really matters is R_{xyxy}. We have:

  $$R_{xyxy} = (F^2 - EG)\cdot K.$$

  A coordinate-free form is

  $$Riem(W,X,Y,Z) = K\cdot\big(g(X,Y)g(W,Z) - g(X,Z)g(W,Y)\big).$$

- The Ricci curvature of a surface satisfies $Ric = K\cdot g$ (both Ric and g are symmetric 2-tensors).
- The scalar curvature of the surface is exactly twice the Gaussian curvature.

Definition. Suppose S is a surface, and suppose we have a curve $\gamma : I \to S$ that has a unit speed parametrization, i.e. $|\gamma'(t)|_g \equiv 1$. Since T_pS is a 2-dimensional inner product space, and since we have the tangent vector $\gamma'(t) \in T_{\gamma(t)}S$, there is a well-defined (up to sign) unit normal vector $\mathbf{n}(t) \in T_{\gamma(t)}S$ to γ. The *signed geodesic curvature* κ_g of γ is

$$\kappa_g(t) = g\big(D_{\gamma'}\gamma'(t), \mathbf{n}_t\big).$$

It is known that $D_{\gamma'}\gamma'(t) = \kappa_g(t)\mathbf{n}_t$.

The *unsigned geodesic curvature* is $k_g := |\kappa_g|$.

Theorem 2.4.23 (Gauss-Bonnet Theorem)**.** *Suppose we have a compact surface S with boundary ∂S. Then,*

$$\int_S K dA + \int_{\partial S} \kappa_g ds = 2\pi\chi(S)$$

where χ is the Euler characteristic of S and dA is the skew-symmetrization of the 2-tensor g (if $g = Edx^2 + 2Fdx \cdot dy + Gdy^2$, then $dA = \sqrt{EG - F^2}dx \wedge dy$). Here, κ_g is the signed geodesic curvature of the arc-length parametrization of ∂S with respect to an inward pointing unit normal vector, and the integral is taken with respect to arc length. If S does not have boundary, then we simply have

$$\int_S K dA = 2\pi\chi(S).$$

The Euler characteristic for an orientable compact surface without boundary is $2 - 2g$ where g is the genus. It can be calculated from any triangulation of S.

Example 2.4.24 (Fall 2020 Day 3)**.** Let $\mathbb{H}$ be the upper half-plane with hyperbolic metric $g = y^{-2}dx \cdot dy$. Suppose Γ is a group of isometries of $\mathbb{H}$ such that $S = \mathbb{H}/\Gamma$ is a smooth surface S, with a fundamental domain for the action given by

$$D = \left\{(x, y) \in \mathbb{H} : -\tfrac{3}{2} \leq x \leq \tfrac{3}{2}, (x - c)^2 + y^2 \geq \tfrac{1}{9} \text{ for } c \in \{\pm\tfrac{1}{3}, \pm\tfrac{2}{3}, \pm\tfrac{4}{3}\}\right\}.$$

Compute $\chi(S)$ using Gauss-Bonnet, given that the Gaussian curvature of $\mathbb{H}$ is identically -1.

First note that D is the disjoint union of 6 reflections/translates of

$$D' = \left\{(x, y) \in \mathbb{H} : 0 \leq x \leq \tfrac{1}{2}, (x - \tfrac{1}{3})^2 + y^2 \geq \tfrac{1}{9}\right\}.$$

Thus an integration on D is 6 times the integration on D' (assuming symmetry of the function being integrated). Since S has no boundary, and since the Gaussian curvature is identically -1, the Gauss-Bonnet Theorem yields

$$2\pi\chi(S) = -6\int_{D'} \frac{1}{y^2} dxdy = -6\int_0^{\frac{1}{2}} \int_{\sqrt{\frac{1}{9} - (x - \frac{1}{3})^2}}^{\infty} \frac{1}{y^2} dydx.$$

Using the substitution $x = \frac{1}{3}(1 - \cos\theta)$ so that $dx = \frac{1}{3}\sin\theta$, we have

$$\chi(S) = \frac{-1}{\pi}\int_0^{2\pi/3} \int_{\frac{1}{3}\sin\theta}^{\infty} y^{-2}\sin\theta dyd\theta = \frac{-1}{\pi}\int_0^{2\pi/3} \sin\theta \cdot \frac{3}{\sin\theta} dyd\theta = \frac{-3}{\pi} \cdot \frac{2\pi}{3} = -2.$$

Example 2.4.25 (Fall 2019 Day 1)**.** Equip the standard open unit disk $\mathbb{D}$ with the metric $g = \frac{1}{1-x^2-y^2}(dx^2 + dy^2)$. Find the Riemannian curvature tensor and the Gaussian curvature of $(\mathbb{D}, g)$.

While we could use the above formulae with $E = G = (1 - x^2 - y^2)^{-1}$ and $F \equiv 0$ directly, we compute from scratch. From $g(\partial_x, \partial_y) \equiv 0$ it follows that

$$0 = g(\nabla_{\partial_x}\partial_x, \partial_y) + g(\partial_x, \nabla_{\partial_x}\partial_y) = g(\Gamma^x_{xx}\partial_x + \Gamma^y_{xx}\partial_y, \partial_y) + g(\partial_x, \Gamma^x_{xy}\partial_x + \Gamma^y_{xy}\partial_y) = \frac{\Gamma^y_{xx} + \Gamma^x_{xy}}{1 - x^2 - y^2}$$

and so $\Gamma^y_{xx} = -\Gamma^x_{xy} = -\Gamma^x_{yx}$. Similarly $\Gamma^x_{yy} = -\Gamma^y_{yx} = -\Gamma^y_{xy}$. Also, we have

$$\frac{2x}{(1 - x^2 - y^2)^2} = \nabla_{\partial_x} g(\partial_x, \partial_x) = 2g(\nabla_{\partial_x}\partial_x, \partial_x) = \frac{2\Gamma^x_{xx}}{1 - x^2 - y^2},$$

and so $\Gamma^x_{xx} = x(1 - x^2 - y^2)^{-1}$. Similarly $\Gamma^x_{yx} = y(1 - x^2 - y^2)^{-1}$ by evaluating $\nabla_{\partial_y} g(\partial_x, \partial_x)$, and thus we have

$$\Gamma^x_{xx} = \Gamma^y_{xy} = \Gamma^y_{yx} = -\Gamma^x_{yy} = \frac{x}{1 - x^2 - y^2}, \quad \text{and} \quad \Gamma^y_{yy} = \Gamma^x_{xy} = \Gamma^x_{yx} = -\Gamma^y_{xx} = \frac{y}{1 - x^2 - y^2}.$$

Let $f(x,y) = (1-x^2-y^2)^{-1}$; then $f_x = 2xf^2$ and $f_y = 2yf^2$. Since R_{xyxy} is the only non-zero term (apart from the usual symmetries of the Riemannian curvature tensor), we have

$$\begin{aligned} R_{xyxy} &= g\left(\nabla_{\partial_x}(\nabla_{\partial_y}\partial_x) - \nabla_{\partial_y}(\nabla_{\partial_x}\partial_x), \partial_y\right) = g\left(\nabla_{\partial_x}(yf\partial_x + xf\partial_y) - \nabla_{\partial_y}(xf\partial_x - yf\partial_y), \partial_y\right) \\ &= g\big(2xyf^2\partial_x + xyf^2\partial_x - y^2f^2\partial_y + (f+2x^2f^2)\partial_y + xyf^2\partial_x + x^2f^2\partial_y - 2xyf^2\partial_x - xyf^2\partial_x - \\ &\qquad x^2f^2\partial_y + f\partial_y + 2y^2f^2\partial_y - yxf^2\partial_x + y^2f^2\partial_y, \partial_y\big) \\ &= -y^2f^3 + f^2 + 2x^2f^3 + x^2f^3 - x^2f^3 + f^2 + 2y^2f^3 + y^2f^3 = 2f^2 + 2x^2f^3 + 2y^2f^3 \\ &= \frac{2}{(1-x^2-y^2)^3}. \end{aligned}$$

It follows that

$$Riem = \frac{2}{(1-x^2-y^2)^3}\left(dx\otimes dy\otimes dx\otimes dy - dx\otimes dy\otimes dy\otimes dx - dy\otimes dx\otimes dx\otimes dy + dy\otimes dx\otimes dy\otimes dx\right)$$

Since $K = -(\det g)^{-1}R_{xyxy}$ for any linearly independent X, Y, it follows that the Gaussian curvature is

$$K(x,y) = -\frac{2}{1-x^2-y^2}.$$

2.4.7 Model Riemannian Manifolds

We now compute the Levi-Cevita connection and the various curvatures for Euclidean space, for the sphere, and for hyperbolic space.

Euclidean Space

We have a global frame $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$. The standard metric g on $\mathbb{R}^n$ is simply $g = \sum_i (dx^i)^2$, i.e. $(g_{ij}) = I_n$ is constant. It follows that $\Gamma_{ij}^k = 0$ for all i,j,k. Hence, the Levi-Cevita connection ∇ on $\mathbb{R}^n$ satisfies $\nabla_{\partial/\partial x_i}\frac{\partial}{\partial x_j} = 0$ for all i,j. This also implies that $R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k} = 0$ for all i,j,k. Hence the $(1,3)$-curvature tensor is identically zero, which implies that all the various notions of curvature are also zero.

Spheres

Consider S^n with the usual round metric $g_R = \iota^*(g_E)$, where $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ is the embedding and $g_E = \sum_{i=1}^{n+1}(dx^i)^2$ is the usual Euclidean metric on $\mathbb{R}^{n+1}$. The coordinates on $\mathbb{R}^{n+1}$ will be denoted by $x_1, ..., x_{n+1}$. Let $N = (0, ..., 0, 1)$, and fix $\epsilon \in \{\pm 1\}$. Let $(y_1, ..., y_n)$ be the stereographic projection of S^n from ϵN onto $\mathbb{R}^n$, given by $y_i(p) = \frac{p_i}{1-\epsilon p_{n+1}}$. In local coordinates the embedding ι is given as $x_i \circ \iota(y) = \frac{2y_i}{1+\|y\|^2}$ and $x_{n+1}\circ\iota(y) = \epsilon\frac{\|y\|^2-1}{\|y\|^2+1} = \epsilon - \frac{2\epsilon}{1+\|y\|^2}$. In particular,

$$\iota_*\left(\frac{\partial}{\partial y_i}\right) = \frac{4\epsilon y_i}{(1+\|y\|^2)^2}\frac{\partial}{\partial x_{n+1}} + \sum_{k=1}^n \left(\frac{2\delta_{ik}}{1+\|y\|^2} - \frac{4y_iy_k}{(1+\|y\|^2)^2}\right)\frac{\partial}{\partial x_k},$$

so that

$$\begin{aligned} g_{ij} = g_E(\iota_*\partial_i, \iota_*\partial_j) &= \frac{16y_iy_j}{(1+\|y\|^2)^4} + \sum_{k=1}^n \left(\frac{2\delta_{ik}}{1+\|y\|^2} - \frac{4y_iy_k}{(1+\|y\|^2)^2}\right)\left(\frac{2\delta_{jk}}{1+\|y\|^2} - \frac{4y_jy_k}{(1+\|y\|^2)^2}\right) \\ &= \frac{16y_iy_j}{(1+\|y\|^2)^4} + \frac{4\delta_{ij}}{(1+\|y\|^2)^2} - 16\frac{y_iy_j}{(1+\|y\|^2)^3} + 16\frac{y_iy_j\|y\|^2}{(1+\|y\|^2)^4} = \frac{4\delta_{ij}}{(1+\|y\|^2)^2}. \end{aligned}$$

Thus $g^{ij} = \frac{1}{4}(1+\|y\|^2)^2\delta_{ij}$. To evaluate the Riemann-Christoffel symbols, note that $\frac{\partial g_{ij}}{\partial y_\ell} = -\frac{16\delta_{ij}y_\ell}{(1+\|y\|^2)^3}$ so that

$$\begin{aligned} \Gamma_{ij}^k &= \sum_\ell \frac{1}{2}\left(\frac{(1+\|y\|^2)^2}{4}\delta_{k\ell}\right)\left(-\frac{16\delta_{\ell j}y_i}{(1+\|y\|^2)^3} - \frac{16\delta_{i\ell}y_j}{(1+\|y\|^2)^3} + \frac{16\delta_{ij}y_\ell}{(1+\|y\|^2)^3}\right) \\ &= 2\frac{\delta_{ij}y_k - \delta_{ik}y_j - \delta_{jk}y_i}{1+\|y\|^2}. \end{aligned}$$

We now evaluate curvatures. First, notice that

$$\frac{\partial\Gamma_{jk}^\ell}{\partial x_i} = -4y_i\cdot\frac{\delta_{jk}y_\ell - \delta_{j\ell}y_k - \delta_{k\ell}y_j}{(1+\|y\|^2)^2} + 2\frac{\delta_{jk}\delta_{i\ell} - \delta_{j\ell}\delta_{ik} - \delta_{k\ell}\delta_{ij}}{1+\|y\|^2}$$

so that

$$\frac{\partial \Gamma^\ell_{jk}}{\partial x_i} - \frac{\partial \Gamma^\ell_{ik}}{\partial x_j} = 4\frac{(\delta_{ik}y_j - \delta_{jk}y_i)y_\ell - (\delta_{i\ell}y_j - \delta_{j\ell}y_i)y_k}{(1+\|y\|^2)^2} + 2\frac{\delta_{jk}\delta_{i\ell} - \delta_{j\ell}\delta_{ik} - \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}}{1+\|y\|^2}.$$

On the other hand,

$$\Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm} = \frac{4}{(1+\|y\|^2)^2}\Big((\delta_{jk}\delta_{im} - \delta_{ik}\delta_{jm})y_\ell y_m + (\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell})y_m^2 - (\delta_{km}\delta_{j\ell} + \delta_{jk}\delta_{m\ell})y_i y_m +$$
$$(\delta_{jm}\delta_{i\ell} - \delta_{im}\delta_{j\ell})y_k y_m + (\delta_{km}\delta_{i\ell} + \delta_{ik}\delta_{m\ell})y_j y_m + \delta_{jm}\delta_{m\ell}y_k y_i - \delta_{km}\delta_{im}y_j y_\ell - \delta_{im}\delta_{m\ell}y_k y_j + \delta_{km}\delta_{jm}y_i y_\ell\Big)$$

so that

$$\sum_m(\Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm}) = \frac{4}{(1+\|y\|^2)^2}\Big((\delta_{jk}y_i - \delta_{ik}y_j)y_\ell + (\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell})\|y\|^2 - (\delta_{j\ell}y_k + \delta_{jk}y_\ell)y_i + (\delta_{i\ell}y_j - \delta_{j\ell}y_i)y_k$$
$$+ (\delta_{i\ell}y_k + \delta_{ik}y_\ell)y_j + \delta_{j\ell}y_k y_i - \delta_{ik}y_j y_\ell - \delta_{i\ell}y_k y_j + \delta_{jk}y_i y_\ell\Big).$$

Thus, the Riemannian curvature endomorphism is

$$R^\ell_{ijk} = \frac{4}{(1+\|y\|^2)^2}\left(\delta_{jk}\delta_{i\ell} - \delta_{j\ell}\delta_{ik} - (\delta_{j\ell}y_k + \delta_{jk}y_\ell)y_i + (\delta_{i\ell}y_k + \delta_{ik}y_\ell)y_j + \delta_{j\ell}y_k y_i - \delta_{ik}y_j y_\ell - \delta_{i\ell}y_k y_j + \delta_{jk}y_i y_\ell\right).$$

The Ricci curvature is given by

$$R_{jk} = \frac{4(n-1)\delta_{jk}}{(1+\|y\|^2)^2} = (n-1)g_{jk}.$$

Hence $\mathrm{Ric} = (n-1)g$ as global tensors on S^n. In particular, this implies that the scalar curvature S is simply the constant function $(n-1)$.

Hyperbolic Space

We consider the upper half-space model $\mathbb{H}^n$ of hyperbolic space, with metric $g = \frac{1}{x_n^2}((dx^1)^2 + \cdots + (dx^n)^2)$. Here we have global frames $\frac{\partial}{\partial x_i} =: \partial_i$ with which to calculate. Notice that $g_{ij} = \delta_{ij}x_n^{-2}$, so that $g^{ij} = x_n^2\delta^{ij}$. Hence, the Riemann-Christoffel Symbols are given by

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2}\sum_\ell x_n^2\delta^{k\ell}\left(\frac{\partial \delta_{i\ell}x_n^{-2}}{\partial x_j} + \frac{\partial \delta_{j\ell}x_n^{-2}}{\partial x_i} - \frac{\partial \delta_{ij}x_n^{-2}}{\partial x_\ell}\right)\\
&= \frac{1}{2}x_n^2\left(-2\delta_{ik}\delta_{jn}x_n^{-3} - 2\delta_{jk}\delta_{in}x_n^{-3} + 2\delta_{ij}\delta_{kn}x_n^{-3}\right)\\
&= \frac{\delta_{ij}\delta_{kn} - \delta_{ik}\delta_{jn} - \delta_{jk}\delta_{in}}{x_n}.
\end{aligned}$$

Now, notice that

$$\begin{aligned}
\sum_m\left(\Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm}\right) &= \frac{1}{x_n^2}\sum_m\Big(\big(\delta_{jk}\delta_{mn} - \delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn}\big)\big(\delta_{im}\delta_{\ell n} - \delta_{i\ell}\delta_{mn} - \delta_{m\ell}\delta_{in}\big) -\\
&\qquad \big(\delta_{ik}\delta_{mn} - \delta_{im}\delta_{kn} - \delta_{km}\delta_{in}\big)\big(\delta_{jm}\delta_{\ell n} - \delta_{j\ell}\delta_{mn} - \delta_{m\ell}\delta_{jn}\big)\Big)\\
&= \frac{1}{x_n^2}\left(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell} + \delta_{jk}\delta_{in}\delta_{\ell n} - \delta_{jn}\delta_{ik}\delta_{\ell n} + \delta_{jn}\delta_{i\ell}\delta_{kn} - \delta_{in}\delta_{j\ell}\delta_{kn}\right)
\end{aligned}$$

Hence the $(1,3)$-curvature tensor is

$$\begin{aligned}
R^\ell_{ijk} &= \frac{\partial \Gamma^\ell_{jk}}{\partial x_i} - \frac{\partial \Gamma^\ell_{ik}}{\partial x_j} + \sum_m\left(\Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm}\right)\\
&= \frac{1}{x_n^2}\big(-\delta_{in}\delta_{jk}\delta_{\ell n} + \delta_{in}\delta_{j\ell}\delta_{kn} + \delta_{in}\delta_{k\ell}\delta_{jn} + \delta_{jn}\delta_{ik}\delta_{\ell n} - \delta_{jn}\delta_{i\ell}\delta_{kn} - \delta_{jn}\delta_{k\ell}\delta_{in} + \delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell} +\\
&\qquad \delta_{jk}\delta_{in}\delta_{\ell n} - \delta_{jn}\delta_{ik}\delta_{\ell n} + \delta_{jn}\delta_{i\ell}\delta_{kn} - \delta_{in}\delta_{j\ell}\delta_{kn}\big)\\
&= \frac{1}{x_n^2}\big(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell}\big).
\end{aligned}$$

The $(0,4)$-curvature tensor is simply

$$R_{ijk\ell} = \sum_m g_{\ell m} R^m_{ijk} = \frac{1}{x_n^4}\big(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell}\big).$$

The Ricci curvature is

$$R_{jk} = \sum_\ell R^\ell_{\ell jk} = \frac{1}{x_n^2}\sum_\ell \big(\delta_{\ell k}\delta_{j\ell} - \delta_{jk}\big) = -\frac{(n-1)}{x_n^2}\delta_{jk} = -(n-1)g_{jk}.$$

Hence $Ric = -(n-1)g$ as global tensors on $\mathbb{H}^n$. In particular, we see that the scalar curvature is the constant function $1-n$.

Chapter 3

Complex Analysis

Throughout set $z = x + iy$, and set $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. Set

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}, \quad D^*(z_0, r) = D(z_0, r)\backslash\{x_0\}, \quad \text{and } \bar{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

Also set

$$A(a; r, R) = D(a, R)\backslash\bar{D}(a, r) = \{z \in \mathbb{C} : r < |z - a| < R\}.$$

3.1 Review of Basic Facts of Holomorphic and Meromorphic Functions

Define the differential operators $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

Definition. A function $f : U \subset \mathbb{C} \to \mathbb{C}$ (U open) is *holomorphic* if either of the following equivalent conditions hold:

1. at all $z_0 \in U$, the limit of $\frac{f(z)-f(z_0)}{z-z_0}$ exists as $z \to z_0$;
2. u, v satisfy the *Cauchy-Riemann equations*, namely $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$;
3. $\frac{\partial f}{\partial \bar{z}} \equiv 0$ on U.
4. (Taylor's Theorem) for any $z_0 \in U$, there exists $R > 0$ such that on $D(z_0, R)$, we have the (locally uniformly and absolutely convergent on $D(z_0, R)$) power series expansion

$$f(z) = \sum_{n \geq 0} c_n (z - z_0)^n.$$

A function holomorphic on $\mathbb{C}$ is *entire.*

A function f is *meromorphic* on U if there exists a discrete set of points $S \subset U$ such that $f : U\backslash S \to \mathbb{C}$ is holomorphic and either of the following two equivalent conditions:

1. there exists an open cover $\{V_i\}$ of U such that on each V_i there exists g, h holomorphic on V_i such that $f = g/h$ on $V_i\backslash S$;
2. for each $a \in S$, there exists $r > 0$ and $n > 0$ such that $(z - a)^n f$ is a holomorphic on $D^*(a, r)$.

We also quickly introduce the notion of Riemann surfaces.

Definition. A *Riemann surface* is a 2-dimensional real manifold M covered by *holomorphically compatible* atlas, i.e. there exists an atlas such that for any charts (U, φ) and (V, ψ), the transition map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is bi-holomorphic.

Definition. Suppose X is a Riemann surface. If $Y \subset X$ is open, then a function $f : Y \to \mathbb{C}$ is *holomorphic* if for every chart (U, ψ) with $U \subset Y$, the function $f \circ \psi^{-1} : \psi(U) \to \mathbb{C}$ is holomorphic.

A function $f : X \to Y$ of Riemann surfaces is *holomorphic* if for any chart (U, φ) on X and (V, ψ) on Y with $f(U) \subset V$, the map $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic. A function $f : X \to Y$ is *bi-holomorphic* if it is bijective and both $f : X \to Y$ and $f^{-1} : Y \to X$ are holomorphic.

A *meromorphic map* is a function $f : X\backslash E \to Y$ where $E \subset X$ is a discrete set of points such that for any $p \in E$, we have $\lim_{x \to p} |f(x)| = \infty$.

Important Examples:

1. Open subsets of $\mathbb{C}$.

2. The *Riemann Sphere*, denoted by either of $\mathbb{P}^1_{\mathbb{C}}$ or $\hat{\mathbb{C}}$, is the set $\mathbb{C} \cup \{\infty\}$ (∞ just a symbol not contained in $\mathbb{C}$) equipped with charts $U = \mathbb{P}^1 \backslash \{\infty\}, z = \mathrm{Id} : U \cong \mathbb{C} \to \mathbb{C}$, and $V = \mathbb{P}^1 \backslash \{0\} \to \mathbb{C}, w : V \to \mathbb{C}$ that sends a point $p \in V$ to $1/p$ (we set $1/\infty := 0$). This is homeomorphic to the 2-sphere.

   Any meromorphic function f on $U \subset \mathbb{C}$ can be considered as a holomorphic map $f : U \to \mathbb{P}^1_{\mathbb{C}}$, where poles are sent to ∞. More generally, any meromorphic function $f : X \to \mathbb{C}$ for an arbitrary Riemann surface X can be considered as a holomorphic map $\bar{f} : X \to \mathbb{P}^1_{\mathbb{C}}$, and vice versa. Similarly, any meromorphic function on $\mathbb{C}$ can be considered as a holomorphic map from $\mathbb{P}^1_{\mathbb{C}}$ to $\mathbb{P}^1_{\mathbb{C}}$.

   The meromorphic maps $z \mapsto \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}, ad - bc \neq 0$ induce bi-holomorphisms $\mathbb{P}^1 \to \mathbb{P}^1$.

3. *Torus* T. Fix $\omega_1, \omega_2 \in \mathbb{C}$ which are $\mathbb{R}$-linearly independent. Set $\Gamma := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and consider the quotient $T = \mathbb{C}/\Gamma$ (technically a quotient by the equivalence relation $z \sim z'$ iff $z - z' \in \Gamma$). Charts can be given subsets U of $\mathbb{C}$ such that $z, w \in U$ iff $z - w \in \Gamma$.

Definition. Given a curve $\gamma : (a, b) \to U$, the line integral of f is simply defined by

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

We now list all important facts of holomorphic and meromorphic functions. Throughout, we assume U open and $f : U \to \mathbb{C}$ holomorphic, unless otherwise specified. Also, X denotes an arbitrary Riemann surface, and $f : X \to \mathbb{C}$ a holomorphic map.

- Holomorphic maps on Riemann surfaces are smooth.

- (Cauchy-Hadamard Formula) The radius of convergence of $\sum_n c_n z^n$ is $R = \left(\limsup_{n\to\infty} |c_n|^{1/n} \right)^{-1}$ (convention: $0^{-1} = \infty, \infty^{-1} = 0$).

- If $\sum_n c_n(z - z_0)^n$ is the Taylor expansion of f on $D(z_0, R)$ around the point z_0, then

  $$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi} \int_{\partial D(z_0,r)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi,$$

  where $r \in (0, R)$.

- (Cauchy's Theorem) If f is analytic on a convex open domain U, then for any closed piecewise C^1-path γ on U we have $\int_\gamma f(z)dz = 0$.

- (Cauchy's Integral Theorem) If f is analytic on $D(a, R)$, then for any $0 < r < R$ and any $z \in D(a, r)$ we have

  $$f(z) = \frac{1}{2\pi i} \int_{\partial D(a;r)} \frac{f(\xi)}{\xi - z} d\xi.$$

  More generally, for any $n \in \mathbb{N}$,

  $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D(a;r)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

- (Maximum Principle) Suppose X a connected Riemann surface and $f : X \to \mathbb{C}$ holomorphic. Then the continuous function $|f| : X \to \mathbb{R}$ does not achieve a maximum anywhere on X. Moreover, suppose $G \subset X$ is open such that $\bar{G} \subset X$ is compact. Then for any $z \in \bar{G}$, we have

  $$|f(z)| \leq \sup_{\xi \in \partial G} |f(\xi)|$$

  where if equality holds, then $z \in \partial G$.

- (Cauchy Estimates) If f is analytic on $D(a, R)$, then for any $0 < r < R$ we have

  $$|f^{(n)}(a)| \leq \frac{n!}{r^n} \sup_{\xi \in \partial D(a;r)} |f(\xi)|.$$

- (Liouville's Theorem) Any uniformly bounded entire function must be a constant.
- (Weierstrass' Theorem) The limit of a sequence of holomorphic functions that converge uniformly on all compact subsets must itself be holomorphic. In such a case, the derivatives of the original sequence converge uniformly on compact subsets to the derivative of the limit.
- (Morera's Theorem) Suppose f is a continuous function in an open connected set Ω, and if $\int_\gamma f(z)dz = 0$ for all closed curves γ in Ω, then f is holomorphic in Ω.
- If f is meromorphic on U and $a \in U$, then there exists a unique $n \in \mathbb{Z}$ such that $(z-a)^{-n}f$ is holomorphic in a neighbourhood of a, and $\lim_{z\to a}(z-a)^{-n}f \neq 0$. If $n > 0$ then f has a *zero of order* n; if $n < 0$ then f has a *pole of order* $|n|$. In particular, the zero set as well as the pole set are both discrete. We also say that $ord_a(f) = n$.
- (Laurent Expansions) If f is holomorphic on $A(a; r, R)$, then we have the Laurent expansion

  $$f(z) = \sum_{n\in\mathbb{Z}} c_n(z-a)^n$$

  where for any $r < \delta < R$ we have

  $$c_n = \frac{1}{2\pi i}\int_{\partial D(a,\delta)} \frac{f(\xi)}{(\xi-a)^{n+1}}d\xi$$

  for all $n \in \mathbb{Z}$. In particular, if f is holomorphic on $D^*(a, R)$, then

  1. If $c_n = 0$ for all $n < 0$, then we say that f has a *removable singularity at* a. Equivalently, f has a removable singularity at a if it is bounded in $D^*(a, r)$ for some $r \in (0, R)$.
  2. If there exists $m > 0$ such that $c_n = 0$ for all $n < -m$, then f is meromorphic on $D(a, R)$ with a pole at a.
  3. Otherwise, there is an *essential singularity.*
- (Riemann Removable Singularity Theorem) Suppose f is a meromorphic function on X such that $f : X\backslash\{p\} \to \mathbb{C}$ is holomorphic for some $p \in X$. If f is bounded in a small open (punctured) neighbourhood of p, then f can be uniquely extended to a holomorphic function $f : X \to \mathbb{C}$.
- If f is meromorphic on U, and $a \in U$, write $f(z) = \sum_{n\in\mathbb{Z}} c_n(z-a)^n$. Then, the *residue of* f *at* a is $Res(f; a) = c_{-1}$.

  (Residue Theorem) If f is meromorphic on open connected U, and if G is open with compact closure in U such that ∂G contains no poles of f. Then,

  $$\int_{\partial G} f(\xi)d\xi = 2\pi i \sum_{a\in G} Res(f; a);$$

  this is a finite sum since there are only finitely many poles in G.
- (Argument Principle) If f is meromorphic on open connected U, and if G is open with compact closure in U such that ∂G contains no zeroes or poles of f. Then,

  $$\frac{1}{2\pi i}\int_{\partial G} \frac{f'(\xi)}{f(\xi)}d\xi = \sum_{a\in G} ord_a(f);$$

  this is a finite sum since there are only finitely many zeroes and poles in G.
- (Great Picard Theorem) If f is analytic in $D^*(a, R)$ with an essential singularity at a, then for all $c \in \mathbb{C}\backslash S$ (where $|S| \le 1$ is a subset of $\mathbb{C}$), there are infinitely many solutions in $D^*(a, R)$ to the equation $f(z) = c$.
- (Little Picard Theorem) If f is an entire non-constant function, then either $f(\mathbb{C}) = \mathbb{C}$ or $f(\mathbb{C}) = \mathbb{C}\backslash\{a\}$ for some $a \in \mathbb{C}$.
- (Identity Theorem) Suppose $f : X \to Y$ is a holomorphic map of Riemann surfaces. If for some $q \in Y$ the set $f^{-1}(q)$ has an accumulation point (i.e. there exists $a \in f^{-1}(q)$ such that for any open neighbourhood U of $f^{-1}(q)$, we have $(U \cap f^{-1}(q))\backslash\{a\} \neq \emptyset$) on X, then f is identically zero.

- (Open Mapping Theorem) The image of an open set under a non-constant holomorphic map of Riemann surfaces is also an open set.

  As a corollary, suppose $f : X \to Y$ is a holomorphic map of Riemann surfaces. If X is compact, Y connected, and f non-constant, then Y is compact and f surjective.

- (Schwarz' Lemma) Suppose $f : D(0,1) \to \mathbb{C}$ is holomorphic with $|f(z)| < 1$ and $f(0) = 0$. Then $|f(z)| \le |z|$ for all $z \in D(0,1)$ and $|f'(0)| \le 1$. Moreover, equality holds in either of the two statements iff $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$.

Example 3.1.1 (Fall 2020 Day 3)**.** Prove *Rouche's Theorem*, i.e. if f, g are holomorphic functions on an open neighbourhood of a closed disk D such that $|f(z)| > |g(z)|$ for all $z \in \partial D$, then $f + g$ and f have the same number of zeros (including multiplicity) in D. Using Rouche's Theorem, calculate the number of roots α with $|\alpha| < 1$ (including multiplicity) of $p(z) = z^7 - 2z^5 + 6z^3 - z + 1$.

Since $|f(z)| > |g(z)|$ for all $z \in \partial D$, it follows that $f(z) + tg(z)$ is non-zero on ∂D for all $t \in [0,1]$. By the Argument principle applied to the holomorphic function $f(z) + tg(z)$, the number of zeros $n(t)$ of $f(z) + tg(z)$ is

$$n(t) = \frac{1}{2\pi i}\int_{\partial D} \frac{f'(z) + tg'(z)}{f(z) + tg(z)}dz.$$

This is clearly a continuous function $n(t) : [0,1] \to \mathbb{R}$ whose image is contained in $\mathbb{Z}$. It follows that $n(t)$ is a constant, and therefore f and $f + g$ have the same number of zeros in D.

Finally, note that $|z^7 - 2z^5 - z + 1| \le 5 < 6 = |6z^3|$ for all $|z| = 1$. By Rouche's Theorem, it follows that $p(z) = z^7 - 2z^5 + 6z^3 - z + 1$ has three zeros with multiplicity with $|z| < 1$.

Example 3.1.2 (Fall 2021 Day 2)**.** Let $\Delta \subset \mathbb{C}$ be the open unit disk centred at 0, and let $\Delta^* = \Delta \setminus \{0\}$. Let $f : \Delta^* \to \mathbb{C}$ be a holomorphic function such that, for some $A > 0$, $|f(z)| \le A|z|^{-3/2}$ for all $z \in \Delta^*$. Prove there exists $\alpha \in \mathbb{C}$ such that $f(z) - \alpha z^{-1}$ can be extended to a holomorphic function on Δ.

Consider the holomorphic function $f_1(z) = z^2 f(z)$ on Δ^*. The given inequality becomes $|f_1(z)| \le A\sqrt{|z|}$, and so f_1 is bounded on Δ^*. The Riemann removable singularity theorem implies that f_1 can be extended to a holomorphic function on Δ. Moreover, the inequality $|f_1(z)| \le A\sqrt{|z|}$ then implies that f_1 has a zero at 0. Hence, $f_1 = zf_2$ for some holomorphic function $f_2 : \Delta \to \mathbb{C}$; notice that $f_2(z) = zf(z)$ for $z \in \Delta^*$. Set $\alpha = f_2(0)$, and consider $h(z) = f_2(z) - \alpha$. Then $h : \Delta \to \mathbb{C}$ is a holomorphic function with a zero at 0, and so $h(z) = zg(z)$ for some holomorphic function $g : \Delta \to \mathbb{C}$. It follows that $g(z) = f(z) - \alpha z^{-1}$ on Δ^*, and so g is the required holomorphic extension of $f(z) - \alpha z^{-1}$ to Δ.

Example 3.1.3 (Fall 2020 Day 1)**.** Suppose $U \subset \mathbb{C}$ is open containing the closed unit disk $\bar{D}(0,1)$. Suppose f is a holomorphic function on U except for a simple pole at $z_0 \in S^1$. If $f(z) = \sum_{n\ge 0} a_n z^n$ is the power series of f in $D(0,1)$, then we want to show that $\lim_n \frac{a_n}{a_{n+1}} = z_0$.

Let $b = Res(f, z_0) \neq 0$. Note that the function $g(z) = f(z) - \frac{b}{z - z_0}$ is holomorphic on U, and on $D(0,1)$ has the power series expansion

$$g(z) = \frac{b}{z_0}\frac{1}{1 - (z/z_0)} + \sum_{n\ge 0} a_n z^n = \sum_{n\ge 0}\left(\frac{b}{z_0^{n+1}} + a_n\right)z^n.$$

Here, we use the fact that $|z_0| = 1$ so that $|z/z_0| < 1$. In particular, we get that $\frac{1}{n!}g^{(n)}(0) = \frac{b}{z_0^{n+1}} + a_n$. Now, U is an open set containing $\bar{D}(0,1)$ which is compact. It follows that there exists $R_1 > 1$ such that $\bar{D}(0,R_1) \subset U$. By the Cauchy estimates on g, for any $R < R_1$ we have

$$n! \cdot |b + a_n z_0^{n+1}| = n! \cdot \left|\frac{b}{z_0^{n+1}} + a_n\right| = |g^{(n)}(0)| \le \frac{n!}{R^n}\sup_{|z|=R}|g(z)|.$$

By the maximal principle, the above supremum is bounded above by $\sup_{|z|=R_1}|g(z)| =: C > 0$, and thus

$$|b + a_n z_0^{n+1}| \le \frac{C}{R^n}$$

for all $n \in \mathbb{N}$ and all $R < R_1$. Taking R slightly bigger than 1, we see that $\lim_{n\to\infty} a_n z_0^{n+1} = -b$. Since $b \neq 0$, it then follows that

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = z_0 \lim_{n\to\infty}\frac{a_n z_0^{n+1}}{a_{n+1}z_0^{n+1}} = z_0 \cdot \frac{b}{b} = z_0.$$

Example 3.1.4 (Spring 2020 Day 1; see also Fall 2018 Day 1)**.** Let $\Omega \subset \mathbb{C}$ be a connected open subset. Let $\{f_n\}$ be a sequence of holomorphic functions on Ω converging uniformly on compact sets to f. Suppose $f(z_0) = 0$ for some $z_0 \in \Omega$. Then, we claim that either $f \equiv 0$ on Ω, or there exists a sequence $\{z_n\} \subset \Omega$ such that $z_n \to z_0$ as $n \to \infty$ and for all sufficiently large $n \in \mathbb{N}$, $f_n(z_n) = 0$.

To see this, first note by Weierstrass' Theorem, f is a holomorphic function and moreover f_n' converges uniformly on compact subsets to f'. We may suppose throughout that $f \not\equiv 0$. Suppose first there exists $R > 0$ and there exists a subsequence $\{f_{n_k}\}$ such that the functions f_{n_k} are nowhere zero on $D(z_0, R)$. By the argument principle for f_{n_k} $(k \geq N)$, for any $\delta > 0$ we have $\int_{\partial D(z_0,\delta)} \frac{f_{n_k}'(z)}{f_{n_k}(z)} dz = 0$. By uniform convergence on the compact set $\partial D(z_0, \delta)$, and assuming $f \not\equiv 0$, it follows that $\int_{\partial D(z_0,\delta)} \frac{f'(z)}{f(z)} dz = 0$ for all sufficiently small $\delta > 0$. However, for sufficiently small $\delta > 0$, f has a single zero at z_0 and no poles (as f is holomorphic by Weierstrass' Theorem). The argument principle would then imply that $\int_{\partial D(z_0,\delta)} \frac{f'(z)}{f(z)} dz = 2\pi i \mathrm{ord}_0(f) \neq 0$, a contradiction.

Hence, for any $\epsilon > 0$ and every subsequence $\{f_{n_k}\}$ of $\{f_n\}$, there exists $k \in \mathbb{N}$ and there exists $z \in D(z_0, \epsilon)$ such that $f_{n_k}(z) = 0$. In particular, for any $r > 0$, there exists only finitely many $n \in \mathbb{N}$ such that f_n is nowhere zero on $D(z_0, r)$. If $r > 0$ is such that $D(z_0, r) \subset \Omega$, then for each $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that f_n has a zero in $D(z_0, \frac{r}{k})$ for all $n \geq N_k$. Without loss of generality, suppose the sequence $\{N_k\}$ is increasing. Then, for each $n \geq N_1 =: N$, we set z_n to be the zero of f_n in $D(z_0, \frac{r}{k})$ where k is uniquely defined by $N_k \leq n < N_{k+1}$. Then, we see that $z_n \to z_0$ as $n \to \infty$ and that, for $n \geq N$ we have $f_n(z_n) = 0$.

Example 3.1.5 (Fall 2019 Day 2)**.** Fix $a \in \mathbb{C}$ and $n \geq 2$. Prove that $p(z) = az^n + z + 1$ has a root in $\bar{D}(0, 2)$.

By Vieté's formulae, we known that the product of all roots of p (note that by the fundamental theorem of algebra p has n roots with multiplicity) is $(-1)^n/a$. Hence, if $|a| \geq 2^{-n}$, then the absolute value of the product of all the roots is $\leq 2^n$, and so there must be some root with absolute value ≤ 2. On the other hand, if $|a| < 2^{-n}$, then we have $|az^n| \leq 1 < |z + 1|$ for all $z \in \partial D(0, 1)$. By Rouche's Theorem, it follows that $az^n + z + 1$ has the same number of roots as $z + 1$ in $D(0, 1)$. Hence p has exactly one root in $D(0, 2)$ if $|a| < 2^{-n}$.

Example 3.1.6 (Fall 2021 Day 3)**.** Suppose $a < b$ and f a continuous function on $\bar{S} = \{a \leq \mathrm{Re}\,(z) \leq b\}$ and holomorphic on $S = \{a < \mathrm{Re}\,(z) < b\}$ such that, writing $z = x+iy$, we have the following two growth conditions on f: (1) for any $\epsilon > 0$, $|f(z)| = O(e^{\epsilon|y|})$ as $|y| \to \infty$; and (2) there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \partial S$ and for all $z \in \mathbb{R} \cap S$. Prove that $|f(z)| \leq M$ for all $z \in \bar{S}$.

Set $S^+ = S \cap \mathbb{H}$ and $S^- = S \setminus \bar{\mathbb{H}}$, so that $S^+ \sqcup S^- \sqcup (S \cap \mathbb{R}) = S$. For any $\epsilon > 0$, set $g_\epsilon^+(z) = e^{2\epsilon z} f(z)$ and $g_\epsilon^-(z) = e^{-2\epsilon z} f(z)$. Fix a sign $s \in \{\pm\}$. Then $z \in S^s$ satisfies $s\,\mathrm{Im}\,(z) > 0$. As usual we fix the notation $z = x + iy$.

Let $C_\epsilon^s > 0$ be the constant such that $|f(z)| \leq C_\epsilon^s e^{\epsilon s y}$ as $sy = |y| \to \infty$. Notice that $|g_\epsilon^s(z)| = e^{-2\epsilon s y}|f(z)| \leq C_\epsilon^s e^{-\epsilon s y}$. Suppose $a \in S^s$ is arbitrary; then there exists $R = R_\epsilon^s > \mathrm{Im}\,(a)$ large enough so that $C_\epsilon^s e^{-\epsilon s y} < M$ for all $sy \geq R$. Hence $|g_\epsilon^s(z)| < M$ for all $sy > R$. Also, for the open rectangle $\mathcal{R} = S^s \cap \{|y| < R\}$, we have $|g_\epsilon^s(z)| < M$ on $\partial\mathcal{R} \cap \{y = sR\}$, while on $\partial\mathcal{R} \setminus \{y = sR\} \subset \partial S \cup (S \cap \mathbb{R})$ we have

$$|g_\epsilon^(z)| = e^{-2\epsilon s y}|f(z)| \leq |f(z)| \leq M.$$

Thus g_ϵ^s is bounded by M on $\partial\mathcal{R}$, and so by the maximum principle it follows that g_ϵ^s is bounded by M on $\mathcal{R}$. In particular, noting that $a \in \mathcal{R}$, we have $|f(a)| \leq M e^{2\epsilon s\,\mathrm{Im}(a)}$ for all $\epsilon > 0$. Taking $\epsilon \to 0$ then implies that $|f(a)| \leq M$. Since $a \in S^s$ was arbitrary, it follows that f is bounded by M on S^s for both $s \in \{\pm\}$. Since f was bounded by M on $\bar{S} \setminus (S^+ \cup S^-)$ by assumption, it follows that f is bounded by M on $\bar{S}$ as required.

We end with some examples of integration of special functions on $\mathbb{R}$ by using Residue calculus.

Example 3.1.7. We compute $\int_{\mathbb{R}} \frac{\cos 2x}{x^2+x+1} dx$. Write $S_R^+ = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0, z \in D(0, R)\}$ and $S_R^- = \{z \in \mathbb{C} : \mathrm{Im}\,(z) < 0, z \in D(0, R)\}$, and define $\partial D_R^\pm := [-R, R] \sqcup S_R^\pm$. Then,

$$\int_{-R}^{R} \frac{\cos 2x}{x^2 + x + 1} dx = \mathrm{Re}\left(\int_{-R}^{R} \frac{e^{2ix}}{x^2 + x + 1} dx\right)$$

Suppose $R > 2$. Then,

$$\int_{D_R^+} \frac{e^{2iz}}{z^2 + z + 1} dz = \int_{-R}^{R} \frac{e^{2ix}}{x^2 + x + 1} dx + \int_{S_R^+} \frac{e^{2iz}}{z^2 + z + 1} dz.$$

Since $z^2 + z + 1$ has roots $e^{\pm 2\pi i/3}$, it follows from the Residue theorem that

$$\int_{D_R^+} \frac{e^{2iz}}{z^2 + z + 1} dz = 2\pi i Res\left(\frac{e^{2iz}}{z^2 + z + 1}, e^{2\pi i/3}\right) = 2\pi i \frac{\exp i\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}{e^{2\pi i/3} - e^{-2\pi i/3}} = \frac{2\pi \exp(-\sqrt{3} - i)}{\sqrt{3}}.$$

Now, we see that

$$\left|\frac{e^{2\pi iz}}{z^2+z+1}\right| \leq \frac{e^{-2\pi \operatorname{Im}(z)}}{|z^2+z+1|} \leq \frac{1}{|z|^2-|z|-1} = \frac{1}{R^2-R-1} \to 0$$

as $R \to \infty$. Hence,

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+x+1}dx = \lim_{R\to\infty}\int_{D_R^+} \frac{e^{2iz}}{z^2+z+1}dz = \frac{2\pi\exp(-\sqrt{3}-i)}{\sqrt{3}}.$$

Taking the real part, we have

$$\int_{\mathbb{R}} \frac{\cos 2x}{x^2+x+1}dx = \frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}}\cos 1.$$

Example 3.1.8 (Fall 2020 Day 2)**.** We compute $\int_{\mathbb{R}} \frac{x}{x^2+1}\sin x dx$. Let $a, b, c > 0$ be arbitrary (and sufficiently large enough). Consider the rectangle $R = R(a, b, c) = \{z \in \mathbb{C} : -a < \operatorname{Re}(z) < b, 0 < \operatorname{Im}(z) < c\}$. It is clear that

$$\int_{\partial R} \frac{ze^{iz}}{z^2+1}dz = \int_{-a}^{b} \frac{x}{x^2+1}e^{ix}dx + \int_{b}^{b+ic} \frac{z}{z^2+1}e^{iz}dz - \int_{-a+ic}^{b+ic} \frac{z}{z^2+1}e^{iz}dz - \int_{-a}^{-a+ic} \frac{z}{z^2+1}e^{iz}dz.$$

By the residue theorem, we have

$$\int_{\partial R} \frac{ze^{iz}}{z^2+1}dz = 2\pi i Res\left(\frac{ze^{iz}}{z^2+1}; i\right) = 2\pi i \cdot \frac{ie^{-1}}{2i} = \frac{\pi}{e}i.$$

Now, we have the bound.

$$\left|\int_{-a+ic}^{b+ic} \frac{z}{z^2+1}e^{iz}dz\right| \leq \int_{-a}^{b} \frac{t+ic}{(t+ic)^2+1}e^{-c}e^{it}dt \to 0$$

as $c \to \infty$ (for fixed a, b). Next,

$$\left|\int_{b}^{b+ic} \frac{z}{z^2+1}e^{iz}dz\right| = \int_0^c \frac{|b+it|}{|(b+it)^2+1|}e^{-t}dt \leq \int_0^c \frac{b+t}{b^2+t^2-1}e^{-t}dt \leq \frac{b+c}{b^2} < \frac{A}{b}.$$

Similarly $\left|\int_{-a}^{-a+ic} \frac{z}{z^2+1}e^{iz}dz\right| < \frac{B}{a}$. Hence, we have

$$\left|\int_{-a}^{b} \frac{x}{x^2+1}e^{ix}dx - \frac{\pi}{e}i\right| < \frac{A}{b} + \frac{B}{a}$$

after taking the limit as $c \to \infty$. It follows that $\int_{-\infty}^{\infty} \frac{x}{x^2+1}e^{ix}dx = \frac{\pi}{e}i$. Taking imaginary parts we have

$$\int_{\mathbb{R}} \frac{x}{x^2+1}\sin x dx = \frac{\pi}{e}.$$

Example 3.1.9. We compute $\int_0^{\infty} \frac{x^{a+1}}{(1+x^2)^2}dx$ where $a \in (0, 1)$. Consider the branch of the logarithm with argument from 0 to 2π; the branch cut lies on the positive real axis. Let $0 < r \ll 1 \ll R$. We consider the region $C = C(R, r)$ enclosed by the following segments: the line L_1 from ri to $\sqrt{R^2-r^2}+ri$, the line L_2 from $-ri$ to $\sqrt{R^2-r^2}-ri$ 'below' the branch cut, the circular arc S_1 of $\partial D(0, R)$ between $\sqrt{R^2-r^2}+ri$ and $\sqrt{R^2-r^2}-ri$, and the circular arc $S_2 = \partial D(0, r)$. By the residue theorem, for $R \gg 0$ and $r \ll 1$, we have

$$\int_{\partial C} \frac{z^{a+1}}{(1+z^2)^2}dz = 2\pi i\left(Res\left(\frac{z^{a+1}}{(1+z^2)^2}, i\right) + Res\left(\frac{z^{a+1}}{(1+z^2)^2}, -i\right)\right).$$

The residue of $\frac{z^{a+1}}{(1+z^2)^2}$ at i is

$$\left.\frac{\partial}{\partial z}\frac{z^{a+1}}{(z+i)^2}\right|_{z=i} = \left.\left(\frac{(a+1)z^a}{(z+i)^2} - \frac{2z^{a+1}}{(z+i)^3}\right)\right|_{z=i} = -\frac{a}{4}e^{i\pi a/2}.$$

Similarly the residue at $-i$ is $-\frac{a}{4}e^{i3\pi a/2}$; thus

$$\int_{\partial C} \frac{z^{a+1}}{(1+z^2)^2}dz = -\frac{a\pi i}{2}\left(e^{i3\pi a/2} + e^{i\pi a/2}\right).$$

Now, the integral along L_1 in the limit as $r \to 0$ is

$$I_R := \int_0^R \frac{x^{a+1}}{(1+x^2)^2} dx$$

while the integral along L_2 in the limit as $r \to 0$ is

$$\int_0^R \frac{x^{a+1} e^{2\pi(a+1)i}}{(1+x^2)^2} dx = e^{2\pi a i} I_R.$$

The integral along S_1 may be bounded as follows:

$$\left| \int_{S_1} \frac{z^{a+1}}{(1+z^2)^2} dz \right| \leq 2\pi R \cdot \frac{R^{a+1}}{(R^2-1)^2} \to 0$$

as $r \to 0$ and $R \to \infty$ (independently), since $a < 1$. The integral along S_2 is bounded as

$$\left| \int_{S_2} \frac{z^{a+1}}{(1+z^2)^2} dz \right| < 2\pi r \cdot \frac{r^{a+1}}{(1-r^2)^2} \to 0$$

as $r \to 0$ and $R \to \infty$ (independently). Hence, in the limit as $R \to \infty$ and $r \to 0$, we see that

$$(1 - e^{2\pi i a}) \int_0^\infty \frac{x^{a+1}}{(1+x^2)^2} dx = -\frac{a\pi i}{2} \left(e^{i3\pi a/2} + e^{i\pi a/2} \right).$$

Hence

$$\begin{aligned} \int_0^\infty \frac{x^{a+1}}{(1+x^2)^2} dx &= \frac{a\pi i}{2} \frac{e^{i3\pi a/2} + e^{i\pi a/2}}{e^{2\pi i a} - 1} = \frac{a\pi i}{2} \frac{e^{i\pi a/2} + e^{-i\pi a/2}}{e^{\pi i a} - e^{-\pi i a}} = \frac{a\pi i}{2} \frac{1}{e^{\pi i a/2} - e^{-\pi i a/2}} = \frac{a\pi i}{2} \frac{1}{2i \sin \frac{\pi a}{2}} \\ &= \frac{a\pi}{4 \sin \frac{\pi a}{2}}. \end{aligned}$$

Example 3.1.10 (Fall 2021 Day 1)**.** We evaluate the series $\sum_{n=-\infty}^{\infty} \frac{n^2+1}{n^4+1}$.

Consider $f(z) = \frac{z^2+z+1}{z^4+1} \cot \pi z$. Let C_n be the square in $\mathbb{C}$ whose vertices are $(n+\frac{1}{2})(\pm 1 \pm i)$. Since $\cot \pi(z+1) = \cot \pi z$, and since

$$|\cot \pi z| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| = \frac{e^{-\pi y} + e^{\pi y}}{|e^{-\pi y} - e^{\pi y}|} \to 1$$

as $|y| \to \infty$, and so $|\cot \pi z|$ is uniformly bounded on ∂C_n, independent of n. Since $|\frac{z^2+z+1}{z^4+1}| = O(\frac{1}{n^2})$ while the length of ∂C_n is $O(n)$, it follows that

$$\lim_{n \to \infty} \int_{\partial C_n} f(z) dz = 0.$$

Now, f has simple poles at each $n \in \mathbb{Z}$ with residue

$$Res(f, n) = \lim_{\xi \to 0} \xi f(\xi + n) = \frac{n^2+n+1}{n^4+1} \lim_{\xi \to 0} \frac{\cos \pi(\xi+n)\xi}{\sin \pi(\xi+n)} = \frac{1}{\pi} \cdot \frac{n^2+n+1}{n^4+1}$$

and also has simple poles at $e^{\pi i(2k+1)/4}$ $(0 \leq k \leq 3)$ with residue

$$Res(f, e^{\pi i(2k+1)/4}) = \frac{e^{\pi i(2k+1)/2} + e^{\pi i(2k+1)/4} + 1}{4e^{3\pi i(2k+1)/4}} \cdot \cot \pi e^{\pi i(2k+1)/4}.$$

By the residue theorem, and taking $n \to \infty$, it follows that

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \frac{n^2+n+1}{n^4+1} &= -\frac{\pi}{4} \left(\frac{i + \frac{i+1}{\sqrt{2}} + 1}{\frac{1-i}{\sqrt{2}}} \cot \frac{\pi(1+i)}{\sqrt{2}} + \frac{-i + \frac{1-i}{\sqrt{2}} + 1}{\frac{1+i}{\sqrt{2}}} \cot \frac{\pi(1-i)}{\sqrt{2}} + \frac{i - \frac{i+1}{\sqrt{2}} + 1}{\frac{1-i}{\sqrt{2}}} \cot \frac{-\pi(1+i)}{\sqrt{2}} \right. \\ &\qquad \left. + \frac{-i + \frac{i-1}{\sqrt{2}} + 1}{\frac{1+i}{\sqrt{2}}} \cot \frac{\pi(i-1)}{\sqrt{2}} \right) \\ &= \frac{i\pi}{\sqrt{2}} \left(\cot \frac{\pi(1+i)}{\sqrt{2}} + \cot \frac{\pi(i-1)}{\sqrt{2}} \right) = \frac{\pi\sqrt{2}(e^{\pi\sqrt{2}} - e^{-\pi\sqrt{2}})}{e^{\pi\sqrt{2}} + e^{-\pi\sqrt{2}} - 2\cos \pi\sqrt{2}}. \end{aligned}$$

Example 3.1.11 (Fall 2019 Day 1)**.** Evaluate $I := \int_0^\pi \frac{d\theta}{a-b\cos\theta}$ where $a, b \in \mathbb{R}$ satisfy $a > b > 0$.

Note first that
$$\int_\pi^{2\pi} \frac{d\theta}{a-b\cos\theta} = -\int_\pi^0 \frac{d\theta}{a-b\cos(2\pi-\theta)} = I$$
so that
$$I = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a-b\cos\theta} = \frac{1}{2i}\int_{\partial\Delta} \frac{z^{-1}dz}{a-\frac{b}{2}(z+z^{-1})} = -\frac{1}{bi}\int_{\partial\Delta} \frac{dz}{z^2-\frac{2a}{b}z+1},$$
where Δ is the standard unit disk. The roots of the denominator are $\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2}-1}$, the smaller of which lies inside Δ and the other outside. By the residue theorem, it then follows that
$$I = -\frac{1}{bi}\cdot 2\pi i \mathrm{Res}\left(\frac{1}{z^2-\frac{2a}{b}z+1}, \frac{a}{b}-\sqrt{\frac{a^2}{b^2}-1}\right) = -\frac{2\pi}{b}\cdot\frac{1}{\frac{a}{b}-\sqrt{\frac{a^2}{b^2}-1}-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}} = \frac{\pi}{\sqrt{a^2-b^2}}.$$

3.2 Conformal Maps

Definition. Suppose f is a continuous function defined on a neighbourhood of z_0. Then, f is *conformal at* z_0 if it preserves angles at z_0, i.e. if γ_1 and γ_2 are curves through z_0, then the angle between the tangent lines of γ_1 and γ_2 at z_0 is the same as the angle between the tangent lines of $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(z_0)$.

If f is conformal at each point in U, then it is *conformal on* U.

Definition. A function f is *locally injective* at z_0 if there exists $\delta > 0$ such that for any distinct $z_1, z_2 \in D(z_0, \delta)$, we have $f(z_1) \neq f(z_2)$.

A function that is locally injective at each point in U is *locally injective on* U.

Definition. Suppose $k \in \mathbb{N}$. A function f is a *k-to-1 mapping of* U *to* V if for any $\alpha \in V$, the equation $f(z) = \alpha$ has k roots in U (counting multiplicity).

For instance, the function z^k at 0 multiplies angles by a factor of k, and is a k-to-1 mapping from $D(0, r)$ to $D(0, r^k)$ for any $r > 0$.

Theorem 3.2.1. *Suppose f is a non-constant function that is holomorphic at z_0, and let k be the least positive integer such that $f^{(k)}(z_0) \neq 0$. Then f multiplies angles at z_0 by a factor of k, and there is a sufficiently small open set containing z_0 such that f is a k-to-1 mapping on this open set.*

In particular, if $f'(z_0) \neq 0$ then f is conformal at z_0 and is locally injective at z_0.

As an application, we have the following important result.

Proposition 3.2.2. *If f is an injective holomorphic function on an open connected set U, then $f^{-1} : f(U) \to U$ exists and is holomorphic. Moreover, the maps f and f^{-1} are conformal in D and $f(D)$ respectively.*

More generally, if $F : X \to Y$ is an injective holomorphic mapping of Riemann surfaces, then $F : X \to F(X)$ is a bi-holomorphic map.

Definition. A bijective holomorphic mapping on U is called a *conformal mapping* on U.

Two open connected sets U and V are *conformally equivalent* if there exists a conformal mapping of U onto V.

We list here some important examples of conformal mappings:

1. Linear maps $z \mapsto az + b$ are conformal mappings on $\mathbb{C}$.
2. The function z^α ($\alpha > 0$) is a conformal mapping from circular segments $\{z : \theta_1 < Arg(z) < \theta_2\}$ onto $\{z : \alpha\theta_1 < Arg(z) < \alpha\theta_2\}$ whenever $\theta_2 - \theta_1 \in (0, \frac{2\pi}{\alpha})$. Here, we fix the branch cut of the logarithm to be the negative real axis.
3. The exponential e^z is a conformal mapping from the region $\{z : y_1 < \mathrm{Im}(z) < y_2\}$ (whenever $y_2 - y_1 < 2\pi$) onto the circular segment $\{z : y_1 < Arg(z) < y_2\}$.
4. (*Möbius transformations*) The map $f(z) = \frac{az+b}{cz+d}$ (with $ad - bc \neq 0$) has derivative $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$, and is injective and conformal on its domain. It is in fact a conformal mapping from $\mathbb{C} \setminus \{-\frac{d}{c}\}$ onto $\mathbb{C} \setminus \{\frac{a}{c}\}$. These maps map circles to circles and lines to lines. Apart from the identity map z, Möbius transformations have at most two fixed points. Moreover, for any two sets z_1, z_2, z_3 and w_1, w_2, w_3 of 3 distinct points in $\mathbb{C}$, there exists a unique Möbius transformation sending z_i to w_i for each $i = 1, 2, 3$.

Definition. An automorphism of an open connected set U is a conformal mapping from U onto itself.

We have the following results on automorphisms:

1. If $f : U \to V$ is a conformal map, then every other conformal map $U \to V$ must be of the form $g \circ f : U \to V$, where g is an automorphism of V.
2. If $f : U \to V$ is a conformal map, the map $g \mapsto f^{-1} \circ g \circ f$ is a group isomorphism from the automorphism group of V to the automorphism group of U.
3. The automorphisms of the unit disk $D(0,1)$ are precisely of the form $e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$ where $\theta \in \mathbb{R}$ and $|\alpha| < 1$.
4. All conformal mappings from the upper half-plane $\mathbb{H}$ to $D(0,1)$ are of the form $e^{i\theta}\frac{z-\alpha}{z-\bar{\alpha}}$, where $\alpha \in \mathbb{H}$ and $\theta \in \mathbb{R}$.
5. The automorphisms of the upper half plane $\mathbb{H}$ are of the form $\frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

Theorem 3.2.3 (Riemann Mapping Theorem)**.** *For any two simply connected open sets $U, V \subsetneq \mathbb{C}$, and for any fixed $z_0 \in U, w_0 \in V$, there exists a unique conformal mapping $\varphi : U \to V$ such that $\varphi(z_0) = w_0$ and such that $\varphi'(z_0) \in \mathbb{R}^+$.*

3.3 Special Functions

3.3.1 Harmonic Functions

Definition. A real valued C^2-function $u : U \to \mathbb{R}$ is *harmonic on U* if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \equiv 0$ on U. Equivalently, u is harmonic if $\frac{\partial^2 u}{\partial z \partial \bar{z}} \equiv 0$.

Example 3.3.1. For any holomorphic $f : U \to \mathbb{R}$, a consequence of the Cauchy-Riemann equations is that the smooth functions $\operatorname{Re}(f)$, $\operatorname{Im}(f)$, and $\log|f|^2$ are harmonic. If u is harmonic, then $f = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is holomorphic.

Theorem 3.3.2 (Mean Value Theorem)**.** *Suppose u is continuous on $\bar{D}(z_0, r)$. Then u is harmonic on $D(z_0, r)$ iff*

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = u(z_0)$$

for any sufficiently small $r > 0$.

Theorem 3.3.3 (Poisson's Formula)**.** *Suppose u is harmonic on $D(z_0, r)$ and continuous on $\partial D(z_0, r)$. For any $a \in D(z_0, r)$, writing $a = z_0 + r_0 e^{i\theta_0}$ we have*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 - 2rr_0\cos(\theta - \theta_0) + r_0^2} u(z_0 + re^{i\theta})d\theta.$$

Theorem 3.3.4 (Maximum Principle)**.** *If u is harmonic on the open U, then u achieves neither maximum nor minimum on U. In particular, if u is harmonic in an open neighbourhood of $\bar{G}$, where G is open and $\bar{G}$ is compact, then the maximum of u on G must be obtained on ∂G.*

In particular, if a continuous function is identically zero on ∂G and harmonic on G (where G is open with compact closure), then it is identically zero on G.

Theorem 3.3.5 (Schwarz' Theorem for Harmonic Functions)**.** *Suppose U is any given continuous function on $\partial D(0,1)$. Then, the* Poisson integral of *U,*

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) U(e^{i\theta})d\theta,$$

is a harmonic function on $D(0,1)$ such that $U(a) = \lim_{z \to a} P_U(z)$ for any $a \in \partial D(0,1)$.

Theorem 3.3.6 (Schwarz' Reflection Principle)**.** *Suppose Ω is a symmetric (i.e. $z \in \Omega$ iff $\bar{z} \in \Omega$) connected open set. Set $\Omega^+ := \{z \in \Omega : \operatorname{Im}(z) > 0\}$, $\Omega^- := \{z \in \Omega : \operatorname{Im}(z) < 0\}$, and $\Omega^0 = \Omega \cap \mathbb{R}$.*

- *Suppose v is continuous on $\Omega^+ \cup \Omega^0$, harmonic on Ω^+, and identically zero on Ω^0. Then there exists a harmonic function $\tilde{v}$ on Ω such that $\tilde{v}|_{\Omega^+ \cup \Omega^0} = v$ and such that $\tilde{v}(z) = -v(\bar{z})$ for all $z \in \Omega^-$.*

- *Suppose f is continuous on $\Omega^+ \cup \Omega^0$, holomorphic on Ω^+, and $f(\Omega^0) \subset \mathbb{R}$. Then there exists a holomorphic function $\tilde{f}$ on Ω such that $\tilde{f}|_{\Omega^+ \cup \Omega^0} = f$ and such that $\tilde{f}(z) = \overline{f(\bar{z})}$ for all $z \in \Omega^-$.*

Theorem 3.3.7 (Harnack's Inequality). *If u is an everywhere non-negative continuous function on $\bar{D}(z_0, r)$ that is harmonic on $D(z_0, r)$, then*

$$\frac{r - |z - z_0|}{r + |z - z_0|} \cdot u(z_0) \leq u(z) \leq \frac{r + |z - z_0|}{r - |z - z_0|} \cdot u(z_0)$$

for any $z \in D(z_0, r)$.

In particular, Harnack's Inequality allows us to establish uniform convergence of a sequence of harmonic functions in an open neighbourhood of a point given that the harmonic functions converge at that point. As an example of this, we have the following.

Theorem 3.3.8 (Harnack's Principle). *Suppose $u_n : \Omega_n \to \mathbb{R}$ is a sequence of harmonic functions on the open connected set Ω_n. Let Ω be an open connected subset of $\mathbb{C}$ such that at each $p \in \Omega$ there exists an open neighbourhood U_p such that $U_p \subset \Omega_n$ for all sufficiently large n. Moreover, suppose that these U_p are such that for any $z \in U_p$, $u_n(z) \leq u_{n+1}(z)$ for all sufficiently large n. Then, either $u_n(z)$ tends uniformly to $+\infty$ on all compact subsets of Ω, or $u_n(z)$ tends uniformly on compact sets to a harmonic function u on Ω.*

Definition. A continuous function $v : \Omega \to \mathbb{R}$ (Ω open and connected) is *sub-harmonic in* Ω if for any open connected region $\Omega' \subset \Omega$ and any harmonic function u on Ω', the function $v - u$ also satisfies the maximal principle on Ω' (i.e. does not achieve a maximum in Ω', unless it is constant).

Proposition 3.3.9. *Suppose $v : \Omega \to \mathbb{R}$ is C^2. Then v is sub-harmonic iff $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \geq 0$ on Ω.*

Theorem 3.3.10. *A continuous function $v : \Omega \to \mathbb{R}$ is sub-harmonic iff for any open disk $D(z_0, r) \subset \Omega$, we have*

$$v(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + re^{i\theta}) d\theta.$$

Theorem 3.3.11. *Suppose Ω is an open connected set such that for each $p \in \partial\Omega$, there exists a line segment L with an endpoint at p such that $L \cap \bar{\Omega} = \{p\}$. Let f be any continuous function on $\partial\Omega$. Define $\mathcal{B}(f)$ to be the set of sub-harmonic v on Ω such that* $\limsup_{z \to \xi} \leq f(\xi)$ *for all $\xi \in \partial\Omega$. Then, the function $u : \Omega \to \mathbb{R}$ given by*

$$u(z) = \sup_{v \in \mathcal{B}(f)} v(z)$$

is C^2, harmonic on Ω, and uniquely solves the Dirichlet problem on Ω with boundary conditions f, *i.e. u is the unique harmonic function on Ω such that for each $\xi \in \partial\Omega$ we have $\lim_{z \to \xi} u(z) = f(\xi)$.*

3.3.2 Exponentials, Logarithms, Trigonometric, and Hyperbolic Functions

The exponential function e^z is given by any of the following:

$$e^z = \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^x \cos y + ie^x \sin y.$$

Also, e^z is the unique holomorphic function $f : D(0, R) \to \mathbb{C}$ such that $f'(z) \equiv f(z)$ and $f(0) = 1$.

The *Principal Logarithm* $Log : \mathbb{C} \backslash \{x \in \mathbb{R} : x \leq 0\} \to \mathbb{C}$ is the holomorphic function $Log(z) = \log |z| + i\text{Arg}(z)$ where we choose $\text{Arg}(z)$ to be the unique solution $y \in [-\pi, \pi)$ satisfying $e^{iy} = z/|z|$. Different choices of the range of the argument yield different branches of the logarithm. More generally, a holomorphic function $f : \Omega \to \mathbb{C}$ where Ω is open, connected, and does not contain the origin, is called a *branch of the logarithm on* Ω if $e^{f(z)} = z$ for all $z \in \Omega$. Any branch of the logarithm may be denoted by *log*. If Ω is simply connected, then we can always define a branch of the logarithm on Ω. More precisely, if Ω is simply connected open set not containing 0, and if we fix some point $z_0 \in \Omega$ and also fix a corresponding value of $log(z_0)$, then we can define $log(z) := log(z_0) + \int_{z_0}^{z} \frac{d\zeta}{\zeta}$ where the curve of integration can be any curve from z_0 to z.

By fixing a branch *log* of the logarithm, we can then define complex exponentials $z^c := e^{c log(z)}$ for any c, z.

We also have the trigonometric and hyperbolic functions:

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} = \sum_{k \geq 0} \frac{(-1)^k}{(2k)!} z^{2k}, \qquad \sin z := \frac{e^{iz} - e^{-iz}}{2i} = \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)!} z^{2k+1},$$

$$\cosh z := \frac{e^z + e^{-z}}{2} = \sum_{k \geq 0} \frac{1}{(2k)!} z^{2k}, \qquad \sinh z := \frac{e^z - e^{-z}}{2} = \sum_{k \geq 0} \frac{1}{(2k+1)!} z^{2k+1}.$$

3.3.3 Partial Fractions and Canonical Products

Partial Fractions

Theorem 3.3.12 (Mittag-Leffler). *Suppose $\{b_n\}$ is a discrete sequence of complex numbers with $b_n \to \infty$, and suppose P_n are polynomials with no constant term. Then, there are meromorphic functions on $\mathbb{C}$ with poles precisely at the points b_n with corresponding singular parts $P_n\left(\frac{1}{z-b_n}\right)$. Moreover, the most general form of such a meromorphic function is given by*

$$\sum_{n\geq 1}\left(P_n\left(\tfrac{1}{z-b_n}\right) - p_n(z)\right) + g(z)$$

for some suitable polynomials p_n and some entire function g. Moreover, on any compact set K, by omitting the terms $P_n\left(\frac{1}{z-b_n}\right) - p_n(z)$ for which $b_n \in K$, the series converges absolutely and uniformly on K.

We take some examples:

1. (*Fall 2019 Day 3*) Consider the function $f(z) = \pi^2/\sin^2 \pi z$. It has double poles at precisely the point $z = n$ with $n \in \mathbb{Z}$, and moreover the singular part is precisely $\frac{1}{(z-n)^2}$ at $z = n$. Since the series $\sum_n (z-n)^{-2}$ converges absolutely on compact sets whenever we omit finitely many terms, it follows that

$$f(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} + g(z)$$

for some entire g. Now notice that both f and the above series are periodic with period 1, and moreover that as $|\operatorname{Im}(z)| \to \infty$ both f and the series converge uniformly in $\operatorname{Re}(z)$ to 0. Hence, $g(z) \to 0$ as $\operatorname{Im}(z) \to \pm\infty$ uniformly for $\operatorname{Re}(z)$ in some interval, so that g is bounded in the strip $\{0 < \operatorname{Re}(z) < 2\}$. Since g is periodic with period 1, it follows that g is bounded everywhere and thus by Liouville's theorem is constant. Since $g(z) \to 0$ as $\operatorname{Im}(z) \to \pm\infty$, it follows that $g \equiv 0$. Therefore

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}.$$

2. We integrate both sides of the above identity. On the right we get $-\pi \cot \pi z$, while on the left the general term is $-\frac{1}{z-n}$. Since $\sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right)$ converges (say by comparing with $\sum_n \frac{1}{n^2}$), it follows that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right).$$

If instead we group the terms $\frac{1}{z-n}$ and $\frac{1}{z+n}$ together, the series again converges so that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\geq 1} \frac{2z}{z^2 - n^2}.$$

3. We evaluate $f(z) = \frac{1}{z} + \sum_{n\geq 1} (-1)^n \frac{2z}{z^2-n^2} = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n}$. By writing

$$\sum_{n=-(2k+1)}^{2k+1} \frac{(-1)^n}{z-n} = \sum_{n=-k}^{k} \frac{1}{z-2n} - \sum_{n=-k-1}^{k} \frac{1}{z-1-2n}$$

and comparing with the series for $\pi \cot \pi z$, we see that

$$f(z) = \frac{\pi}{2}\cot\frac{\pi z}{2} - \frac{\pi}{2}\cot\frac{\pi(z-1)}{2} = \frac{\pi}{\sin \pi z}.$$

Therefore

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n\geq 1} (-1)^n \frac{2z}{z^2-n^2} = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n}.$$

4. As another example we compute $f(z) := \sum_n \frac{1}{z^3-n^3}$, which converges absolutely and locally uniformly. Setting $\omega = e^{2\pi i/3}$, we have

$$\frac{1}{z^3-n^3} = \frac{1/3n^2}{z-n} + \frac{1/3n^2}{(z/\omega)-n} + \frac{1/3n^2}{(z/\omega^2)-n},$$

we can rewrite the partial sum as

$$\sum_{n=-m}^{m} \frac{1}{z^3-n^3} = \frac{1}{z^3} + \sum_{n=1}^{m} \frac{2z/3n^2}{z^2-n^2} + \sum_{n=1}^{m} \frac{2z/3n^2}{(z/\omega)^2-n^2} + \sum_{n=1}^{m} \frac{2z/3n^2}{(z/\omega^2)^2-n^2}.$$

It thus suffices to understand $\lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n^2}\frac{2z}{z^2-n^2}$. Notice that the function $\frac{\pi}{z^2 \sin \pi z}$ has simple poles at $z = n$ ($n \in \mathbb{Z} \setminus \{0\}$) with residue $\frac{1}{n^2}$, and moreover at $z = 0$ is of the form

$$\frac{\pi}{z^2 \sin \pi z} = \frac{1}{z^3} \frac{1}{1 - \frac{1}{6}(\pi z)^2 + O(z^4)} = \frac{1}{z^3}\left(1 + \frac{1}{6}(\pi z)^2 + O(z^4)\right) = \frac{1}{z^3} + \frac{\pi^2/6}{z} + O(z).$$

It follows that

$$\frac{\pi}{z^2 \sin \pi z} - \frac{1}{z^3} - \frac{\pi^2/6}{z} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n^2} \frac{2z}{z^2-n^2}.$$

Hence

$$\begin{aligned} f(z) &= \frac{1}{z^3} + \frac{1}{3}\left(\frac{\pi}{z^2 \sin \pi z} - \frac{1}{z^3} - \frac{\pi^2/6}{z} + \frac{\pi}{(\frac{z}{\omega})^2 \sin \frac{\pi z}{\omega}} - \frac{1}{(z/\omega)^3} - \frac{\pi^2/6}{z/\omega} + \frac{\pi}{(\frac{z}{\omega^2})^2 \sin \frac{\pi z}{\omega^2}} - \frac{1}{(z/\omega^2)^3} - \frac{\pi^2/6}{z/\omega^2}\right) \\ &= \frac{1}{3}\left(\frac{\pi}{z^2 \sin \pi z} + \frac{\pi\omega}{z^2 \sin \pi z\omega} + \frac{\pi\omega^2}{z^2 \sin \pi z\omega^2}\right). \end{aligned}$$

We therefore have the identity

$$\frac{\pi}{3z^2 \sin \pi z} + \frac{\pi\omega}{3z^2 \sin \pi z\omega} + \frac{\pi\omega^2}{3z^2 \sin \pi z\omega^2} = \sum_{n\in\mathbb{Z}} \frac{1}{z^3-n^3}.$$

Canonical Products

Recall that an infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges (absolutely) iff the series $\sum_{n=1}^{\infty} Log(1+a_n)$ converges (absolutely), where we assume none of the a_n are -1. Moreover, the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges if the series $\sum_{n=1}^{\infty}|a_n|$ converges. If u_k are a sequence of holomorphic functions on an open connected set U such that $\sum_{k=1}^{\infty}|u_k(z)|$ converges uniformly on compact sets, then the product $\prod_{k=1}^{\infty}(1+u_k(z))$ converges uniformly on compact sets to an analytic function on U.

Now, a holomorphic function f on a simply connected open set Ω (for instance $\Omega = \mathbb{C}$) which is nowhere 0 has a well-defined logarithm, in the sense that the function f'/f is also holomorphic function on Ω. By simple-connectedness of Ω, we can integrate f'/f to get a well-defined holomorphic function on Ω satisfying $g' = f'/f$, so that fe^{-g} is identically a constant, and hence $f(z) = e^{g(z)}$ where we may absorb any constant into g.

Now suppose f is an entire function with a zero of order m at $z = 0$, and has remaining zeros given by $a_1, ..., a_n \in \mathbb{C} \setminus \{0\}$ (multiple zeros are repeated here); then for some holomorphic function g we have

$$f(z) = z^m e^{g(z)} \prod_{i=1}^{m}\left(1 - \frac{z}{a_n}\right).$$

More generally, if the entire f has a zero of order m at $z = 0$, and has remaining zero $\{a_n\} \subset \mathbb{C} \setminus \{0\}$ (multiple zeros listed repeatedly), then for some holomorphic function g we have

$$f(z) = z^m e^{g(z)} \prod_{n\geq 1}\left(1 - \frac{z}{a_n}\right)$$

assuming the infinite product converges locally uniformly, which is true iff $\sum_{n=1}^{\infty} \frac{1}{|a_n|}$ converges.

However, if this does not hold, we can introduce error terms, as follows.

Proposition 3.3.13 (Weierstrass Theorem on Entire Products). *Given any sequence $\{a_n\} \subset \mathbb{C}$ of distinct non-zero complex numbers with prescribed multiplicities $m_n \in \mathbb{N}$, and given a prescribed multiplicity $m \in \mathbb{Z}_{\geq 0}$ for a zero at $z = 0$, any entire function f with precisely these prescribed zeros is of the form*

$$f(z) = z^m e^{g(z)} \prod_{n\geq 1} \left(1 - \frac{z}{a_n}\right)^{m_n} \exp\left(m_n \tfrac{z}{a_n} + \frac{1}{2} m_n \left(\tfrac{z}{a_n}\right)^2 + \frac{1}{3} m_n \left(\tfrac{z}{a_n}\right)^3 + \cdots + \frac{1}{k_n} m_n \left(\tfrac{z}{a_n}\right)^{k_n}\right)$$

for some integers k_n and some entire function g. Here, this product converges locally uniformly and absolutely.

Moreover, if there exists an integer h such that $\sum_{n=1}^{\infty} \frac{m_n}{|a_n|^{h+1}}$, then we may pick all of the $k_n = h$. By picking h to be smallest possible such that the series converges, then the above representation is unique and is called the canonical product of f. *The integer h is called the* genus of f.

Example 3.3.14. Consider $\sin \pi z$. Since the zeros of $\sin \pi z$ are all simple and precisely $\mathbb{Z}$, and the sum $\sum_{n\geq 1} \frac{1}{n}$ diverges whereas $\sum_{n\geq 1} \frac{1}{n^2}$ converges, it follows that $\sin \pi z$ has genus 1 and that

$$\sin \pi z = e^g z \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1 - \frac{z}{n}\right) e^{z/n}$$

for some holomorphic g. Taking logarithmic derivatives yields

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{z-n} + \frac{1}{n}\right)$$

We thus see that $g' \equiv 0$ so that e^g is a constant. Since $\frac{\sin \pi z}{z} \to \pi$ as $z \to 0$, we see that this constant must be π. Therefore, the canonical product for $\sin \pi z$ is

$$\sin \pi z = \pi z \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

By bracketing terms corresponding to n and $-n$, it follows that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Example 3.3.15.

3.3.4 Riemann Zeta Function and the Gamma Function

Gamma Function

Definition. *Euler's Gamma function* is the meromorphic function on $\mathbb{C}$ given by the product

$$\Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

where γ is the Euler-Mascheroni constant given by

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdot + \frac{1}{n} - \log n\right) \approx 0.57722.$$

We have the following properties of the Gamma function:

1. The Gamma function has no zeroes on $\mathbb{C}$, and has simple poles precisely at $\mathbb{Z}_{\leq 0} = \{0, -1, -2, -3, ...\}$. In particular, $1/\Gamma$ is an entire function whose only zeros are simple zeroes at $\mathbb{Z}_{\leq 0}$. We have $Res(\Gamma, -n) = \frac{1}{n!}$ for all $n \geq 0$.
2. We have the functional equations $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$; both of these may be obtained by considering the canonical products of both sides of the equation.
3. We have $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
4. $\displaystyle \frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$

5. (*Legendre's Duplication Formula*) $\Gamma(2z) = \pi^{-\frac{1}{2}} 2^{2z-1} \Gamma(z)\Gamma(z+\frac{1}{2})$.

6. (*Gauss' Formula*) $\Gamma(z) = (2\pi)^{(1-n)/2} n^{z-\frac{1}{2}} \Gamma\left(\frac{z}{n}\right)\Gamma\left(\frac{z+1}{n}\right)\cdots\Gamma\left(\frac{z+n-1}{n}\right)$.

7. (*Stirling's Formula*) Define the function $J(z) := -\frac{1}{\pi}\int_0^\infty \frac{z}{\eta^2+z^2}\log\left(1-e^{-2\pi\eta}\right)d\eta$ where $\mathrm{Re}\,(z) > 0$. This is holomorphic in z, and converges to 0 as $|z| \to +\infty$ and $\mathrm{Re}\,(()\,z) \gg 0$. Then, we have

$$\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} e^{J(z)}$$

where we use the principal branch of the logarithm to define $z^{z-\frac{1}{2}} = \exp\left((z-\frac{1}{2})Log(z)\right)$.

The above function J is in fact asymptotically given as follows: for any $k \in \mathbb{N}$,

$$J(z) = \frac{B_1}{1\cdot 2}\frac{1}{z} - \frac{B_2}{3\cdot 4}\frac{1}{z^3} + \cdots + (-1)^{k-1}\frac{B_k}{(2k-1)2k}\frac{1}{z^{2k-1}} + J_k(z)$$

where $J_k(z)$ is a holomorphic function on $\{\mathrm{Re}\,(z) > 0\}$ such that, for any $c > 0$, we have $z^{2k}J_k(z) \to 0$ as $|z| \to \infty$ in the half plane $\{\mathrm{Re}\,(z) > c\}$. Here $\{B_n\}$ are the *Bernoulli numbers* given by

$$\frac{z}{e^z-1} =: 1 - \frac{1}{2}z + \sum_{n\geq 1}(-1)^{n-1}\frac{B_n}{(2n)!}z^{2n}.$$

8. If $\mathrm{Re}\,(z) > 0$ then $\Gamma(z) = \displaystyle\int_0^\infty e^{-t}t^{z-1}dt$.

Riemann Zeta Function

Definition. The *Riemann Zeta Function* is the holomorphic continuation to $\mathbb{C}\backslash\{1\}$ of the holomorphic function defined on $\{\mathrm{Re}\,(z) > 1\}$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Properties of the Riemann Zeta Function:

1. The analytic continuation is given by

$$\zeta(s) = -\frac{1}{2\pi i}\Gamma(1-s)\int_C \frac{(-z)^{s-1}}{e^z-1}dz,$$

the branch of the logarithm is fixed with $-\pi < \mathrm{Im}\,(log(-z)) < \pi$ so that $(-z)^{s-1} = \exp\left((s-1)log(-z)\right)$. Here, the curve of integration C is taken to be any curve consisting of the two lines $\{\mathrm{Re}\,(z) \geq \sqrt{1-r^2},\ \mathrm{Im}\,(z) = ir\}$ and $\{\mathrm{Re}\,(z) \geq \sqrt{1-r^2},\ \mathrm{Im}\,(z) = -ir\}$, and the greater arc of the circle $\partial D(0,r)$ between $\sqrt{1-r^2}+ir$ and $\sqrt{1-r^2}-ir$, where $r \in (0,1)$ is arbitrary.

2. For $\mathrm{Re}\,(z) > 1$, $\zeta(s) = \displaystyle\prod_p \frac{1}{1-p^{-s}}$, the product taken over all primes.

3. (functional equation) $\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)$.

   Stated another way, the function $\xi(s) = \frac{1}{2}s(1-s)\pi^{-s/2}\Gamma(\frac{s}{2})\cdot\zeta(s)$ is an entire function satisfying $\xi(s) = \xi(1-s)$.

4. We have $\zeta(0) = -\frac{1}{2}$, $\zeta(-2m) = 0$ for $m \in \mathbb{Z}$, and $\zeta(1-2m) = (-1)^m\frac{1}{2m}B_m$, where B_m are the Bernoulli numbers defined previously.

5. The only pole of $\zeta(s)$ is the simple pole at $s = 1$, with residue 1. The only zeroes of ζ outside the *critical strip* $\{0 < \mathrm{Re}\,(s) < 1\}$ are the negative even integers.

3.3.5 Elliptic Functions

We now study meromorphic functions $f : T \to \mathbb{C}$ where T is the compact torus $\mathbb{C}/L$, $L := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for ω_1, ω_2 $\mathbb{R}$-linearly independent complex numbers. This is equivalent to studying meromorphic functions $f : \mathbb{C} \to \mathbb{C}$ that are *doubly-periodic* with *periods* ω_1, ω_2, i.e. such that $f(z + \omega_1) = f(z)$ and $f(z + \omega_2) = f(z)$ for all $z \in \mathbb{C}$ (barring poles). Such doubly-periodic functions are also called *elliptic functions*.

Fix the lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. A *fundamental parallelogram* is any parallelogram $P = \{\lambda + r\omega_1 + s\omega_2 : r, s \in [0, 1)\}$ where $\lambda \in \mathbb{C}$; it is fundamental in the sense that if $z, w \in P$ with $z - w \in L$, then $z = w$.

We list some elementary properties of elliptic functions:

- A holomorphic elliptic function must be constant.
- Elliptic functions form an $\mathbb{R}$-algebra. Moreover, f' is elliptic if f is elliptic.
- The sum of residues of an elliptic function inside any fundamental parallelogram is zero.
- Counting multiplicities, the number of zeros of an elliptic function in a fundamental parallelogram is equal to the number of poles in the fundamental parallelogram.
- If $a_1, ..., a_m$ are the zeros and $b_1, ..., b_m$ are the poles of an elliptic function, with higher order zeros and poles counted with multiplicity, then $(a_1 + \cdots + a_m) - (b_1 + \cdots + b_m) \in L$.

We now briefly describe the Weierstrass theory of elliptic functions. Throughout we fix the period lattice L.

Definition. The *Weierstrass $\wp$-function* is the unique elliptic function with period lattice L whose only pole is a double pole on all lattice points, normalized so that for any $\omega \in L$ we have $\wp(z) - \frac{1}{(z-\omega)^2} \to 0$ as $z \to \omega$.

We have the following properties of the Weierstrass $\wp$-function.

1. $\wp(z) = \dfrac{1}{z^2} + \displaystyle\sum_{\omega \in L\setminus 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$, where the series converges absolutely and locally uniformly.
2. $\wp'(z) = -2 \displaystyle\sum_{\omega \in L} \frac{1}{(z-\omega)^3}$.
3. We have the differential equation $\wp'(z)^2 = 4\wp(z)^3 - 60G_2\wp(z) - 140G_3$, where $G_k := \displaystyle\sum_{\omega \in L\setminus 0} \frac{1}{\omega^{2k}}$.
4. Since $\wp'$ is an elliptic function with only a triple pole in a fundamental parallelogram, it follows that it has 3 zeros (including multiplicities). Oddness of $\wp'$ implies that these zeros are in fact all simple zeros, and moreover the zeros are $z = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. Moreover, the polynomial $4x^3 - 60G_2x - 140G_3 = 0$ has the three roots $\wp(\frac{\omega_1}{2}), \wp(\frac{\omega_2}{2}), \wp(\frac{\omega_1+\omega_2}{2})$. These roots are *distinct*.

The anti-derivative of $\wp$ is usually denoted by $-\zeta$, which is then normalized so that it is odd. We have

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in L\setminus 0} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

As usual convergence is absolute and locally uniform. Its only poles are simple poles at all lattice points (with residue 1). We have the following properties:

1. By definition $\zeta' = -\wp$ and $\zeta(-z) = -\zeta(z)$.
2. ζ is never elliptic. However $\zeta(z+\omega_i) - \zeta(z)$ is a constant (has zero derivative); define $\eta_i := \zeta(z+\omega_i) - \zeta(z)$ for $i = 1, 2$. By evaluating the integral of ζ along the boundary of a fundamental parallelogram, we get *Legendre's relation*, namely $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$.

Now define the function σ given by the canonical product

$$\sigma(z) = z \prod_{\omega \in L\setminus 0} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}.$$

This is an entire function whose only zeroes are simple zeroes at all lattice points. It also satisfies

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z).$$

We have the identities

$$\sigma(z+\omega_1) = -\sigma(z)e^{\eta_1(z+\frac{1}{2}\omega_1)} \quad \text{and} \quad \sigma(z+\omega_2) = -\sigma(z)e^{\eta_2(z+\frac{1}{2}\omega_2)}.$$

The importance of σ lies in the fact that every elliptic function with period lattice L is of the form

$$C\prod_{n=1}^{m}\frac{\sigma(z-a_n)}{\sigma(z-b_n)}$$

where $a_1, ..., a_m$ are the zeroes and $b_1, ..., b_m$ are the poles of the elliptic function (counted with multiplicity).

We have a bunch of identities due to the simple fact that holomorphic elliptic functions are necessarily constant, along with some simple algebraic manipulations and/or logarithm differentiations.

- $\wp(z) - \wp(u) = -\dfrac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$.
- $\zeta(z+u) = \zeta(z) + \zeta(u) + \dfrac{1}{2}\dfrac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}$.
- $\wp(z+u) = -\wp(z) - \wp(u) + \dfrac{1}{4}\left(\dfrac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}\right)$.
- $\wp(2z) = \dfrac{1}{4}\left(\dfrac{\wp''(z)}{\wp'(z)}\right) - 2\wp(z)$.
- $\wp'(z) = -\dfrac{\sigma(2z)}{\sigma(z)^4}$.

Chapter 4

Real Analysis

4.1 Functional Analysis

4.1.1 Banach Spaces

Recall that a normed space is a vector space V equipped with a norm $\|.\| : V \to \mathbb{R}$ that is positive definite, satisfies the triangle inequality, and $\|c \cdot v\| = |c|\|v\|$. We work only with vector space defined over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A *Banach Space* is a normed space that is complete under the metric topology induced by the norm. Two norms are *equivalent* if they define the same metric topology, i.e. if $c\|.\|_1 < \|.\|_2 < C\|.\|_2$ for some $c, C > 0$.

We have some basic properties:

1. Banach subspaces of a normed space must be closed subsets. Closed subspaces of Banach spaces are Banach.
2. All norms on a finite dimensional normed space are equivalent. Moreover, any finite dimensional normed space is Banach.
3. A normed space $(X, \|.\|)$ is Banach iff for all sequences $\{v_n\} \subset X$ such that $\sum_n \|v_n\| < \infty$, the series $\sum_n v_n$ converges in X.
4. (Riesz Lemma) Suppose Y is a proper closed linear subspace of a normed space $(X, \|.\|)$, and suppose $\alpha \in (0, 1)$. Then, there exists a unit vector $x_\alpha \in X$ such that $\|x_\alpha - y\| > \alpha$ for all $y \in Y$.
5. If X is infinite dimensional, then neither $\bar{B}_X(0, 1)$ nor $S_X(0, 1)$ are compact though they are closed and bounded spaces.
6. Every normed space X has a unique (up to isometry) completion $\tilde{X}$, i.e. such that X is dense in $\tilde{X}$ and $\tilde{X}$ is Banach. Moreover, $\tilde{X}$ is isometric to the vector space of all equivalence classes of Cauchy sequences in X, where the equivalence is $[\{x_n\}] \sim [\{y_n\}]$ iff $\lim \|x_n - y_n\| = 0$. The norm on the space of equivalence classes is given by $\|[\{x_n\}]\| := \lim_n \|x_n\|$.
7. Assuming Zorn's Lemma, every non-trivial linear space has a *Hamel Basis* (B is a Hamel Basis for X iff B is linearly independent and $\text{Span}(B) = X$).
8. A *Schauder basis* of normed space X is a countable set of unit vector $\{e_n\}$ such that for any $x \in X$, we have $x = \sum_{n=1}^{\infty} \alpha_n e_n$ for some unique sequence of scalars $\{\alpha_n\}$. If a normed space has a Schauder basis, then it is *separable*, i.e. has a countable dense subset.

Definition. A linear operator $T : X \to Y$ of normed spaces is *continuous/bounded* if it satisfies any of the following equivalent statements:

1. T is continuous with respect to the metric space topology.
2. T is continuous at 0.
3. T is bounded on $\bar{B}(0, 1)$.
4. there exists $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in X$.
5. T is Lipschitz continuous from the metric space X to the metric space Y.

The *operator norm* $\|T\|$ of a bounded linear operator T is defined equivalently by any of the following

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| = \inf\{M > 0 : \|Tx\| \leq M\|x\| \; \forall x \in X\}.$$

With the operator norm, the space $B(X, Y)$ *of all bounded linear operators from X to Y* is a normed space.
The space $X^* := B(X, \mathbb{K})$ is called the *dual space* of X, and elements of X^* are called *linear functionals*.
An *isometry* is a linear operator T such that $\|Tx\| = \|x\|$ for all $x \in X$.
Two normed spaces X and Y are *isometrically isomorphic* if there is an isometry from X onto Y.

Basic properties:

1. The operator norm satisfies $\|Tx\| \leq \|T\| \cdot \|x\|$.
2. If $T : X \to Y$ with X finite dimensional, then T is bounded.
3. Isometries are injective, and if $X \neq \{0\}$ they have operator norm 1.
4. If X is normed and Y Banach, then $B(X, Y)$ is Banach. In particular, X^* is *always* Banach.
5. If $f : X \to \mathbb{K}$ is linear and $\ker f$ is a closed subspace of X, then $f \in X^*$.
6. Given $T \in B(X, Y)$, the *operator adjoint* is the bounded linear operator $T^* \in B(Y^*, X^*)$ given by $T^*(f) = f \circ T$. We have $\|T^*\| = \|T\|$.
7. If $T \in B(X, Y)$ and $S \in B(Y, Z)$, then $S \circ T \in B(X, Z)$ and moreover $\|S \circ T\| \leq \|S\| \cdot \|T\|$. We usually denote $S \circ T$ by ST. In particular, $B(X)$ is a *normed $\mathbb{K}$-algebra* (it is a $\mathbb{K}$-algebra equipped with a norm such that $\|ST\| \leq \|S\|\|T\|$).

   Elements of $B(X)$ that have an inverse in the $\mathbb{K}$-algebra $B(X)$ are called *invertible*. Usual ring-theoretic properties apply.
8. Suppose X is normed, and $\{T_n\}, \{S_n\}$ are sequences in $B(X)$ such that $T_n \to T$ and $S_n \to S$. Then $S_n T_n \to ST$. Moreover, if all the S_ns as well as S are invertible, then $S_n^{-1} \to S^{-1}$.
9. If X is Banach and $T \in B(X)$ with $\|T\| < 1$, then $(I - T) \in B(X)$ is invertible with inverse $(I - T)^{-1} = \sum_{n\geq 0} T^n$. In particular, if X is Banach then the set of all invertible elements of $B(X)$ is open in $B(X)$.

Theorem 4.1.1 (Uniform Boundedness Principle)**.** *Suppose X is Banach and Y normed. Let $\{T_\delta : \delta \in D\} \subset B(X, Y)$ be any collection of bounded operators such that for each $x \in X$, the set $\{\|T_\delta x\| : \delta \in D\}$ is bounded in $\mathbb{K}$. Then, the collection $\{T_\delta\}_{\delta \in D}$ is bounded in $B(X, Y)$, i.e.*

$$\sup_{\delta \in D} \|T_\delta\| < \infty.$$

Corollary 4.1.1.1 (Banach-Steinhaus Theorem)**.** *Suppose $\{T_n\}$ is a sequence in $B(X, Y)$ where X Banach and Y normed. If $\{T_n(x)\} \subset Y$ converges in Y for all $x \in X$, then the map $T : X \to Y, Tx := \lim_n T_n x \in Y$ is a well-defined bounded linear operator. Moreover, $\|T\| \leq \liminf_n \|T_n\| < \infty$.*

Theorem 4.1.2 (Hahn-Banach Extension Theorem)**.** *Suppose X is a normed space with linear subspace M. If $f \in M^*$ (and assuming Zorn's Lemma if $\dim X$ is uncountable), there exists $\tilde{f} \in X^*$ such that $\|\tilde{f}\| = \|f\|$ and $\tilde{f}|_M = f$.*

Proposition 4.1.3. *Suppose X is normed. If X^* is separable then so is X.*

Theorem 4.1.4 (Hahn-Banach Separation Theorem)**.** *Suppose X is normed over $\mathbb{R}$, and A, B are disjoint non-empty convex subsets of X.*

- *If A is open, then there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(a) < \alpha \leq f(b)$ for all $a \in A, b \in B$.*
- *If A is compact and B closed, then there exists $f \in X^*$ and $\alpha, \beta \in \mathbb{R}$ such that $f(a) \leq \alpha < \beta \leq f(b)$ for all $a \in A, b \in B$.*

Proposition 4.1.5. *We have an isometry $\iota_X : X \to (X^*)^*$, called the* canonical embedding*, which sends $x \in X$ to the linear functional on X^* given by $f \mapsto f(x)$. In particular, if $X \neq \{0\}$, then*

$$\|x\| = \sup_{f \in X^*, \|f\|=1} |f(x)|.$$

Definition. A normed space is *reflexive* if ι_X is an isometric isomorphism of X and $(X^*)^*$. Notice that reflexive spaces are necessarily Banach.

Proposition 4.1.6. *Suppose X reflexive and Y a proper closed linear subspace. Assuming Hahn-Banach Extension Theorem, Y is also reflexive.*

Definition. Suppose (M, d) is a metric space. Suppose $A \subset M$.

- A is *nowhere dense* if $\bar{A}$ has empty interior.
- A is of the *first Baire category in M* if A is the union of a sequence of nowhere dense subsets in M.
- A is of the *second Baire category in M* if it is not of the first.

Clearly all subsets of first Baire category sets (in M) are also of the first Baire category (in M).

Theorem 4.1.7 (Baire category Theorem)**.** *In a complete metric space, the intersection of countably many open dense sets is dense.*

Consequently, any non-empty complete metric space M and all of its subsets with non-empty interior are of the second Baire category in M. In particular, Banach spaces are of second Baire category in themselves.

Theorem 4.1.8 (Open Mapping Theorem)**.** *Suppose X is Banach and Y normed. For any $T \in B(X, Y)$, either $T(X) \subset Y$ is of first category in Y, or T is an open surjective map.*

Theorem 4.1.9 (Inverse Mapping Theorem)**.** *If X and Y are Banach spaces and $T \in B(X, Y)$ bijective, then there is a unique $S \in B(Y, X)$ such that $ST = I_X$ and $TS = I_Y$, i.e. T is bijective iff T is a linear homeomorphism from X to Y.*

Definition. If X and Y are normed spaces and $T : X \to Y$ linear, the *graph of T* is the set $\mathcal{G}(T) = \{(x, Tx) \in X \times Y : x \in X\}$. If T is bounded, then $\mathcal{G}(T)$ is closed in $X \times Y$ (the latter equipped with the norm $(x, y) \mapsto \|x\| + \|y\|$).

Theorem 4.1.10 (Closed Graph Theorem)**.** *If X and Y are Banach and $T : X \to Y$ is linear with $\mathcal{G}(T)$ closed in $X \times Y$, then $T \in B(X, Y)$.*

Proposition 4.1.11. *If X and Y are normed spaces and $T \in B(X, Y)$ a linear homeomorphism. Then $T^* \in B(Y^*, X^*)$ is also a linear homeomorphism.*

Proposition 4.1.12. *If X and Y are normed spaces such that there exists a linear homeomorphism $X \to Y$. Then X^* is reflexive iff Y^* is reflexive.*

Proposition 4.1.13. *Suppose X is Banach and $T \in B(X)$. Then, T is invertible iff $T(X)$ is dense and T is* bounded below, *i.e. there exists $\alpha > 0$ such that $\|Tx\| \geq \alpha \|x\|$ for all $x \in X$.*

Lemma-Definition. A linear operator $T : X \to Y$ of normed spaces X, Y is a *compact operator* if any of the following equivalent statements hold:

1. T maps bounded subsets of X to subsets of Y with compact closure.
2. $\overline{T(B_X(0, 1))}$ is compact.
3. for any bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ in Y contains a subsequence that converges to some point in Y.

The space of compact linear operators is denoted by $K(X, Y)$.

Basic properties of compact operators:

1. All compact operators are continuous, so that $K(X, Y) \subseteq B(X, Y)$. All linear operators with finite rank are compact.
2. The range of any compact operator is separable.
3. If Y is Banach, then $T \in K(X, Y)$ iff the image of any bounded subset of X is totally bounded in Y.
4. If X and Y are Banach, then $K(X, Y)$ is a closed subspace of $B(X, Y)$.
5. If X and Y are Banach, then $T \in B(X, Y)$ is compact iff its adjoint $T^* \in B(Y^*, X^*)$ is compact.

4.1.2 Hilbert Spaces

An inner product $\langle , \rangle : X \times X \to X$ is a bilinear conjugate-symmetric positive-definite map. Inner product spaces induce a norm $\|.\|$ on X given by $\|x\|^2 = \langle x, x \rangle$. A *Hilbert Space* is an inner product space that is Banach under the induced norm. A collection of vectors $\{x_\alpha\}$ is an *orthonormal set* if $\langle x_\alpha, x_\beta \rangle = \delta_{\alpha,\beta}$. Orthonormal sets are always linearly independent. The *orthogonal complement* $A^\perp$ to a non-empty subset $A \subset X$ is the closed linear subspace

$$A^\perp = \{x \in X : \langle x, a \rangle = 0 \ \forall a \in A\}.$$

Basic properties:

- (Cauchy-Schwarz Inequality) $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ for all $x, y \in X$, with equality iff $\{x, y\}$ is linearly dependent.
- Inner products are continuous in either argument.
- (Parallelogram Law) Suppose $(X, \|.\|)$ is normed. Then, $\|.\|$ is induced by an inner product $\langle , \rangle$ on X iff $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in X$. Moreover, we have the *polarization identity*

  $$\langle x, y \rangle = \frac{1}{4} \sum_{\xi \in S} \xi \|x + \xi y\|^2$$

  where $S = \{\pm 1\}$ if $\mathbb{K} = \mathbb{R}$ and $S = \{\pm 1, \pm i\}$ if $\mathbb{K} = \mathbb{C}$.
- If $\emptyset \neq A \subseteq X$, then $A \cap A^\perp \subseteq \{0\}$ and $A \subseteq (A^\perp)^\perp$. If $\emptyset \neq B \subseteq A \subseteq X$, then $A^\perp \subseteq B^\perp$. Also, if X_i are linear subspaces of X, then

  $$(X_1 + \cdots + X_n)^\perp = \bigcap_{i=1}^n X_i^\perp \quad \text{and} \quad \left(\bigcup_{n \geq 1} (X_1 + \cdots + X_n) \right)^\perp = \bigcap_{i \geq 1} X_i^\perp.$$

- If $Y \subseteq X$ is a linear subspace, then $Y^\perp = \{x \in X : \|x - y\| \geq \|x\| \ \forall y \in Y\}$.
- If A is a non-empty closed convex subset of a Hilbert space H, and if $p \in H$, then there exists a unique $q \in A$ such that $\|p - q\| = \inf\{\|p - a\| : a \in A\}$. If A is a closed subspace of H, then the map $p \mapsto q$ gives a well-defined bounded linear operator $\mathrm{Pr}_A : H \to A$ such that $\mathrm{Pr}_A^2 = \mathrm{Pr}_A$.
- If H is a Hilbert space and Y a closed sub-space of H, then $H = Y \oplus Y^\perp$ where in fact $Y^\perp = \ker \mathrm{Pr}_Y$ and $Y = \ker(I - \mathrm{Pr}_Y)$. Moreover, $\|x\|^2 = \|\mathrm{Pr}_Y(x)\|^2 + \|x - \mathrm{Pr}_Y(x)\|^2$.
- If Y is a linear subspace of a Hilbert space H, then $\bar{Y} = (Y^\perp)^\perp$.

Example 4.1.14 (Spring 2020 Day 3)**.** We prove that the projection onto a closed subspace of a Hilbert space is uniquely defined. Moreover, we show that the condition that H is a complete inner product space is necessary.

Suppose H is a Hilbert space, K a closed linear subspace, and $x \in H$ is fixed. Since $\|x - y\| > 0$ for all $y \in K$, the infimum d of $\{\|x - y\| : y \in K\}$ exists. Then, there exists a sequence $y_n \subset K$ such that $\|x - y_n\| < d + \frac{1}{n}$. By the parallelogram law, we have for any $m \geq n \geq 2$,

$$\|y_m - y_n\|^2 + 4d^2 \leq \|y_m - y_n\|^2 + \|2x - y_m - y_n\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2 \leq 4d^2 + \frac{4d}{m} + \frac{2}{m^2} + \frac{4d}{n} + \frac{2}{n^2} \leq 4d^2 + \frac{8d+2}{n}$$

and so $\|y_m - y_n\| \leq \sqrt{8d+2}\, n^{-1/2}$ for all $m \geq n$. Thus $\{y_n\}$ is a Cauchy sequence, and as H is a Hilbert space, it must converge to some y. Since K is closed, $y \in K$. From $d \leq \|x - y_n\| < d + \frac{1}{n}$ and the continuity of the norm, it then follows that $\|x - y\| = d$.

Now suppose that this y is not unique, i.e. both y_1 and y_2 minimize $\|x - y\|$. Then, by the parallelogram law we have

$$2\|x - y_1\|^2 + 2\|x - y_1\|^2 = \|y_2 - y_1\|^2 + \|2x - y_1 - y_2\|^2 \geq 4 \left\| x - \frac{y_1 + y_2}{2} \right\|^2$$

where equality holds iff $y_1 = y_2$. Since both y_1 and y_2 are supposed to minimize $\|x - y\|$ for $y \in K$, and since $\frac{y_1 + y_2}{2} \in K$, this is a contradiction unless $y_1 = y_2$. Hence the above y is unique.

To see that completeness is necessary, consider the non-Hilbert space $H = C^2([-1,1])$ with the L^2-norm. Now, in the space $L^2([-1,1])$, we know that the function $f \mapsto f\chi_{[-1,0]}$ is a bounded linear operator. Consider the kernel K of this bounded linear functional in $L^2([-1,1])$. Then, $K \cap H$ is the space of continuous functions

f such that $f|_{[-1,0]}$ is identically zero. It is clear that K is closed. Let $f \in H$ be the constant function 1. For any $g \in K$, we have

$$\|f-g\|_2^2 = \int_{-1}^{0} |1-0|^2 dt + \int_{0}^{1} |1-g(x)|^2 dx \geq 1,$$

and so $1 \leq \inf\{\|f-g\| : g \in K\}$. Moreover, notice that equality holds iff $|1-g(x)|^2 = 0$ almost everywhere, i.e. g is identically 1 on $[0,1]$. By uniqueness in $L^2([-1,1])$, it then follows that the function $g = \chi_{[0,1]}$ is the unique minimizer of $\|f-h\|$ for $h \in K$. However $g \notin H$, and so $\|f-h\|$ does not achieve a minimum for $h \in H \cap K$.

Now, suppose X is an inner product space and $\{u_n\}$ an orthonormal sequence in X.

Lemma 4.1.15 (Fourier Coefficients)**.** *If $x \in X$ is such that $x = \sum_{i\geq 1} \alpha_i u_i$ for some $\alpha_i \in \mathbb{K}$, then $\alpha_i = \langle x, u_i \rangle$.*

Proposition 4.1.16 (Bessel's Inequality)**.** *For any $x \in X$, we have $\displaystyle\sum_{i=1}^{\infty} |\langle x, u_i \rangle|^2 \leq \|x\|^2$.*

Proposition 4.1.17. *If $\{u_\alpha\}$ is an orthonormal set in an inner product space X, then for any $x \in X$ the set $E_x := \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$ is countable. If we write $E_x = \{u_1, u_2, ...\}$, then $\lim_n \langle x, u_n \rangle = 0$. Moreover, for any fixed $x \in X$, the following two are equivalent.*

- *(*Orthonormal Representation*) $x = \displaystyle\sum_{u \in E_x} \langle x, u \rangle u$;*
- *(*Parseval's Identity*) $\|x\|^2 = \displaystyle\sum_{u \in E_x} |\langle x, u \rangle|^2$.*

An *orthonormal basis* if an orthonormal sequence $\{u_n\}$ of X such that $x = \sum_n \langle x, u_n \rangle u_n$ for all $x \in X$.

Theorem 4.1.18 (Riesz-Fisher Theorem)**.** *Suppose $\{u_n\}$ is an orthonormal sequence in a Hilbert space $(H, \langle, \rangle)$, and suppose $\{\xi_n\} \in \ell_2$. Then, $\sum_n \xi_n u_n$ converges.*

Combining Bessel's Inequality and the Riesz-Fisher Theorem, we have the following:

Corollary 4.1.18.1. *If $\{u_\alpha\}$ is a maximal orthonormal set in a Hilbert space H, then for any $x \in H$ we have*

$$x = \sum_{u \in E_x} \langle x, u \rangle u.$$

Theorem 4.1.19. *Suppose H is a Hilbert space. Then the following are equivalent:*

1. *H is isometrically isomorphic to ℓ_2;*
2. *H is separable; and*
3. *H has a countable maximal orthonormal set $\{u_n\}$.*

Moreover, if any one of these holds, then the isometric isomorphism from H to ℓ_2 is given by $x \mapsto \{\langle x, u_n \rangle\}$.

Lemma 4.1.20. *If X is an inner product space with $q \in X$, then the function $x \mapsto \langle x, q \rangle$ is a linear functional on X with operator norm $\|q\|$.*

Theorem 4.1.21 (Riesz Representation Theorem)**.** *Suppose H is a Hilbert space. Then, for any $f \in H^*$ there exists a unique $q \in H$ such that $\|f\| = \|q\|$ and $f(x) = \langle x, q \rangle$ for all $x \in H$.*

Moreover, if $T : H^* \to H$ denotes the map $T(f) = q$, then T is a conjugate linear bijection. The inner product $\langle, \rangle_*$ on H^* induced by T, given by $\langle f, g \rangle_* = \langle Tg, Tf \rangle$ for all $f, g \in H^*$, makes H^* a Hilbert space.

Corollary 4.1.21.1. *Every Hilbert space is reflexive.*

Proposition 4.1.22. *Suppose $T \in B(H, K)$ where H, K are Hilbert spaces. There is a unique bounded linear operator $T^* \in B(K, H)$ satisfying $\langle Tx, y \rangle = \langle x, T^* y \rangle$. This map T^* is called the* Hilbert adjoint *of T. It satisfies $\|T^*\| = \|T\|$ and $\|T^*T\| = \|TT^*\| = \|T\|^2$.*

We have some basic properties of Hilbert-adjoints:

1. The map $B(H, K) \to B(K, H), T \mapsto T^*$ is a norm-preserving conjugate-linear map. We also have $(SR)^* = R^* S^*$.
2. If $T \in B(H)$ is invertible then T^* is invertible with inverse $(T^{-1})^*$.

3. We have $\ker(T^*) = (T(H))^\perp$ and $\ker T = (T^*(K))^\perp$.
4. If H is a Hilbert space, then $T \in B(H)$ is invertible iff T^* is injective and T is bounded below.
5. Suppose H a Hilbert space with M a closed linear subspace, and $T \in B(H)$. If M and $M^\perp$ are T-invariant (i.e. $T(M) \subseteq M, T(M^\perp) \subseteq M^\perp$), then they are also invariant under T^*. Moreover, $(T|_M)^* = (T^*)|_M$ and $(T|_{M^\perp})^* = (T^*)|_{M^\perp}$.

Definition. $T \in B(H)$ is *normal* if $TT^* = T^*T$. It is *self-adjoint* if $T = T^*$. T is *unitary* if $TT^* = T^*T = I$

Basic properties (throughout, assume $T \in B(H)$ with H Hilbert):

1. If T is normal, then $\|Tx\| = \|T^*x\|$ for all $x \in H$.
2. A normal operator is invertible iff it is bounded below.
3. If M and $M^\perp$ are T-invariant closed subspaces, and if T is normal, then $T|_M$ and $T|_{M^\perp}$ are normal.
4. For any $T \in B(H)$, TT^* and T^*T are self-adjoint. Moreover, for any $T \in B(H)$ there exist unique self-adjoint operators $R, J \in B(H)$ such that $T = R + \sqrt{-1}J$ and $T^* = R - \sqrt{-1}J$.
5. $T \in B(H)$ is unitary iff it is an isometry from H onto H. In particular, $\|T\| = 1$.
6. (*Schur's Decomposition*)For any square matrix A, there exists a unitary matrix U and an upper triangular matrix T such that $A = UTU^*$. Moreover, T is diagonal iff A is normal.

4.1.3 Spectral Theory

Throughout this section we assume base field is $\mathbb{C}$, and that the normed space under consideration is Banach.

Definition. Throughout, suppose $T \in B(X)$ where X is Banach.

1. *Spectrum:* $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not invertible}\}$.
2. *Resolvent:* $\rho(T) = \mathbb{C} \setminus \sigma(T)$.
3. *Approximate Point Spectrum:* $\sigma_{ap}(T) = \{\lambda \in \sigma(T) : T - \lambda I \text{ not bounded below}\}$.
4. *Point Spectrum/Set of Eigenvalues:* $\sigma_p(T) = \{\lambda \in \sigma(T) : \ker(T - \lambda I) = 0\}$.
5. *Residual Spectrum:* $\sigma_r(T) = \{\lambda \in \sigma(T) \setminus \sigma_p(T) : \overline{(T - \lambda I)(X)} \neq X\}$.
6. *Continuous Spectrum:* $\sigma_c(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T))$.

Basic properties:

- $\lambda \in \rho(T)$ iff $T - \lambda I$ bijective.
- *Spectral Mapping Theorem For Polynomials* If $p \in \mathbb{C}[z]$, then $\sigma(p(T)) = p(\sigma(T))$.
- *Spectral Mapping Theorem For Inverses* If T invertible, then $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.
- The spectrum is always a non-empty compact subset of $\mathbb{C}$, contained in $\bar{B}(0, \|T\|)$. The *spectral radius* of T is $r_\sigma(T) := \sum_{\lambda \in \sigma(T)} |\lambda|$.
- *Gelfand's Spectral Radius Formula* $r_\sigma(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$.
- If H is a Hilbert space and $T \in B(H)$ with Hilbert-adjoint T^*, and if $A^* := \{\bar{z} : z \in A\}$ for all $A \subseteq \mathbb{C}$, then
    1. $\rho(T)^* = \rho(T^*)$, $\sigma(T)^* = \sigma(T^*)$, $\sigma_c(T)^* = \sigma_c(T^*)$;
    2. $\sigma_r(T) = \sigma_p(T^*)^* \setminus \sigma_p(T)$ and $\sigma_r(T^*) = \sigma_p(T)^* \setminus \sigma_p(T^*)$;
    3. $\sigma_c(T) = \sigma(T) \setminus \big(\sigma_p(T) \cup \sigma_p(T^*)^*\big)$.

If $T \in B(H)$ is a normal operator (H Hilbert), then we have the extra properties:

- $\ker(T - \lambda I) = \ker(T^* - \bar{\lambda} I)$ and so $\sigma_p(T^*) = \sigma_p(T)^*$ and $\sigma_r(T) = \emptyset$.
- $r_\sigma(T^n) = \|T\|^n$ for all $n \in \mathbb{N}$.

- $\ker(T-\lambda I) \perp \ker(T-\lambda' I)$ for all $\lambda, \lambda' \in \sigma_p(T)$ with $\lambda \neq \lambda'$.
- If T is in addition self-adjoint, then $\sigma(T) \subseteq \mathbb{R}$ and $\{\pm\|T\|\} \cap \sigma(T) \neq \emptyset$.

If $T \in K(H)$ is a compact operator on H (Hilbert), then we have the following extra properties:

- $\ker(T-\lambda I)$ is finite dimensional for all $\lambda \neq 0$.
- The set $\sigma_p(T)$ is countably infinite. Moreover, we can write $\sigma_p(T) \setminus \{0\} = \{\lambda_i\}$ so that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$. If moreover $\sigma_p(T)$ is an infinite set, then $\lambda_n \to 0$ as $n \to \infty$.
- $\sigma_a p(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.
- If T is a compact normal operator, then $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$ and $|\lambda| = \|T\|$ for some $\lambda \in \sigma_p(T)$. Moreover, writing $\sigma_p(T) = \{\lambda_i\}$ so that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$, we have

$$\ker(T)^\perp = \overline{T(H)} = \left(\bigcup_{n \geq 1} \bigoplus_{i=1}^{n} \ker(T - \lambda_i I) \right).$$

Theorem 4.1.23 (Spectral Theorem for Compact Normal Operators)**.** *Suppose T is a compact normal operator such that $\sigma_p(T) \setminus \{0\} = \{\lambda_i\}$ so that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$. Let P_i be the orthogonal projection onto $\ker(T - \lambda_i I)$. Then, $T = \sum_j \lambda_{j \geq 1} P_j$.*

Example 4.1.24 (Fall 2021 Day 2)**.** Let $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^2$ be the torus, let $a \in C(\mathbb{T}, \mathbb{R})$. Prove that the $\mathbb{R}$-space of solutions of the PDE $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = af$ is finite dimensional.

Let the space of solutions be denoted by V. Let $H = L^2(\mathbb{T}, \mathbb{C})$, with orthonormal basis $\{e^{2\pi i(mx+ny)}\}_{(m,n)\in\mathbb{Z}^2}$. Then $X = \mathrm{Span}_{\mathbb{R}}\{e^{2\pi i(mx+ny)}\}_{(m,n)\in\mathbb{Z}^2}$ is a dense linear subspace of H. Consider the Laplacian operator $\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$. Notice that $\Delta(e^{2\pi i(mx+ny)}) = -4\pi^2(m^2+n^2)e^{2\pi i(mx+ny)}$. Thus $1-\Delta$ is a well-defined invertible linear operator (not bounded) on X since it is diagonalizable in the Fourier basis and all of its eigenvalues are ≥ 1. Consider $T := (1-\Delta)^{-1}$; notice that T is continuous since T maps an orthonormal basis into $\bar{B}_X(0,1)$. It can thus be uniquely extended to a continuous operator on H. Moreover, if $P_{m,n} \in B(H)$ is the projection onto $\mathbb{R} \cdot e^{2\pi i(mx+ny)}$, then notice that

$$T = \sum_{(m,n)\in\mathbb{Z}^2} \big(1 + 4\pi^2(m^2+n^2)\big) P_{m,n}$$

where each of the $P_{m,n}$ are finite rank operators. Hence, T is a compact operator on H.

Now, notice that $V = \{f \in H : (1-a)f = (1-\Delta)f\} = \{f \in H : T(1-a)f = f\}$. Since multiplication by $1-a$ is a bounded linear operator (its operator norm is bounded by $1 + \|a\|_\infty$ which is finite since $\mathbb{T}$ is compact and a continuous), and since T is compact, it follows that $T(1-a)$ is a compact linear operator. Hence V is an 1-eigenspace for a compact linear operator, which implies that V is finite.

4.2 Measure Theory

4.2.1 Measures

Measures on Measurable Spaces

Definition (σ-Algebras)**.** Let X be a set. A non-empty subset $\mathcal{M} \subset \mathcal{P}(X)$ is called a σ-algebra if it is

1. closed under complement, i.e. $E^c = X \backslash E \in \mathcal{M}$ for all $E \in \mathcal{M}$;
2. closed under countable union, i.e. $\{E_j\}_{j\in\mathbb{N}} \subset \mathcal{M} \implies \bigcup_j E_j \in \mathcal{M}$.

Lemma 4.2.1 (Properties of σ-Algebras)**.** *If $\mathcal{M}$ is a σ-algebra on X, then*

1. *$\emptyset, X \in \mathcal{M}$,*
2. *$\bigcap_j E_j \in \mathcal{M}$ for any sequence $\{E_j\}$.*
3. *If $\mathcal{M}_\alpha$, $\alpha \in I$, is some collection of σ-algebras on X, then $\bigcap_{\alpha\in I} \mathcal{M}_\alpha$ is a σ-algebra on X.*

Definition (Measurable Space)**.** A set X with a σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$ is called a *measurable space*, and elements in $\mathcal{M}$ are called *measurable sets.*

Definition (Measurable Functions and Isomorphisms)**.** Suppose $f : X \to Y$ where $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces. Then f is said to be a *measurable function* if for all $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$. We say that f is an *isomorphism* if f is bijective and f and f^{-1} are both measurable.

Definition (σ-Algebra Generated by $\mathcal{E} \subset \mathcal{P}(X)$)**.** Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$. Let $\mathcal{M}(\mathcal{E})$ be the intersection of all σ-algebras $\mathcal{M}$ containing $\mathcal{E}$. Then $\mathcal{M}(\mathcal{E})$ is the smallest σ-algebra on X containing $\mathcal{E}$, and is called the *σ-algebra generated by $\mathcal{E}$.*

Note that $\mathcal{P}(X)$ is itself a σ-algebra, and so the collection of σ-algebras containing $\mathcal{E}$ is non-empty. Some examples are as follows:

1. If $(X, \mathcal{T})$ is a topological space, then $\mathcal{M}(\mathcal{T}) =: B_X$ is called the *Borel σ-algebra on X*, and the elements of this σ-algebra are called the Borel sets.
2. Taking $\mathbb{R}$ with the usual topology, then $B_{\mathbb{R}}$ contains the open intervals (a, b), closed intervals $[a, b]$, half-closed/open and half-open/closed intervals $[a, b)$ and $(a, b]$, etc.
3. Take $X = [-\infty, \infty] \supset \mathbb{R}$. We define a topology on X by taking the usual open sets of $\mathbb{R}$, and declaring that $(a, \infty], [-\infty, a)$, and $\mathbb{R}$ are all open sets in X. We can then consider the Borel σ-algebra of X.

Proposition 4.2.2. *Suppose $f : (X, \mathcal{M}) \to (Y, \mathcal{N}(\mathcal{E}))$ is some map. Then f is measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.*

Corollary 4.2.2.1. *Continuous maps between topological spaces are measurable, where the σ-algebra on the domain and co-domain is the Borel σ-algebra.*

Definition (Product σ-algebra)**.** Suppose $(X_\alpha, \mathcal{M}_\alpha)$, $\alpha \in A$, is some collection of measurable spaces, then the *product σ-algebra* is given by

$$\left(\prod_{\alpha \in A} X_\alpha, \bigotimes_{\alpha \in A} \mathcal{M}_\alpha\right).$$

It is defined to be the smallest σ-algebra on the Cartesian product such that for any $\alpha \in A$, the projection maps

$$\pi_\alpha : \prod_{\beta \in A} X_\beta \to X_\alpha$$

are measurable. This σ-algebra is generated by

$$\mathcal{E} = \left\{\pi_\alpha^{-1}(E_\alpha) \subseteq \prod_{\alpha \in A} X_\alpha : \alpha \in A, E_\alpha \in \mathcal{M}_\alpha\right\}.$$

Definition (Measures)**.** A *measure* μ on a measurable space $(X, \mathcal{M})$ is a map $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$,
2. (Countable Additivity) If $\{E_j\}_{j=1}^\infty$ is a sequence of mutually disjoint measurable subsets, then

$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \mu(E_j).$$

The sum here is the limit of $\sum_{j=1}^k \mu(E_j)$. If this sum diverges, then the measure of the union is ∞.

The triple $(X, \mathcal{M}, \mu)$ is called a *measure space.* The measure μ is *finite* if $\mu(X) < \infty$. The measure μ is *σ-finite* if $\mu(X) = \bigcup_{j=1}^\infty E_j$ for $E_j \in \mathcal{M}$ with $\mu(E_j) < \infty$ for all j. If X is a topological space, then any measure on (X, B_X) is called a *Borel measure.*

Examples:

1. Suppose $(X, \mathcal{P}(X) = \mathcal{M})$. We define the *counting measure* $\mu(E) := |E|$ (i.e., if E is an infinite subset, then the measure is ∞. If E is finite, it is the cardinality).
2. The *Lebesgue measure* on $(\mathbb{R}^n, B_{\mathbb{R}^n})$ satisfies $\mu\left(\prod_{j=1}^n [a_j, b_j]\right) = \prod_{j=1}^n (b_j - a_j)$. The measure space $(\mathbb{R}, B_{\mathbb{R}})$ with the Lebesgue measure is σ-finite.

Lemma 4.2.3 (Properties of a Measure)**.** *Suppose $(X, \mathcal{M})$ is a measurable space with measure μ.*

1. *(Finite additivity) If $E_1, ..., E_k$ are mutually disjoint measurable subsets, then* $\mu\left(\bigcup_{j=1}^{k} E_j\right) = \sum_{j=1}^{k} \mu(E_j)$.

2. *(Sub-additivity) If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$, then* $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$.

3. *(Monotonicity) If $E, F \in \mathcal{M}$ with $E \subset F$, then $\mu(E) \leq \mu(F)$.*

4. *(Continuity from below) If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ such that $E_j \subset E_{j+1}$ for all j, then*
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

5. *(Continuity from above) If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ such that $E_j \supset E_{j+1}$ for all j and $\mu(E_1) < \infty$, then*
$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

Continuity from above need not hold if $\mu(E_1) = \infty$. Counter-example: Let $E_j = (j, \infty)$ in $(\mathbb{R}, B_{\mathbb{R}})$. Then $\mu(E_j) = \infty$ for all j, but $\bigcap_j E_j = \emptyset$.

Definition (Null Sets)**.** Suppose $(X, \mathcal{M}, \mu)$ is a measure space. A subset $E \in \mathcal{M}$ with $\mu(E) = 0$ is called a *null set.*

Lemma-Definition (Completion of Measure Space)**.** Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Define

$$\bar{\mathcal{M}} := \{E \cup F \subset X : E \in \mathcal{M}, F \subset N \text{ where } \mu(N) = 0\},$$
$$\bar{\mu}(E \cup F) := \mu(E)$$

where $E \in \mathcal{M}$ and F is any subset of any null set. Then $\bar{\mathcal{M}}$ is a σ-algebra and $\bar{\mu}$ defines a measure on $(X, \bar{\mathcal{M}})$. The measure space $(X, \bar{\mathcal{M}}, \bar{\mu})$ is called the *completion* of $(X, \mathcal{M}, \mu)$.

Definition. A property of points $x \in X$ is said to hold *almost everywhere* (abbreviated a.e.) if there exists a null set $E \in \mathcal{M}$ such that the property holds on $X \backslash E$.

Example: If $f, g : (X, \mathcal{M}, \mu) \to \mathbb{R}$, then $f = g$ holds almost everywhere if $F := \{x \in X : f(x) = g(x)\}$ has measure zero (possibly in the completion of μ).

Measurable Functions

From here onwards, we consider measurable functions $f : X \to [-\infty, \infty]$ (where we have the Borel measure on $[-\infty, \infty]$). Recall also that for a sequence $\{f_k\}$ of functions, we have

$$\limsup f_n := \lim_{k\to\infty} \sup_{n\geq k} f_n;$$

this limit always exists (possibly $-\infty$), since $\sup_{n\geq k} f_n$ is a decreasing function of k.

Proposition 4.2.4 (Properties of Measurable Functions)**.** *Suppose $(X, \mathcal{M})$ is a measurable space. Suppose $f, g : X \to \mathbb{R}$ are measurable, and $\phi : \mathbb{R} \to \mathbb{R}$ is continuous.*

1. *$\phi \circ f : X \to \mathbb{R}$ is measurable.*

2. *$f + g, f \cdot g : X \to \mathbb{R}$ (defined point-wise) are measurable.*

3. *Suppose $f_1, f_2, ... : X \to [-\infty, \infty]$ is a sequence of measurable functions. Then*
$$\left(\sup_n f_n\right)(x) := \sup\{f_n(x) : n \in \mathbb{N}\}$$
is measurable. Similarly,
$$\left(\inf_n f_n\right)(x) := \inf\{f_n(x) : n \in \mathbb{N}\}$$
is measurable.

4. *Suppose $f_1, f_2, ... : X \to [-\infty, \infty]$ is a sequence of measurable functions. Then*

$$\liminf f_n \quad \text{and} \quad \limsup f_n$$

are measurable.

5. *Suppose $f_1, f_2, ... : X \to [-\infty, \infty]$ is a sequence of measurable functions that converges point-wisely to f. Then f is measurable.*

Let $A = \{f : X \to \mathbb{R} | f \text{ measurable}\}$; then by the previous proposition it is a ring under point-wise addition and multiplication. It is also a vector space over $\mathbb{R}$ (the map $x \mapsto cx$ is continuous, and so the composition cf for any measurable function f is measurable as well).

4.2.2 Integration

Basic Definitions

Definition (Characteristic Functions)**.** Suppose X is a non-empty set and $E \subset X$ any non-empty subset. Then, the *characteristic function* χ_E of E is the map $\chi_E : X \to \{0, 1\} \subset \mathbb{R}$ such that

$$\chi_E(x) := \begin{cases} 1 & x \in E, \\ 0 & x \notin E. \end{cases}$$

Definition (Simple Functions)**.** If $(X, \mathcal{M})$ is a measurable space, a *simple function* on X is a finite (real) linear combination of χ_E for $E \in \mathcal{M}$.

Lemma 4.2.5.
- *If $(X, \mathcal{M})$ is a measurable space, then $\chi_E : X \to \mathbb{R}$ is a measurable function for any $E \in \mathcal{M}$.*
- *If f is a non-negative simple function, then*

$$f = b_1\chi_{F_1} + \cdots + b_k\chi_{F_k}$$

where $F_i \cap F_j = \emptyset$ for $i \neq j$, and $b_i \in \mathbb{R}_{\geq 0}$.

Definition (Integration of Non-Negative Simple Functions)**.** Suppose $(X, \mathcal{M}, \mu)$ is a measure space, and suppose $\phi = \sum_{j=1}^n c_j\chi_{E_j}$ is a simple function with $c_j \geq 0$ distinct and E_j disjoint. Then

$$\int_X \phi d\mu := \sum_{j=1}^n c_j\mu(E_j) \in [0, \infty].$$

It is easy to see that this is well-defined. Indeed, for any simple function ϕ, since the range of ϕ must be finite, we have $E_j = \phi^{-1}(\{c_j\})$ and $\phi(X) = \{c_1, ..., c_n\}$ and so the representation of ϕ chosen in the above definition is uniquely defined. Such a representation is known as the *standard* representation.

Proposition 4.2.6. *Suppose $f : X \to [0, \infty]$ is measurable. Then, there exists an increasing sequence $\{\phi_n\}$ of simple functions that converges pointwisely to f. If f is bounded, then the convergence is uniform.*

Let $L^+(X) := \{f : X \to [0, \infty] : f \text{ measurable}\}$.

Definition (Integration of Non-Negative Measurable Functions)**.** Suppose $(X, \mathcal{M}, \mu)$ is a measure space, and $f : X \to [0, \infty)$ is measurable. Define

$$\int_X f = \int_X f d\mu := \sup\left\{\int_X \phi d\mu : 0 \leq \phi \leq f, \phi \text{ simple}\right\} \in [0, \infty].$$

Lemma 4.2.7. *If f and g are measurable functions such that $f \leq g$, then $\int_X f \leq \int_X g$.*

Proposition 4.2.8. *If $f \in L^+$, then $\int f = 0$ if and only if $f = 0$ almost everywhere.*

Definition (Integrable Functions)**.** The space of integrable functions is denoted by

$$L^1(X) = \left\{f : X \to [-\infty, \infty] \mid f \text{ measurable and } \int_X |f| < \infty\right\}.$$

Here $|f|$ is measurable as f is measurable and $|.| : \mathbb{R} \to \mathbb{R}$ is continuous.

Remark 4.2.9. If $f \in L^1(X)$, then $\mu(f^{-1}(\pm\infty)) = 0$. If $f, g \in L^1(X)$, then $f + g$ is well-defined outside a set of measure zero (it is not well-defined at points x where $f(x) = -g(x) = \pm\infty$). Similarly $f = g$ in $L^1(X)$ if $f = g$ almost everywhere. In particular, $L^1(X)$ can be considered as a set of equivalence classes of functions that agree outside a set of measure zero. With this convention, $L^1(X)$ is an $\mathbb{R}$-vector space.

Definition (Integration over X)**.** Suppose $f \in L^1(X)$. Write $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then,

$$\int_X f = \int_X f d\mu := \int_X f^+ - \int_X f^-.$$

Note that $\max\{\cdot, 0\} : \bar{\mathbb{R}} \to [0, \infty]$ is continuous, and so f^+, f^- are non-negative measurable functions (and thus in L^+).

Definition (Integration over Measurable Sets)**.** Suppose $f \in L^1(X)$ and $E \in \mathcal{M}$. Then,

$$\int_E f = \int_E f d\mu := \int_X f \cdot \chi_E.$$

Theorem 4.2.10. *Suppose $(X, \mathcal{M}, \mu)$ is a measure space.*

1. *The map $\int_X : L^1(X) \to \mathbb{R}$ is linear.*
2. *$f, g \in L^+ \implies \int(f + g) = \int f + \int g$.*
3. *$f \in L^1(X) \implies |\int f| \leq \int |f|$.*
4. *$f, g \in L^1$ or $f, g \in L^+$. Then $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ if and only if $\int_X |f - g| = 0$ if and only if $f = g$ almost everywhere.*
5. *(*Jensen's Inequality*) Suppose $\mu(X) < \infty$, $f \in L^+ \cap L^1$, and $\varphi : [0, \infty) \to \mathbb{R}$ is a convex function. Then,*

$$\varphi\left(\frac{1}{\mu(X)} \int_X f\right) \leq \frac{1}{\mu(X)} \int_X \varphi \circ f.$$

Theorem 4.2.11 (Monotone Convergence Theorem)**.** *Suppose we have a sequence $\{f_n\} \subset L^+(X)$ such that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$. Let $f := \lim f_n$ be the point-wise limit of the sequence. Then,*

$$\int f = \lim_{n\to\infty} \int f_n,$$

i.e. we can interchange the limit and the integral sign.

Remark 4.2.12. The increasing condition on $\{f_n\}$ is necessary. For instance,

$$f_n = \begin{cases} 1 & x \in [n, n+1] \\ 0 & \text{otherwise} \end{cases}.$$

Then $f = \lim f_n \equiv 0$. However, $\int f_n = 1$ for all $n \in \mathbb{N}$.

Corollary 4.2.12.1. *Suppose $\{f_n\} \subset L^+(X)$ and $f := \sum_{n=1}^\infty f_n$. Then*

$$\int f = \sum_{n=1}^\infty \int f_n.$$

Corollary 4.2.12.2 (Fatou's Lemma)**.** *Given $\{f_n\} \subset L^+$. Then*

$$\int \liminf f_n \leq \liminf \int f_n.$$

Corollary 4.2.12.3. *If $\{f_n\} \subset L^+$ and $f \in L^+$ such that $f_n \to f$ almost everywhere, then*

$$\int f \leq \liminf \int f_n.$$

Proposition 4.2.13. *Suppose $f \in L^+ \cap L^1$, then $\mu(x : f(x) = \infty) = 0$, and $\{x : 0 < f(x) < \infty\}$ is σ-finite, i.e. it is the countable union of finite measure sets.*

Theorem 4.2.14 (Dominated Convergence Theorem)**.** *Suppose $\{f_n\} \subset L^1$ is such that $\{f_n\} \to f$ almost everywhere, and there exists a non-negative $g \in L^1$ such that $|f_n| \leq g$ almost everywhere for all $n \in \mathbb{N}$. Then*

$$\int f = \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n.$$

Theorem 4.2.15. *Suppose $\{f_n\} \subset L^1$ such that $\sum_{n=1}^{\infty} \int |f_n| < \infty$. Then the series $\sum_{n=1}^{\infty} f_n$ converges almost everywhere to some $f \in L^1$, and*

$$\int f = \sum_{n=1}^{\infty} \int f_n.$$

The space $L^1(X)$ is a normed vector space with

$$\|f\| = \int_X |f| d\mu.$$

Lemma 4.2.16. *Let $L^1(X)$ be the space of equivalence classes of integrable functions that agree almost everywhere. It is a Banach space under the norm $\|.\|$ given by $\|f\| := \int_X |f| d\mu$.*

Theorem 4.2.17. *Suppose $f : X \times [a,b] \to \mathbb{R}$ where $(X, \mathcal{M}, \mu)$ is a measure space such that for each fixed $t \in [a,b]$ the function $f(\cdot, t) \in L^1(X)$. Let*

$$F(t) := \int_X f(x,t) d\mu(x).$$

Then,

1. *Suppose there exists $g \in L^1(X)$ such that $|f(x,t)| \leq g(x)$ for all x, t. If $\lim_{t\to t_0} f(x,t) = f(x,t_0)$ for all x, then*
$$\lim_{t\to t_0} F(t) = F(t_0).$$
In particular, if $f(x,\cdot)$ is continuous on $[a,b]$ for all fixed $x \in X$, then F is continuous on $[a,b]$.

2. *Suppose $\frac{\partial f}{\partial t}$ exists, and there exists $g \in L^1(X)$ such that $|\frac{\partial f}{\partial t}(x,t)| \leq g(x)$ for all x, t. Then F is differentiable and*
$$F'(t) = \int_X \frac{\partial f}{\partial t}(x,t) d\mu(x)$$

Remark 4.2.18. In (1), if we "replace" $[a,b]$ by a sequence $\{t_n\} \to t_0$, then the statement is just the dominated convergence theorem. In (2), the functions $\frac{\partial f}{\partial t}$ if they exist are measurable, since they are the limit of a quotient of measurable functions.

Modes of Convergence

Definition (Modes of Convergence)**.** Suppose we have a measure space $(X, \mathcal{M}, \mu)$, and suppose $\{f_n : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ is a sequence of measurable functions.

- **Uniform Convergence:** The sequence converges *uniformly* to f if for all $\epsilon > 0$, there exists N such that for all $n \geq N$ and all $x \in X$, we have $|f_n(x) - f(x)| < \epsilon$.

- **Point-wise Convergence (a.e.):** The sequence converges *point-wisely* to f almost everywhere if there exists $N \in \mathcal{M}$ with $\mu(N) = 0$ such that for all $x \in X \backslash N$, we have $\lim_n f_n(x) = f(x)$.

- **Convergence in L^1:** The sequence $\{f_n\} \subset L^1(X)$ converges to f *in* L^1 if
$$\lim_{n\to\infty} \|f_n - f\| = \lim_{n\to\infty} \int_X |f_n - f| = 0.$$

- **Convergence in Measure:** The sequence converges to f *in measure* if for all $\epsilon > 0$,
$$\lim_{n\to\infty} \mu\left(\{x \in X : |f_n(x) - f(x)| > \epsilon\}\right) = 0.$$
Equivalently, for all $\epsilon, \delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have
$$\mu\left(x \in X : |f_n(x) - f(x)| > \epsilon\right) < \delta.$$

For all modes of convergence, we can talk about an equivalent notion of a "Cauchy" sequence. In particular, we have the following definition.

Definition (Cauchy Sequence in Measure)**.** Suppose we have a measure space $(X, \mathcal{M}, \mu)$, and suppose $\{f_n : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ is a sequence of measurable functions. Then this sequence is *Cauchy in measure* if for all $\epsilon > 0$,

$$\lim_{n,m\to\infty} \mu\left(\{x \in X : |f_n(x) - f_m(x)| > \epsilon\}\right) = 0.$$

Equivalently, for all $\epsilon, \delta > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $n, m \geq N$, we have

$$\mu\left(x \in X : |f_n(x) - f_m(x)| > \epsilon\right) < \delta.$$

Note that in the previous subsection, we have shown that $L^1(X)$ is complete, and so Cauchy in L^1 is equivalent to convergent in L^1. Similarly Cauchy and convergence are equivalent for point-wise (a.e.) convergence as well as for uniform convergence. It will be shown that Cauchy and convergence are equivalent in measure as well.

Remark 4.2.19. Suppose

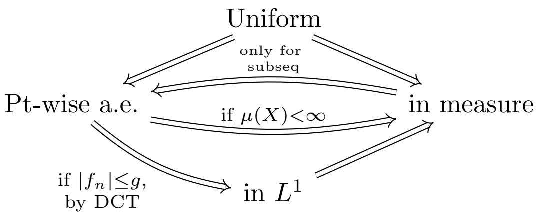
$$f_n(x) : \begin{cases} n & \text{if } x \in [0, \frac{1}{n}], \\ 0 & \text{otherwise,} \end{cases} : [0,1] \to \mathbb{R}.$$

Then $f_n \xrightarrow{\text{in measure}} 0$, as

$$\mu\left(x \in [0,1] : |f_n(x)| > \epsilon\right) = \frac{1}{n}$$

for all $\epsilon > 0$. However $f_n \not\to 0$ in L^1.

It will be shown that



with counterexamples for all of the implications not shown, and for those implications that required added conditions.

Remark 4.2.20 (Convergence in $L^1 \not\Rightarrow$ convergence point-wise a.e.). Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n = \begin{cases} 1 & \text{if } x \in \left[1 + \frac{1}{2} + \cdots + \frac{1}{n}, 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right] \pmod 1, \\ 0 & \text{otherwise.} \end{cases}.$$

Then $f_n = 1$ only on an interval of length $\frac{1}{n+1}$, and so $\int f_n = \frac{1}{n+1}$. Thus $f_n \to 0$ in L^1. However, $f_n \not\to 0$ point-wisely,

Remark 4.2.21 (Convergence pointwise a.e. $\not\Rightarrow$ convergence in L^1). Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n = \begin{cases} n & \text{if } x \in \left[0, \frac{1}{n}\right], \\ 0 & \text{otherwise.} \end{cases}.$$

Then $f_n \to 0$ point-wisely, but $\int f_n = 1$.

Proposition 4.2.22. *If $\{f_n\} \to f$ in L^1, then $f_n \to f$ in measure.*

Theorem 4.2.23. *Suppose $\{f_n\}$ is a Cauchy sequence in measure. Then there exists a measurable function f such that*

1. *$f_n \to f$ in measure.*
2. *There exists a subsequence $\{f_{n_j}\} \to f$ point-wisely almost everywhere.*

Moreover, this f is uniquely defined up to a null set.

Corollary 4.2.23.1. *If $\{f_n\}$ converges in measure, then there exists a subsequence that converges point-wise almost everywhere.*

Theorem 4.2.24 (Egoroff's Theorem)**.** *If $\mu(X) < \infty$ and $\{f_n\} \to f$ point-wisely, then for all $\epsilon > 0$, there exists $K \subset X$ such that $f_n|_K \to f|_K$ uniformly on K, and $\mu(X \backslash K) < \epsilon$.*

Corollary 4.2.24.1. *If $\mu(X) < \infty$, then $\{f_n\} \to f$ point-wisely almost everywhere implies $\{f_n\} \to f$ in measure.*

4.2.3 Construction of Measures

Outer Measures and Product Measures

Definition (Outer Measure)**.** For a set X, consider $\mathcal{P}(X)$. Suppose $\mathcal{E} \subset \mathcal{P}(X)$ such that $\emptyset, X \in \mathcal{E}$. Consider any function $\rho : \mathcal{E} \to [0, \infty]$ such that $\rho(\emptyset) = 0$. Then, for any subset $E \subset X$, define the *outer measure* of E as

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \rho(A_i) \middle| \{A_i\}_{i=1}^{\infty} \subset \mathcal{E} \text{ such that } E \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$

Proposition 4.2.25 (Properties of the Outer Measure)**.** *Suppose we construct μ^* as above. Then,*

1. *$\mu^*(\emptyset) = 0$.*
2. *If $E \subset F \subset X$, then $\mu^*(E) \leq \mu^*(F)$.*
3. *If $\{E_n\}_{n=1}^{\infty} \subset \mathcal{P}(X)$, then*
$$\mu^* \left(\bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \mu^*(E_n).$$

We can similarly define an inner measure.

Remark 4.2.26. If $A \in \mathcal{E}$, then $\mu^*(A) \neq \rho(A)$ in general.

Definition (μ^*-measurable sets)**.** A subset $E \subset X$ is μ^*-measurable if for all $F \subset X$,

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \backslash E).$$

The vague idea is that the outer measure should be equal to the "inner measure". To check that E is μ^*-measurable, it suffices to assume that $\mu^*(F) < \infty$ and verify that

$$\mu^*(F) \geq \mu^*(F \cap E) + \mu^*(F \backslash E)$$

for any F. Easy to see that $\emptyset$ and X are μ^*-measurable.

Theorem 4.2.27 (Caratheodory's Theorem)**.** *If μ^* is the outer measure associated to some $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$, then*

1. *$\mathcal{M} = \{E \subset X : E \text{ is } \mu^*\text{-measurable}\}$ is a σ-algebra.*
2. *$\mu^* : \mathcal{M} \to [0, \infty]$ is a complete measure (i.e. if $N \in \mathcal{M}$ with $\mu^*(N) = 0$, then all $F \subset N$ satisfy $F \in \mathcal{M}$).*

Pre-measures

Definition (Algebra of sets)**.** A subset $\mathcal{A} \subset \mathcal{P}(X)$ is an *algebra* if

1. $\emptyset, X \in \mathcal{A}$,
2. $E \in \mathcal{A} \implies X \backslash E \in \mathcal{A}$,
3. $E_1, ..., E_n \in \mathcal{A} \implies \bigcup_{i=1}^{n} E_i \in \mathcal{A}$.

Lemma 4.2.28. *Suppose $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra. Then $\mathcal{A}$ is a σ-algebra if for any disjoint countable sequence of subsets $\{E_i\} \subset \mathcal{A}$, we have $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.*

Corollary 4.2.28.1. *Suppose $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra. Then $\mathcal{A}$ is a σ-algebra if for any countable increasing chain of subsets $\{E_i\} \subset \mathcal{A}$, we have $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.*

Definition (Pre-measure)**.** Suppose $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra. Then, a function $\mu_0 : \mathcal{A} \to [0, \infty]$ is a *pre-measure* if (1) $\mu_0(\emptyset) = 0$; and (2) if $\{A_i\} \subset \mathcal{A}$ is a disjoint sequence such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, then

$$\mu_0 \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu_0(A_i).$$

Remark 4.2.29. As an example, let $X = \mathbb{R}$ and $\mathcal{A}$ is the collection of finite union of disjoint half-open intervals of the form $[a_i, b_i)$, then we can define $\mu_0([a, b)) = b - a$ and extend (finite) additively. It will be shown later that μ_0 is a pre-measure.

Remark 4.2.30. A pre-measure is always finite additive, i.e. if $A_1, ..., A_n \in \mathcal{A}$ are disjoint, then

$$\mu_0\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu_0(A_i).$$

This also implies that if $A, B \in \mathcal{A}$ with $B \subset A$, then $\mu_0(B) \leq \mu_0(A)$.

Definition (σ-finite)**.** A pre-measure μ_0 is σ-finite if $X = \bigcup_{i=1}^{\infty} A_i$, where $A_i \in \mathcal{A}$ such that $\mu_0(A_i) < \infty$ for all i.

Theorem 4.2.31. *Suppose $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, and $\mathcal{M} = \mathcal{M}(\mathcal{A})$ the σ-algebra generated by $\mathcal{A}$. Suppose $\mu_0 : \mathcal{A} \to [0, \infty]$ is a σ-finite pre-measure. Then μ_0 has a unique extension to a measure $\mu : \mathcal{M} \to [0, \infty]$.*

This extension is given as follows: consider the outer measure μ^* associated to μ_0, and let $\mathcal{M}^*$ be the σ-algebra of μ^*-measurable sets. Then $\mu^*|_{\mathcal{A}} = \mu_0$ and $\mathcal{M} \subset \mathcal{M}^*$, so the extension is $\mu = \mu^*|_{\mathcal{M}}$.

Existence holds for arbitrary measure spaces, where μ_0 need not be σ-finite.

Corollary 4.2.31.1. *Suppose we have two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ where μ and ν are both σ-finite. Consider the product $(X \times Y, \mathcal{M} \otimes \mathcal{N})$; then there exists a unique measure π on $X \times Y$ such that $\pi(A \times B) = \mu(A) \cdot \nu(B)$ for all $A \in \mathcal{M}$ and all $B \in \mathcal{N}$. Here, we use the convention that $\infty \cdot 0 = 0$.*

The measure π is also denoted by $\pi = \mu \times \nu$.

Fubini-Tonelli Theorem

We have a nice result on the integration over product measures. Suppose $E \subset X \times Y$. If $x \in X$ and $y \in Y$, define

$$E_x := \{y \in Y : (x, y) \in E\} \quad \text{and} \quad E^y := \{x \in X : (x, y) \in E\}.$$

Then, given a function $f : E \subset X \times Y \to [-\infty, \infty]$, define $f_x : E_x \to [-\infty, \infty]$ and $f^y : E^y \subset X \to [-\infty, \infty]$ by $f_x(y) = f(x, y) = f^y(x)$. In particular,

$$(\chi_E)_x = \chi_{E_x} \quad \text{and} \quad (\chi_E)^y = \chi_{E^y}.$$

Proposition 4.2.32. *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces.*

1. *If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$ for all $x \in X, y \in Y$.*
2. *If f is measurable on $X \times Y$, then f_x is measurable on Y and f_y is measurable on X.*

Theorem 4.2.33 (Fubini-Tonelli)**.** *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.*

1. *If $E \in \mathcal{M} \otimes \mathcal{N}$, then $\nu(E_x) : X \to [0, \infty]$ and $\mu(E^y) : Y \to [0, \infty]$ are measurable functions, and*
$$(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu = \int_Y \mu(E^y) d\nu. \tag{$\star$}$$
2. *(Tonelli) If $f \in L^+(X \times Y)$, then*
$$g(x) := \int_Y f(x, y) d\nu \in L^+(X) \quad \text{and} \quad h(y) := \int_X f(x, y) d\mu \in L^+(Y).$$
Moreover,
$$\begin{aligned} \int_{X \times Y} f d(\mu \times \nu) &= \int_X g d\mu = \int_X \left(\int_Y f(x, y) d\nu\right) d\mu \\ &= \int_Y h d\nu = \int_Y \left(\int_X f(x, y) d\mu\right) d\nu. \end{aligned} \tag{$*$}$$
3. *(Fubini) If $f \in L^1(X \times Y)$, then for almost every fixed $x \in X$, the function $y \mapsto f(x, y)$ is in $L^1(Y)$. Similarly, for almost every fixed $y \in Y$, the function $x \mapsto f(x, y)$ is in $L^1(X)$. Moreover, the almost everywhere defined functions $g(x)$ and $h(y)$ (defined above) are also in $L^1(X)$ and $L^1(Y)$ respectively, and $(*)$ holds.*

Lebesgue Measure on $\mathbb{R}^n$

By taking the product measure, we can construct measures on $\mathbb{R}^n$ for all $n \geq 2$ from a measure on $\mathbb{R}$. We thus focus on constructing a measure on $\mathbb{R}$. Note first that, *a priori*, we already have a Borel σ-algebra on $\mathbb{R}$.

In light of Theorem 4.2.31, we need only construct an algebra $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ and a pre-measure $\mu : \mathcal{A} \to [0, \infty]$. Take $\mathcal{A}$ to be the collection of finite disjoint union of half intervals of the form $(a, b]$, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$ and $a < b$. Define $\mu : \mathcal{A} \to [0, \infty]$ by setting $\mu(a, b] = b - a$ and then extending (finite) additively.

Lemma 4.2.34. *The function $\mu : \mathcal{A} \to [0, \infty]$ is a pre-measure.*

We thus have an outer measure $\mu^* : \mathcal{M}^* \to [0, \infty]$, where $\mathcal{M}^*$ is the collection of μ^*-measurable sets. Now, notice that $\mathcal{M}(\mathcal{A}) = B_{\mathbb{R}}$, and so we have $B_{\mathbb{R}} \subset \mathcal{M}^*$. It is known that $B_{\mathbb{R}} \subsetneq \mathcal{M}^* \subsetneq \mathcal{P}(\mathbb{R})$. Moreover, since $B_{\mathbb{R}} \subset \mathcal{M}^*$, $\mathfrak{m} := \mu^*|_{B_{\mathbb{R}}}$ yields a measure, called the *Lebesgue Measure on* $\mathbb{R}$.

Lemma 4.2.35. *Let $\mathfrak{m}$ be the Lebesgue measure on $\mathbb{R}$ defined above.*

1. $\mathfrak{m}\left((a, b)\right) = b - a$.
2. $\mathfrak{m}(S) = 0$ *for S countable.*

Remark 4.2.36. The converse of (2) in the lemma above is false. Consider the *Cantor set*, which is constructed as follows. Let $I_0 = [0, 1]$, $I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and each successive I_k is formed from the previous by taking out the middle third of each closed interval, so that

$$I_k = \bigcup_{i=0}^{3^{k-1}-1} \left(\left[\frac{3i+0}{3^k}, \frac{3i+1}{3^k} \right] \cup \left[\frac{3i+2}{3^k}, \frac{3i+3}{3^k} \right] \right).$$

Then the Cantor set is $C = \bigcap_{k=0}^{\infty} I_k$. Note that $C \in B_{\mathbb{R}}$. Moreover, since for each I_k we have $\mathfrak{m}(I_k) = \frac{2^k}{3^k}$, it follows that $\mathfrak{m}(C) = 0$. Moreover, it can be checked easily that C is uncountable by a standard diagonal argument.

Before the next proposition, note that any compact subset of $\mathbb{R}$ is contained in $B_{\mathbb{R}}$.

Proposition 4.2.37. *The measure space $(\mathbb{R}, B_{\mathbb{R}}, \mathfrak{m})$ is* regular*, i.e.*

1. *If $K \subset \mathbb{R}$ is compact, then $\mathfrak{m}(K) < \infty$.*
2. *(Outer regular) If $E \in B_{\mathbb{R}}$, then*
$$\mathfrak{m}(E) = \inf\{\mathfrak{m}(U) : E \subset U, U \text{ open}\}.$$
3. *(Inner regular) If $E \in B_{\mathbb{R}}$, then*
$$\mathfrak{m}(E) = \sup\{\mathfrak{m}(K) : E \supset K, K \text{ compact}\}.$$

Note that to construct the Lebesgue measure, we had the outer measure

$$\mathfrak{m}^*(E) = \inf\left\{ \sum_{i=1}^{\infty}(b_i - a_i) : E \subset \bigcup_{i=1}^{\infty}(a_i, b_i] \right\} = \inf\left\{ \sum_{i=1}^{\infty}(b_i - a_i) : E \subset \bigcup_{i=1}^{\infty}(a_i, b_i) \right\}$$

for any $E \subset \mathbb{R}$. Recall also that an $\mathfrak{m}^*$-measurable set is any subset $E \subset \mathbb{R}$ such that

$$\mathfrak{m}^*(F) = \mathfrak{m}^*(F \cap E) + \mathfrak{m}^*(F \backslash E)$$

for all $F \subset \mathbb{R}$. Define $\mathcal{G}_\delta$ be the collection of all countable intersections of open sets, and $\mathcal{F}_\delta$ to be the collection of all countable union of closed sets.

Proposition 4.2.38 (Characterization of $\mathfrak{m}^*$-measurable sets)**.** *Let $E \subset \mathbb{R}$. Then, the following are equivalent:*

(i) E is $\mathfrak{m}^*$-measurable.

(ii) For all $\epsilon > 0$, there exists open $U \supset E$ such that $\mathfrak{m}^*(U \backslash E) < \epsilon$.

(iii) For all $\epsilon > 0$, there exists closed $F \subset E$ such that $\mathfrak{m}^*(E \backslash F) < \epsilon$.

(iv) There exists $G \in \mathcal{G}_\delta \subset B_{\mathbb{R}}$ such that $E \subset G$ and $\mathfrak{m}^*(G \backslash E) = 0$.

(v) *There exists* $H \in \mathcal{F}_\delta \subset B_\mathbb{R}$ *such that* $H \subset E$ *and* $\mathfrak{m}^*(E \backslash H) = 0$.

If $\mathfrak{m}^*(E) < \infty$, *then the above statements are also equivalent to*

(vi) *For all* $\epsilon > 0$, *there exists a finite union of open intervals* U *such that* $\mathfrak{m}^*(E\backslash U \sqcup U \backslash E) < \epsilon$.

Corollary 4.2.38.1. *If* $\mathcal{M}$ *is the collection of* $\mathfrak{m}^*$*-measurable sets, then*

$$\mathcal{M} = \overline{B_\mathbb{R}} = \{E \cup F : E \in B_\mathbb{R}, \text{ and } F \subset N \in B_\mathbb{R} \text{ where } \mathfrak{m}(N) = 0\}.$$

Proposition 4.2.39 (Steinhaus Theorem)**.** *Suppose* $X \subset \mathbb{R}$ *is a measurable set such that* $\mathfrak{m}(X) > 0$. *Then, the set* $X - X = \{x - y : x, y \in X\}$ *contains an open neighbourhood of 0.*

Proof. We give two proofs. Both use the fact that $y \notin X - X$ iff $y + X \cap X = \emptyset$, where $y + X = \{y + x : x \in X\}$.

1. Let $\epsilon > 0$ be arbitrary. Then, using the fact that $\mathfrak{m}(X) > 0$, there exists an open set U such that $X \subset U$ and $\mathfrak{m}(U \setminus X) < \epsilon\mathfrak{m}(X)$. Notice that U is a countable union of disjoint open intervals I_n. It follows that

   $$\sum_n \mathfrak{m}(I_n \setminus X) < \epsilon \sum_n \mathfrak{m}(X \cap I_n).$$

   Thus there exists n such that $\mathfrak{m}(I_n \setminus X) < \epsilon\mathfrak{m}(X \cap I_n)$. Since X is measurable, we have $\mathfrak{m}(I_n) = \mathfrak{m}(X \cap I_n) + \mathfrak{m}(I_n \setminus X)$, and so

   $$\mathfrak{m}(I_n) - \mathfrak{m}(X \cap I_n) < \epsilon\mathfrak{m}(I_n \backslash X).$$

   Hence (given $\epsilon > 0$), there exists an open interval $I = (a, b)$ such that $\mathfrak{m}(X \cap I) > \frac{b-a}{1+\epsilon}$. Let $\alpha \in (\frac{1}{2}, 1)$, and take $\epsilon = \frac{1}{\alpha} - 1 > 0$, and let I be the above open interval corresponding to ϵ. Set $E := X \cap I$, so that $\mathfrak{m}(E) > \alpha(b - a)$.

   Set $\delta := (\alpha - \frac{1}{2})(b - a)$. We claim that $(-\delta, \delta) \subset E - E \subset X - X$. Suppose not, then there exists x such that $|x| < \delta$ and $(x + E) \cap E = \emptyset$. It follows that

   $$2\mathfrak{m}(E) = \mathfrak{m}\big((x + E) \cup E\big) \leq \mathfrak{m}\big((x + I) \cup I\big)$$

   Now, notice that $(x+I) \cup I = (a+x, b+x) \cup (a, b) \subset (a - |x|, b + |x|)$, and so $m\big((x+I) \cup I\big) \leq b - a + 2|x|$. It follows that

   $$2\mathfrak{m}(E) < b - a + (2\alpha - 1)(b - a) = 2\alpha(b - a) < 2\mathfrak{m}(E),$$

   an obvious contradiction.

2. Suppose the statement is false; then there exists a sequence $\{a_n\} \subset \mathbb{R}$ such that $a_n \to 0$ and $(a_n + X) \cap X = \emptyset$. For each $a_n + X$, there exists an open subset U_n such that $a_n + X \subset U_n$ and $\mathfrak{m}(U_n \backslash (a_n + X)) < \frac{1}{2^{n+1}}\mathfrak{m}(X)$ (here we use the fact that $\mathfrak{m}(X) > 0$). It follows that

   $$\mathfrak{m}\left(\bigcup_n U_n \setminus (a_n + X)\right) \leq \sum_n \frac{1}{2^{n+1}}\mathfrak{m}(X) < \mathfrak{m}(X).$$

   Now, notice that $X \cap U_n \subset U_n \setminus (a_n + X)$ since $X \cap (a_n + X) = \emptyset$. Setting $U = \bigcup_n U_n$, it follows that $X \cap U \subset \bigcup_n U_n \setminus (a_n + X)$ and so

   $$\mathfrak{m}(X \cap U) < \mathfrak{m}(X).$$

   Since U is a countable union of disjoint open intervals, the boundary ∂U is countable and so $\mathfrak{m}(X \cap \partial U) = 0$. Since U is open so that $U \cap \partial U = \emptyset$, it follows that

   $$\mathfrak{m}(X \cap \bar{U}) < \mathfrak{m}(X)$$

   where $\bar{U}$ is the closure of U in $\mathbb{R}$. However, for any $x \in X$, we have $a_n + x \in a_n + X \subset U_n \subset U$ for all n, and so taking $n \to \infty$ it follows that $x \in \bar{U}$. Thus $X \subset \bar{U}$ and so $\mathfrak{m}(X \cap \bar{U}) = \mathfrak{m}(X)$, contradicting the previous inequality.

□

Example 4.2.40 (Spring 2018 Day 1)**.** Suppose $X \subset [0, 1]$ is such that (a) for any $r \in \mathbb{R}$, there exists $x \in X$ such that $r - x \in \mathbb{Q}$, and (b) for any $x, y \in X$ with $x \neq y$, we have $x - y \notin \mathbb{Q}$. Prove that X is not Lebesgue measurable.

We assume that X is Lebesgue measurable. Suppose first that $\mathfrak{m}(X) = 0$. Consider $q + X$ for all $q \in X$; if $q + X \cap q' + X \neq \emptyset$, there exist $x, y \in X$ such that $q + x = q' + y$, and so $x - y = q' - q \in \mathbb{Q}$. By property (b), it follows that $x = y$ and so $q = q'$. Hence, $q + X \cap q' + X \neq \emptyset$ for all $q \neq q'$. On the other hand, property (a) implies that $\mathbb{R} = \bigcup_{q\in\mathbb{Q}}(q + X)$. Since $\mathbb{Q}$ is countable and $\mathfrak{m}(q + X) = \mathfrak{m}(X) = 0$, it follows that

$$\mathfrak{m}(\mathbb{R}) = \sum_{q\in\mathbb{Q}} \mathfrak{m}(q + X) = 0$$

an obvious contradiction. Thus $\mathfrak{m}(X) > 0$, and since X is Lebesgue measurable, Steinhaus' Theorem implies that $X - X$ contains an open interval I of the origin. However, property (b) implies that $(X - X) \cap \mathbb{Q} = \{0\}$ and so $I \cap \mathbb{Q} = \{0\}$, contradicting the dense nature of $\mathbb{Q}$ in $\mathbb{R}$. Therefore X cannot be Lebesgue measurable.

Theorem 4.2.41 (Littlewood's First Principle)**.** *Every Lebesgue measurable set E of finite measure is "nearly" a finite union of open intervals. More precisely, if $\mathfrak{m}(E) < \infty$, then for any $\epsilon > 0$, there exists a disjoint finite union of open intervals $J_\epsilon = \bigsqcup_i (a_i, b_i)$ such that*

$$\mathfrak{m}\left(J_\epsilon \backslash E \sqcup E \backslash J_\epsilon\right) < \epsilon.$$

This follows from (i) implies (vi) in the above characterization theorem.

Corollary 4.2.41.1 (Riemann-Lebesgue Lemma)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is in $L^1(\mathbb{R})$, then*

$$\lim_{k\to\infty} \int_{\mathbb{R}} f(x)e^{ikx}dx := \left(\lim_{k\to\infty} \int_{\mathbb{R}} f(x)\cos kx dx\right) + i\left(\lim_{k\to\infty} \int_{\mathbb{R}} f(x)\sin kx dx\right) = 0.$$

Theorem 4.2.42 (Littlewood's Third Principle)**.** *Every convergent sequence of measurable functions $\{f_n\} \to f$ converging point-wise almost everywhere is "nearly" uniformly convergent.*

More precisely, if $X \subset \mathbb{R}$ with $\mathfrak{m}(X) < \infty$ and $\{f_n : X \to \mathbb{R}\} \to f : X \to \mathbb{R}$ point-wise almost everywhere, then for all $\epsilon > 0$, there exists compact subset $E \subset X$ such that $\mathfrak{m}(X \backslash E) < \epsilon$ and $\{f_n|_E\}$ converges to $f|_E$ uniformly.

Littlewood's Third Principle is Egoroff's Theorem, specialized to $\mathbb{R}$. We can take E to be compact since

$$\mathfrak{m}(E) = \sup\{\mathfrak{m}(K) : K \subset E, K \text{ compact}\}.$$

Theorem 4.2.43 (Littlewood's Second Principle, aka Lusin's Theorem)**.** *Every Lebesgue measurable function is "nearly" continuous.*

More precisely, if $X \subset \mathbb{R}$ and $\mathfrak{m}(X) < \infty$, and if $f : X \to \mathbb{R}$ is measurable, then for all $\epsilon > 0$ there exists compact $F \subset X$ such that $\mathfrak{m}(X \backslash F) < \epsilon$ and $f|_F$ is continuous (with respect to the relative topology on F).

Corollary 4.2.43.1. *The space of continuous integrable functions on $\mathbb{R}^d$ is dense in $L^1(\mathbb{R}^d)$.*

4.2.4 Differentiation

The Fundamental Theorem of Calculus

Definition (Lebesgue Set)**.** Let $f \in L^1(\mathbb{R}^d)$. The *Lebesgue set* L_f of f is

$$L_f := \left\{x \in \mathbb{R}^d : f(x) \in \mathbb{R} \text{ and } \lim_{x\in B, \mathfrak{m}(B)\to 0} \frac{1}{\mathfrak{m}(B)} \int_B |f(t) - f(x)|dt = 0\right\}$$

where B runs through open balls containing x.

Definition (The Hardy Littlewood Maximal Function)**.** Suppose $f \in L^1(\mathbb{R}^d)$. Then the *Hardy Littlewood Maximal Function* of f is

$$f^*(x) := \sup_{x\in B} \left(\frac{1}{\mathfrak{m}(B)} \int_B |f(y)|dy\right) : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\},$$

where B ranges over all open balls containing x.

Theorem 4.2.44. *If $f \in L^1(\mathbb{R}^d)$, then*

1. *f^* is measurable*
2. *$f^*(x) < \infty$ for almost all x.*
3. *Let $E_\alpha := \left\{x \in \mathbb{R}^d : f^*(x) > \alpha\right\}$ for $\alpha \in \mathbb{R}$. Then, $\mathfrak{m}\left(E_\alpha\right) \leq \frac{3^d}{\alpha}\|f\|_{L^1(\mathbb{R}^d)}$ for all $\alpha > 0$.*

Here $\|f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)|dx$ *is the* L^1 *norm of* f.

Theorem 4.2.45 (Lebesgue Differentiation Theorem)**.** *Suppose* $f \in L^1_{\text{loc}}(\mathbb{R}^d)$*, i.e. for all* $x \in \mathbb{R}^d$*, there exists an open ball* $B \ni x$ *such that* $f|_B \in L^1(B)$*. Then*

$$\lim_{x\in B, \mathfrak{m}(B)\to 0} \frac{1}{\mathfrak{m}(B)} \int_B f(y)dy = f(x)$$

for almost all $x \in \mathbb{R}^d$*.*

Corollary 4.2.45.1. *Suppose* $f \in L^1_{\text{loc}}(\mathbb{R}^d)$*. Then the Lebesgue set* $L_f = \mathbb{R}^d$ *almost everywhere, i.e.* $\mathbb{R}^d \backslash L_f$ *has measure zero.*

Now suppose $f \in L^1(\mathbb{R})$, and fix $a \in \mathbb{R}$. Consider the function

$$F(x) = \int_a^x f(y)dy : \mathbb{R} \to \mathbb{R}.$$

Then, notice that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{\mathfrak{m}(I)\to 0} \frac{1}{\mathfrak{m}(I)} \int_I f(y)dy.$$

We ask whether $F' = f$. This statement is equivalent to

$$\lim_{\mathfrak{m}(I)\to 0} \underbrace{\frac{1}{\mathfrak{m}(I)} \left| \int_I (f(y) - f(x))dy \right|}_{LHS} = 0.$$

However, notice that

$$LHS \leq \frac{1}{\mathfrak{m}(I)} \int_I |f(y) - f(x)|dy \leq \frac{2}{\mathfrak{m}(I \cup J)} \int_{I\cup J} |f(y) - f(x)|dy,$$

where J is another interval with x as an endpoint with the same length as I, and such that J and I are on opposite sides of x. However, $I \cup J \cup \{x\}$ is a ball centered at x with radius $\mathfrak{m}(I)$, and so by Corollary 4.2.45.1 it follows that $LHS \to 0$ as $\mathfrak{m}(I) \to 0$. We have thus proved the fundamental theorem of calculus.

Theorem 4.2.46 (Fundamental Theorem of Calculus)**.** *Suppose* $f \in L^1(\mathbb{R})$*, and fix* $a \in \mathbb{R}$*. Then, the function*

$$F(x) := \int_a^x f(y)dy : \mathbb{R} \to \mathbb{R}$$

is differentiable almost everywhere, and moreover $F' = f$ *almost everywhere.*

Absolute Continuity and Functions of Bounded Variation

Proposition 4.2.47. *Suppose* X *is a measure space and* $f \in L^1(X)$ *is a real-valued function. Then, for any* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for any* $E \subset X$ *with* $\mu(E) < \delta$*, we have*

$$\int_E |f| < \epsilon.$$

Definition (Absolute Continuity)**.** A function $F : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{k=1}^N |F(b_k) - F(a_k)| < \epsilon$$

whenever $\sum_{k=1}^N (b_k - a_k) < \delta$, where $N \in \mathbb{N}$ and $(a_1, b_1), ..., (a_N, b_n)$ are disjoint open intervals contained in $[a, b]$.

Here, it is possible that $b_i = a_{i+1}$ and so on. It is easy to see that absolute continuity implies uniform continuity on $[a, b]$, since we can simply take $N = 1$. In particular, if F is absolutely continuous then it is continuous.

Corollary 4.2.47.1. *Suppose* $f \in L^1([a, b])$*. Then,* $F(x) := \int_a^x f(y)dy : [a, b] \to \mathbb{R}$ *is absolutely continuous.*

Definition (Bounded Variation)**.** A function $F : [a,b] \to \mathbb{R}$ is said to be of *bounded variation* if there exists some $M > 0$ such that

$$\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| \leq M$$

for any partition $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ of $[a,b]$.

Note that a function of bounded variation is, in general, not continuous.

Remark 4.2.48. Some examples of functions of bounded variation are as follows.

1. If $F : [a,b] \to \mathbb{R}$ is monotonous, then it is of bounded variation, since the sum in the above definition reduces to simply $|F(b) - F(a)|$.

2. If $F : [a,b] \to \mathbb{R}$ is absolutely continuous, then F is of bounded variation.

3. If $F : [a,b] \to \mathbb{R}$ is continuous and is differentiable on (a,b) such that $|F'(x)| \leq C$ on (a,b), then F is of bounded variation. This follows easily from the mean value theorem, and we can take $M = C(b-a)$.

4. Suppose $F : [0,1] \to \mathbb{R}$ is continuous such that $F(\frac{1}{n}) = (-1)^{n+1}\frac{1}{n}$, say $F(x) = -x\cos(\frac{\pi}{x})$. Then this function F is not of bounded variation, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

The special property of functions F of bounded variation is that they can be written as $F = g - h$ where g and h are increasing. If F is absolutely continuous, then it can be shown that g and h are continuous.

Definition (Total Variation)**.** The *total variation* of $F : [a,b] \to \mathbb{R}$ in $[a,b]$ is

$$T_F(a,b) := \sup_{a=t_0<t_1<\cdots<t_N=b} \sum_{i=1}^{N} |F(t_i) - F(t_{i-1})|,$$

where the supremum is taken over all partitions of $[a,b]$.

If F is of bounded variation, then $T_F(a,b)$ is finite. In particular, $T_F(a,x)$ is also finite for all $x \in [a,b]$. It is also easy to show that $T_F(a,b) = T_F(a,x) + T_F(x,b)$ for all $x \in [a,b]$ if F is of bounded variation. This defines a function

$$T_F(a,x) : [a,b] \to \mathbb{R}.$$

Definition (Positive and Negative Variation)**.** The *positive variation* on $[a,x]$ of a function $F : [a,b] \to \mathbb{R}$ is

$$P_F(a,x) := \sup_{\text{partitions of } [a,x]} \sum_{i=1}^{N} \max\{0, F(t_i) - F(t_{i-1})\}.$$

Similarly, the *negative variation* on $[a,x]$ is

$$N_F(a,x) := \sup_{\text{partitions of } [a,x]} \left(-\sum_{i=1}^{N} \min\{0, F(t_i) - F(t_{i-1})\} \right).$$

Note that $T_F(a,x), P_F(a,x), N_F(a,x)$ are all increasing functions of x. The next lemma then allows us to express a function of bounded variation as the difference of increasing functions.

Lemma 4.2.49. *If $F : [a,b] \to \mathbb{R}$ is of bounded variation, then $F(x) = F(a) + P_F(a,x) - N_F(a,x)$ and $T_F(a,x) = P_F(a,x) + N_F(a,x)$.*

Theorem 4.2.50. *Suppose $F : [a,b] \to \mathbb{R}$. Then, F is a function of bounded variation if and only if $F = g - h$ such that $g, h : [a,b] \to \mathbb{R}$ are increasing.*

Proposition 4.2.51. *Suppose $F : [a,b] \to \mathbb{R}$ is continuous and of bounded variation. Then, $T_F(a,x) : [a,b] \to \mathbb{R}$ is also continuous.*

Corollary 4.2.51.1. *If $F : [a,b] \to \mathbb{R}$ is of bounded variation and continuous, then F is the difference of continuous increasing functions.*

Theorem 4.2.52. *Suppose $F : [a,b] \to \mathbb{R}$ is of bounded variation and continuous. Then, F is differentiable almost everywhere on $[a,b]$.*

Theorem 4.2.53. *1. Suppose $F : [a,b] \to \mathbb{R}$ is absolutely continuous. Then*

(a) F *is differentiable almost everywhere, and* $F' \in L^1([a,b])$.

(b) *If* $F' = 0$ *almost everywhere, then* F *is a constant.*

(c) $F(x) - F(a) = \displaystyle\int_a^x F'(y)dy$ *for all* $x \in [a,b]$. *Hence* $F(b) - F(a) = \int_a^b F'(y)dy$.

2. *Conversely, if* $f \in L^1([a,b])$, *then* $F(x) := \int_a^x f(y)dy$ *is absolutely continuous and* $F' = f$ *almost everywhere on* $[a,b]$.

Consider the vector space V of all absolutely continuous functions on $[a,b]$, and consider the vector space $L^1([a,b])$. The above theorem then implies that the sequence

$$0 \to \mathbb{R} \to V \underset{\frac{d}{dx}}{\to} L^1([a,b]) \to 0$$

is exact. The middle arrow follows from 1(b), while the right arrow follows from part 2.

4.2.5 Signed Measures

Basics

Definition (Signed Measure)**.** Suppose $(X, \mathcal{M})$ is a measurable space. A *signed measure* is a map $v : \mathcal{M} \to [-\infty, \infty]$ such that (i) $v(\emptyset) = 0$, (ii) $\{-\infty, \infty\} \not\subset v(\mathcal{M})$ (i.e. it can only hit one of ∞ or $-\infty$), and (iii) if $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ is a disjoint sequence, then

$$v\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} v(E_j)$$

where the sum converges absolutely if the left hand side is finite.

Remark 4.2.54. 1. If $v : \mathcal{M} \to [0, \infty]$ is a measure in the usual sense (i.e. a *positive* measure), then v can also be considered a signed measure.

2. Suppose $(X, \mathcal{M}, \mu)$ is a measure space with μ a positive measure. Suppose $f \in L^1(X)$. Then $v(E) := \int_E f d\mu$ is a signed measure. This measure is usually written as $v = f d\mu$.

3. If v_1 and v_2 are positive measures on $(X, \mathcal{M})$, and if v_2 is finite (i.e. $v_2(X) < \infty$), then both $v = v_1 - v_2$ and $v' = v_2 - v_1$ are signed measures.

Proposition 4.2.55 (Properties of Signed Measure)**.** *Let* v *be a signed measure on* $(X, \mathcal{M})$.

1. *If* $E_1 \subset E_2 \subset E_3 \subset \cdots$ *in* $\mathcal{M}$, *then*

$$v\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} v(E_j).$$

2. *If* $E_1 \supset E_2 \supset E_3 \supset \cdots$ *in* $\mathcal{M}$, *and if* $v(E_1)$ *is finite, then*

$$v\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} v(E_j).$$

Definition (Positive, Negative, and Null sets)**.** Let v be a signed measure on $(X, \mathcal{M})$. A set $E \in \mathcal{M}$ is called *positive* (resp. *negative*, *null*) if for any $F \in \mathcal{M}$ and $F \subset E$, we have $v(F) \geq 0$ (resp. $v(F) \leq 0$, $v(F) = 0$).

For instance, $\emptyset$ is positive, negative, and null. Note that the definition of a null set is much stronger than the condition that the measure equals zero.

Lemma 4.2.56. 1. *Suppose* E *is positive, and* $F \subset E$ *is measurable as well. Then* F *is also positive. Similarly,* F *is negative if* E *is negative, and is null if* E *is as well.*

2. *Consider an arbitrary* $\{E_n\}$ *sequence. If all of the* E_n *are positive sets, then* $\bigcup_{n\geq 1} E_n$ *is positive. Similarly if they are all negative or all null.*

Theorem 4.2.57 (The Hahn Decomposition Theorem)**.** *Let v be a signed measure on $(X, \mathcal{M})$. Then, there exists $P, N \in \mathcal{M}$ where P is positive and N is negative, such that $X = P \sqcup N$. Moreover, if we have another such pair (P', N'), then $P \triangle P'$ and $N \triangle N'$ are both null sets. (Here, $A \triangle B := (A \backslash B) \sqcup (B \backslash A)$.)*

Definition (Hahn Decomposition)**.** Such a decomposition given in the Hahn Decomposition Theorem is called a *Hahn Decomposition* for the signed measure v.

Notice that if $X = P \sqcup N$ is a Hahn decomposition for v, then we can write $v = v_1 - v_2$, where $v_1(E) := v(E \cap P)$ and $v_2 := -v(E \cap N)$ for any $E \in \mathcal{M}$. It is easy to check that v_1 and v_2 are both positive measures.

Definition (Mutually Singular)**.** Two signed measures μ and ν on $(X, \mathcal{M})$ are *mutually singular*, or that ν is singular with respect to μ, or vice versa, if there exists $E, F \in \mathcal{M}$ such that $X = E \sqcup F$, E is null for μ, and F is null for ν. In such a case, we write $\mu \perp \nu$. Here, we say that μ lives on F and ν lives on E.

Theorem 4.2.58 (The Jordan Decomposition Theorem)**.** *If ν is a signed measure on X, then there exists unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. Moreover, one of ν^+ or ν^- is a finite measure.*

Definition (Jordan Decomposition and Positive, Negative, and Total Variation)**.** The positive measures μ^+ (resp. μ^-) is called the *positive* (resp. *negative*) variation of ν, and $\nu = \nu^+ - \nu^-$ is called the *Jordan decomposition* of ν.

The *total variation of* ν, denoted by $|\nu|$, is the positive measure $|\nu| := \nu^+ + \nu^-$.

Remark 4.2.59.

1. A subset $E \in \mathcal{M}$ is ν-null if and only if it is a $|\nu|$-null set.
2. $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Definition (Integration over Signed Measures)**.** Set $L^1(\nu) := L^1(\nu^+) \cap L^1(\nu^-)$. For $f \in L^1(\nu)$, we define

$$\int f d\nu := \int f d\nu^+ - \int f d\nu^-.$$

Definition (Finite Signed Measures)**.** A signed measure ν is called finite if $|\nu|$ is finite.

Definition (Complex Measure)**.** Suppose $(X, \mathcal{M})$ is a measurable space. A *complex measure* is a map $v : \mathcal{M} \to \mathbb{C}$ such that (i) $v(\emptyset) = 0$ and (ii) if $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ is a disjoint sequence, then

$$v\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} v(E_j)$$

where the sum converges absolutely.

Note that complex measures are necessarily finite. Thus a signed measure is a complex measure iff it is finite.

Absolute Continuity of Measures

Suppose $(X, \mathcal{M})$ is a measurable space, and suppose ν is a signed measure and μ a positive measure on X.

Definition (Absolute Continuity of Signed Measures)**.** We say that ν is *absolutely continuous* with respect to μ, and write $\nu \ll \mu$, if for all $E \in \mathcal{M}$ such that $\mu(E) = 0$, we have $\nu(E) = 0$.

Remark 4.2.60. Note that if $\mu(E) = 0$, then E is a μ-null set, and so for every $F \subseteq E$ we have $\mu(F) = 0$. Then, if $\nu \ll \mu$, we have $\nu(F) = 0$ for all $F \subseteq E$. Hence E is a ν-null set, and thus also a $|\nu|$-null set. Therefore $|\nu| \ll \mu$.

Proposition 4.2.61. *Suppose $\nu \perp \mu$ and $\nu \ll \mu$. Then $\nu = 0$.*

We now show an equivalent condition for absolute continuity that justifies the usage of the term.

Theorem 4.2.62. *Suppose ν is a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then, $\nu \ll \mu$ if and only if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $E \in \mathcal{M}$ is such that $\mu(E) < \delta$, then $|\nu(E)| < \epsilon$.*

Remark 4.2.63. As an example, let $f \in L^1(\mu)$ where μ is positive. If we define a new measure $\nu := f d\mu$, where

$$\nu(E) = \int_E f d\mu,$$

then ν is a signed measure such that $\nu \ll \mu$ (since the integral is zero if $\mu(E) = 0$). It is also easy to see that $|\nu| = |f| d\mu$.

Corollary 4.2.63.1. *Let $f \in L^1(\mu)$. Then, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\mu(E) < \delta$, then*

$$\left|\int_E f d\mu\right| < \epsilon \quad \text{and} \quad \int_E |f| d\mu < \epsilon.$$

Lemma 4.2.64. *Suppose ν and μ are both finite positive measures on $(X, \mathcal{M})$. Then, either $\nu \perp \mu$, or there exists $\epsilon \in (0,1]$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\nu \geq \epsilon\mu$ on E, i.e. the signed measure $\nu - \epsilon\mu$ is positive on E.*

Definition (σ-finite Measures)**.** A signed measure ν is *σ-finite* if $X = \bigsqcup_{n\geq 1} X_n$ and ν is finite on each X_n.

Definition (Extended μ-integrability)**.** If μ is a positive measure, then $f : X \to \mathbb{R}$ is said to be *extended μ-integrable* if at least one of f^+ or f^- is in $L^1(\mu)$. (Recall $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.)

Theorem 4.2.65 (The Lebesgue-Radon-Nikodym Theorem)**.** *Let ν be a σ-finite signed measure (resp. complex measure) and μ a σ-finite positive measure on $(X, \mathcal{M})$. Then, there exists a unique σ-finite signed measure (resp. complex measure) λ and a μ-measurable function $f : X \to \mathbb{R}$ that is extended μ-integrable such that $\nu = \lambda + f\mu$ and $\lambda \perp \mu$. Moreover, any such f are equal almost everywhere (with respect to μ).*

Remark 4.2.66. For a rough idea behind the proof, suppose that $X = \mathbb{R}$, and $\nu = hdx, \mu = gdx$ where g, h are both positive functions and dx is the Lebesgue measure. In this case, note that

$$\nu = hdx = \underbrace{h\chi_{\{g=0\}}dx}_{\lambda} + \underbrace{h\chi_{\{g>0\}}dx}_{\rho}.$$

Notice that $\rho = \frac{h}{g}\chi_{\{g>0\}}d\mu$, with the convention that $\frac{h}{g}\chi_{\{g>0\}} = 0$ if $g = 0$. Moreover, it is easy to see that $\lambda \perp \mu$.

4.3 L^p Spaces

4.3.1 Basic Results

Suppose $(X, \mathcal{M}, \mu)$ is a measure space, and suppose $f : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is a measurable function on X. For any $0 < p < \infty$, define

$$\|f\|_p := \left(\int_X |f|^p d\mu\right)^{1/p}$$

and $L^p(X) = \{f : X \to \bar{\mathbb{R}} \text{ measurable} : \|f\|_p < \infty\}$. Note that $L^1(X)$ is exactly the same space considered previously.

Lemma 4.3.1. *$L^p(X)$ is a vector space for $p > 0$.*

Note that $\|\lambda f\|_p = |\lambda|\|f\|_p$, and $\|f\|_p = 0$ implies $f = 0$ almost everywhere.

Definition (Conjugate Exponents)**.** Suppose $1 < p < \infty$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. This q is unique, satisfies $1 < q < \infty$, and is called the *conjugate exponent of* p.

The conjugate exponent of 2 is 2 itself. If $p \to 1^+$, then $q \to \infty$, and vice versa.

Theorem 4.3.2 (Hölder's Inequality)**.** *If $f, g : X \to \bar{\mathbb{R}}$ are measurable and $p \in (1, \infty)$, then*

$$\|fg\|_1 \leq \|f\|_p\|g\|_q,$$

where q is the conjugate exponent of p. Thus if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$. Moreover, if the right hand side is finite, then the equality holds if and only if $|f|^p$ and $|g|^q$ are linearly dependent almost everywhere (here, we use the convention $0 \cdot \infty = 0$).

Theorem 4.3.3 (Minkowski's Inequality)**.** *If $1 \leq p < \infty$, and $f, g \in L^p$, then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.*

Corollary 4.3.3.1. *$L^p(X)$ is a normed vector space for $p \geq 1$.*

Theorem 4.3.4. *For $1 \leq p < \infty$, then $L^p(X)$ is a Banach space.*

Proposition 4.3.5. *For $1 \leq p < \infty$, the set of simple functions $f = \sum_{j=1}^n a_j\chi_{E_j}$ with $\mu(E_j) < \infty$ for all j, is dense in $L^p(X)$.*

Let us now consider $p = \infty$. Suppose $f : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is a measurable function. The L^∞-norm is

$$\|f\|_\infty = \inf\{a \geq 0 : \mu(x \in X : |f(x)| > a) = 0\}$$

This is also called the essential supremum of $|f|$. For instance, if

$$f = \begin{cases} \infty & x \in \mathbb{Q} \\ 1 & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases},$$

then $\|f\|_\infty = 1$. Define

$$L^\infty(X) = \{f : X \to \bar{\mathbb{R}} \text{ measurable} : \|f\|_\infty < \infty\}.$$

Lemma 4.3.6. *A measurable function $f \in L^\infty(X)$ if and only if there exists a bounded measurable function g such that $f = g$ almost everywhere.*

Proposition 4.3.7.

1. *If f and g are measurable on X, then $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$ (this can be considered as the Hölder inequality for $p = 1$). If $f \in L^1$ and $g \in L^\infty$, then equality holds if and only if $|g(x)| = \|g\|_\infty$ almost everywhere on points where $f(x) \neq 0$.*
2. *The map $\|.\|_\infty : L^\infty(X) \to [0, \infty)$ defines a norm on L^∞. Thus $L^\infty(X)$ is a normed vector space.*
3. *A sequence $\{f_n\}$ satisfies $\|f_n - f\|_\infty \to 0$ if and only if there exists a measurable set E such that $\mu(X\backslash E) = 0$ and $\{f_n\}$ converges to f uniformly on E.*
4. *The space $L^\infty(X)$ is a Banach space.*
5. *The simple functions are dense in $L^\infty(X)$.*
6. *If X is a compact topological space, then the space $C(X)$ of continuous functions on X to $\mathbb{R}$ (or $\mathbb{C}$) is a closed subspace of $L^\infty(X)$, and hence $(C(X), \|.\|_\infty)$ is a Banach Space.*

We can thus consider ∞ as the conjugate exponent of 1, and we have analogous results for this conjugate pair.

We also have some relations between the various L^p spaces. Some examples are given in the following propositions.

Proposition 4.3.8. *Suppose $0 < p < q < r \leq \infty$. Then $L^q \subset L^p + L^r$, i.e. each $f \in L^q$ is a sum $f = g + h$ where $g \in L^p$ and $h \in L^r$.*

Proposition 4.3.9. *If $0 < p < q < r \leq \infty$, then $L^p \cap L^r \subset L^q$. Moreover, $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$ where $\lambda \in (0,1)$ is given by $\frac{1}{q} = \lambda \cdot \frac{1}{p} + (1-\lambda) \cdot \frac{1}{r}$, i.e. $\lambda = \frac{q^{-1}-r^{-1}}{p^{-1}-r^{-1}}$ (if $r = \infty$, then $\lambda = \frac{p}{q}$).*

Proposition 4.3.10. *If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^q \subset L^p$, and $\|f\|_p \leq \|f\|_q \cdot \mu(X)^{\frac{1}{p}-\frac{1}{q}}$.*

Proposition 4.3.11. *If X is discrete (i.e. $\inf\{\mu(A) : A \subseteq X\} > 0$), then $L^p(X) \subseteq L^q(X)$.*

Example 4.3.12 (Fall 2018 Day 1)**.** If (K, μ) a finite measure space, and $f : K \to \mathbb{R}$ any measurable function, then show that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

We present two solutions:

1. On the one hand, notice that $|f| > \|f\|_\infty$ on a set of measure zero (by definition), so that

$$\|f\|_p^p = \int_K |f|^p d\mu \leq \mu(K)\|f\|_\infty^p.$$

Hence $\|f\|_p \leq (\mu(K))^{1/p}\|f\|_\infty \to \|f\|_\infty$ for $p \to \infty$. On the other hand, for any $\epsilon > 0$, there exists a measurable set $A \subset K$ with $\mu(A) > 0$ such that $|f(x)| \geq (1-\epsilon)\|f\|_\infty$ for all $x \in A$. In this case, we have

$$\|f\|_p^p = \int_K |f|^p \geq \int_A |f|^p \geq \mu(A)(1-\epsilon)^p\|f\|_\infty.$$

Taking p'th roots and then taking $p \to \infty$ (and using the fact that $\mu(A) > 0$), we see that $(1-\epsilon)\|f\|_\infty \leq \lim_p \|f\|_p$. Since $\epsilon > 0$ was arbitrary, it follows that $\|f\|_\infty \leq \lim_p \|f\|_p \leq \|f_\|\infty$. Therefore $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$, as required.

2. Let $\varphi_f : [1,\infty) \to \mathbb{R}$ be given by $\varphi_f(p) = \|f\|_p \mu(K)^{-1/p}$. The same argument as above implies that $\varphi_f(p) \leq \|f\|_\infty$ for all $p \in [1,\infty)$. Also, for any $0 < p < q$, Hölder's inequality for the exponent $\frac{q}{p} > 1$ implies that

$$\|f\|_p^p = \||f|^p\|_1 \leq \||f|^p\|_{q/p}\mu(K)^{1-\frac{p}{q}}$$

and so $\|f\|_p \leq \|f\|_q \mu(K)^{\frac{1}{p}-\frac{1}{q}}$. This implies that $\varphi_f(p)$ converges to some limit $L(f) \in [0, \|f\|_\infty]$. Since $\mu(K)^{1/p} \to 1$ as $p \to \infty$, it thus suffices to show that $L(f) = \|f\|_\infty$ for all measurable functions f.

We first consider functions in $L^\infty(X)$. Using Minkowski's inequality namely $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, we see that $L(f+g) \leq L(f) + L(g)$ for all measurable functions f, g. Hence $|L(f) - L(g)| \leq L(f-g) \leq \|f-g\|_\infty$ for all measurable functions f, g. This implies that $L : L^\infty(K) \to \mathbb{R}$ is a Lipschitz continuous function of metric spaces (where $f, g \in L^\infty(K)$). Consider now any simple function $\phi = \sum_{i=1}^k a_i \chi_{E_i}$ where WLOG the a_i are pairwise distinct and all non-zero, and the E_i are pairwise disjoint measurable sets with positive measure. Then $\|\phi\|_\infty = \max_j |a_j| =: a_i$ (say), where this i is unique. Thus, we have

$$\frac{\|\phi\|_p^p}{\|\phi\|_\infty^p} = \sum_{j=1}^k \frac{|a_j|^p}{|a_i|^p}\mu(E_i).$$

Taking $p \to \infty$ on the right, and noting that $|a_j/a_i|^p \to \delta_{ij}$ (the Kronecker delta), we see that

$$\lim_{p\to\infty} \left(\frac{\|\phi\|_p}{\|\phi\|_\infty}\right)^p = \mu(E_i) > 0.$$

Now, if for some $r \in (0,1)$ we have $\|\phi\|_p / \|\phi\|_\infty < r$ for all p large enough, then we would have

$$0 < \mu(E_i) = \lim_{p\to\infty} \left(\frac{\|\phi\|_p}{\|\phi\|_\infty}\right)^p \leq \lim_{p\to\infty} r^p = 0$$

a clear contradiction. Hence it follows that $\|\phi\|_p \to \|\phi\|_\infty$ as $p \to \infty$, for an arbitrary simple function ϕ. Therefore $L = \| \bullet \|_\infty$ on a dense subset of $L^\infty(K)$, and as L is continuous, it follows that $L(f) = \|f\|_\infty$ for all $f \in L^\infty(K)$.

Finally, suppose $\|f\|_\infty = \infty$. Then, for any $R > 0$ large enough, there exists a simple function ϕ such that $|\phi(x)| \leq |f(x)|$ and $\|\phi\|_\infty \geq R$ almost everywhere (concretely, take $\phi = R\chi_{\{x \in K : |f(x)| \geq R\}}$ for instance). Since $|\phi(x)| \leq |f(x)|$ except on a set of measure zero, it follows that $\|\phi\|_p \leq \|f\|_p$ for all finite $p > 1$, and thus $L(\phi) \leq L(f)$ upon taking $p \to \infty$. However, we have $L(\phi) = \|\phi\|_\infty \geq R$, and so $L(f) \geq R$. As $R > 0$ was arbitrary, it follows that $L(f) = \infty = \|f\|_\infty$.

Now, suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space, and suppose $1 \leq p, q \leq \infty$ are conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's inequality, we have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

This defines a map $\Phi : L^q \to (L^p)^*$ (recall $(L^p)^*$ is the space of continuous linear maps from L^p to $\mathbb{R}$), where

$$g \mapsto \Phi_g(f) := \int_X fg d\mu \in \mathbb{R}.$$

Here, finiteness as well as continuity follows from Hölder's inequality. In particular, it is bounded (and thus continuous) since $|\Phi_g(f)| \leq \|g\|_q \cdot \|f\|_p$ where $\|g\|_q \in \mathbb{R}$ is a constant. It is easy to see that Φ is injective: indeed, if $g \neq 0$ on a set of positive measure, then $\Phi_g(f) \neq 0$ for $f = \chi_{\{g \neq 0\}}$.

Proposition 4.3.13. *The operator norm* $\|\Phi_g\| = \|g\|_q$ *for* $1 < q < \infty$.

Theorem 4.3.14. *If* $(X, \mathcal{M}, \mu)$ *is* σ*-finite and* $1 \leq p < \infty$*, then* $L^q \xrightarrow{\Phi} (L^p)^*$ *is bijective.*

Corollary 4.3.14.1. *For* $1 \leq p < \infty$*, the dual space of* L^p *is isometrically isomorphic to* L^q *via the map* $g \mapsto \Phi_g$ *given above. Moreover, if* $1 < p < \infty$*, then* $L^p \cong ((L^p)^*)^*$ *and the map is canonical. In other words,* L^p *is reflexive for* $1 < p < \infty$*.*

Corollary 4.3.14.2. *The space* $L^2(X)$ *is a Hilbert space under the inner product* $\langle f, g \rangle = \int_X f\bar{g}$*.*

We have two important examples:

1. $X \subset \mathbb{R}$ with the Lebesgue measure. Notice that the subspaces $C^p(X)$ of continuous L^p functions on X are not closed in $L^p(X)$ for all $1 \leq p < \infty$. However, $(C^\infty(X), \|.\|_\infty)$ (the space of all bounded continuous functions) is a closed subspace of $L^\infty(X)$, and thus is Banach.

2. $X = \mathbb{N}$ equipped with the counting measure; this is clearly σ-finite. In this case, real (resp. complex valued) measurable functions are simply sequences $a = \{a_n\}$ of real (resp. complex) numbers. The integral is clearly $\int a := \sum_{n=1}^\infty a_n$. In such a case, we denote $L^p(\mathbb{N})$ for $1 \leq p \leq \infty$ by ℓ_p. These are sequence spaces. We always have $\ell_p \subset \ell_q$ for $1 \leq p \leq q \leq \infty$. Each ℓ_p for $1 \leq p < \infty$ has a Schauder basis given by $\{\mathbf{e}^{(n)}\}$, where $\mathbf{e}^{(n)}$ is the sequence whose only non-zero term is a 1 in the n'th position. In particular, ℓ_p is separable for $1 \leq p < \infty$. However, ℓ_∞ is not separable.

   There are other distinguished sequence spaces. Denote by c_0 the space of all sequences that converge to 0; we have $\ell_p \subset c_0 \subset \ell_\infty$ for all $1 \leq p < \infty$. Denote by c_{00} the space of all sequences that have only finitely many non-zero terms; we have $c_{00} \subset \ell_p$ for all $p \geq 1$.

   By the above theorem, we have $(\ell_p)^* \cong \ell_q$ for all $1 \leq p < \infty$.

Example 4.3.15 (Spring 2020 Day 1)**.** We study the linear functional $\varphi(f) = \int_0^1 t^{-1/2} f(t)dt$, $\varphi : L^p([0,1]) \to \mathbb{R}$. In particular, we find those p for which φ is a well-defined linear functional, and for such p we evaluate its norm.

Now, as $(L^p)^*$ is isometrically isomorphic to L^q for $1 \leq p < \infty$ (q the conjugate exponent of p), it follows that φ is a well-defined linear functional iff there exists $g \in L^q([0,1])$ such that

$$\int_0^1 t^{-1/2} f(t)dt = \varphi(f) = \int_0^1 f(t)\overline{g(t)}dt.$$

In particular, we require that $\int_0^1 f(t)\overline{(t^{-1/2} - g(t))}dt = 0$ for all $f \in L^p([0,1])$. This can occur iff $t^{-1/2} = g(t)$ almost everywhere on $[0,1]$. Since $g \in L^q([0,1])$, it follows that φ is a linear functional iff $t^{-1/2} \in L^q([0,1])$. Notice however that

$$\int_0^1 |t^{-1/2}|^q dt = \int_0^1 t^{-q/2} dt = \begin{cases} \infty & q \geq 2, \\ \frac{2}{2-q} & 1 \leq q < 2. \end{cases}$$

It follows that φ is a well-defined bounded linear functional iff $p > 2$. Now, the isometric isomorphism between $(L^p)^*$ and L^q implies that

$$\|\varphi\| = \|t^{-1/2}\|_q = (\frac{2}{2-q})^{1/q} = (\frac{2p-2}{p-2})^{1-1/p}.$$

Example 4.3.16 (Fall 2020 Day 2)**.** Suppose $\{f_n\} \to f$ converges point-wise everywhere on $X = (0,1)$, where $f_n \in L^2(X)$ satisfy $\sup_n \|f_n\|_2 \leq M$ for some fixed $M > 0$. Prove or disprove: (1) $f \in L^2(X)$?; (2) $f_n \to f$ in L^2?; (3) $f_n \to f$ in L^p for all $1 < p < 2$; (4) under the extra assumption that $\|f_n\|_2 \to \|f\|_2$ as $n \to \infty$, does $f_n \to f$ in L^2?

1. We show that $f \in L^2$. Indeed, by Fatou's Lemma, we have

$$\int |f|^2 = \int \lim_{n\to\infty} |f_n|^2 = \int \liminf_{n\to\infty} |f_n|^2 \leq \liminf_{n\to\infty} \int |f_n|^2 = \liminf_{n\to\infty} \|f_n\|_2^2 \leq M^2$$

   and hence $f \in L^2$ with $\|f\|_2 \leq M = \sup_n \|f_n\|_2$.

2. This is false. Indeed, let $f_n = \sqrt{n}\chi_{(0,\frac{1}{n})}$. It is easy to see that $\|f_n\|_2 = 1$, that $f_n \to 0$ pointwise on X so that $f = 0$. Thus $\|f_n - f\|_2 = \|f_n\|_2 = 1 \not\to 0$.

3. This is true. Fix $p \in (1,2)$. Let $\delta > 0$ be arbitrary, and set $E_{\delta,n} = \{x \in X : |f_n(x) - f(x)| > \delta\}$. Since $\mu(X) = 1 < \infty$ and $f_n \to f$ pointwise, it follows as a consequence of Egoroff's Theorem that $f_n \to f$ in measure, i.e. for any $\epsilon, \delta > 0$ there exists $N \in \mathbb{N}$ such that $\mu(E_{\delta,n}) < \epsilon$ for all $n \geq N$. By Hölder's Inequality applied to $\frac{2}{p} > 1$, and letting $q > 2$ be the conjugate exponent of $\frac{2}{p}$, we have

$$\int |f_n - f|^p \chi_{E_{\delta,n}} \leq \left(\int (|f_n - f|^p)^{2/p}\right)^{p/2} \mu(E_{\delta,n})^q = \|f_n - f\|_2^p \cdot \epsilon^q \leq (2M)^p \epsilon^q$$

   for all $n \geq N$. Hence $\|(f_n - f)\chi_{E_{\delta,n}}\|_p \leq 2M\epsilon^{q/p} = 2M\epsilon^{2p/(2-p)}$ for all $n \geq N$. On the other hand, we have

$$\|(f_n - f)\chi_{X \setminus E_{\delta,n}}\|_p \leq \delta.$$

   Therefore, for all $\epsilon, \delta > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, we have

$$\|f_n - f\|_p \leq \delta + 2M\epsilon^{2p/(2-p)}.$$

   It follows that $\|f_n - f\|_p \to 0$ as $n \to \infty$ for all $p \in (1,2)$.

4. This is true. We have,

$$\int |f_n - f|^2 = \int f_n^2 + \int f^2 - 2\int f_n f = \|f_n\|_2^2 + \|f\|_2^2 - 2\int f_n f.$$

Consider $E_{\delta,N} = \{x \in X : |f_n(x) - f(x)| > \delta \forall n \geq N\}$; we have $E_{\delta,N} \subseteq E_{\delta,N+1}$, and as above for any $\epsilon, \delta > 0$ there exists $N \in \mathbb{N}$ such that $\mu(E_{\delta,N}) < \epsilon$. Let $F_k = \{x \in X : |f(x)| > k\}$ for each $k \in \mathbb{N}$; clearly $F_k \supseteq F_{k+1}$ and $\bigcap_k F_k = f^{-1}(\infty)$, so that $\mu(F_k) \to 0$ as $k \to \infty$. Then for any $x \in (X \setminus F_k) \cap (X \setminus E_{\delta,N}) = X \setminus (F_k \cup E_{\delta,N})$, we have $|f_n(x)f(x)| \leq |f(x)^2| + |f_n(x) - f(x)||f(x)| \leq k^2 + k\delta$ for all $n \geq N$. Since $k^2 + k\delta \in L^1(X)$, the dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_{X\setminus(F_k \cup E_{\delta,N})} f_n f = \int_X (1 - \chi_{F_k \cup E_{\delta,N}})|f|^2 = \|f\|_2^2 - \|f\chi_{F_k \cup E_{\delta,N}}\|_2^2.$$

On the other hand, Hölder's inequality implies that

$$-\int f_n f \chi_{F_k \cup E_{\delta,N}} \leq \|f_n\|_2 \cdot \|f\chi_{F_k \cup E_{\delta,N}}\|_2$$

for all $n \in \mathbb{N}$. Since $\|f_n\|_2 \to \|f\|_2$, we have

$$-\limsup_{n\to\infty} \int f_n f \chi_{F_k \cup E_{\delta,N}} \leq \|f\|_2 \cdot \|f\chi_{F_k \cup E_{\delta,N}}\|_2.$$

Putting it all together, we have

$$-\limsup_{n\to\infty} 2\int f_n f \leq -2\|f\|_2 + 2\|f\chi_{F_k \cup E_{\delta,N}}\|_2^2 + 2\|f\|_2 \cdot \|f\chi_{F_k \cup E_{\delta,N}}\|_2$$

and thus

$$\limsup_{n\to\infty} \int |f_n - f|^2 \leq 2\|f\chi_{F_k \cup E_{\delta,N}}\|_2^2 + 2\|f\|_2 \cdot \|f\chi_{F_k \cup E_{\delta,N}}\|_2 \leq 4\|f\|_2\|f\chi_{F_k \cup E_{\delta,N}}\|_2$$

for all $k, N \in \mathbb{N}$ and all $\delta > 0$. However, as $\mu(F_k \cup E_{\delta,N})$ goes to 0 as $k \to \infty, N \to \infty$, and $\delta \to 0$, it follows that $\limsup_n \int |f_n - f|^2 = 0$ and hence $f_n \to f$ in L^2.

4.3.2 Classical Theorems on $(C(X), \|.\|_\infty)$

If X is a compact Hausdorff space, then there are many simple dense subsets of $C(X, \mathbb{R})$, given by the Stone-Weierstrass theorem. Note first that $C(X, \mathbb{R})$ is an associative $\mathbb{R}$-algebra.

Theorem 4.3.17 (Stone-Weierstrass)**.** *Suppose X is an arbitrary compact Hausdorff space, and suppose A is a subalgebra of $C(X, \mathbb{R})$ which contains a non-zero constant function. Then, A is dense in $C(X, \mathbb{R})$ under the sup norm iff for all $x, y \in X$ with $x \neq y$, there exists $p \in A$ such that $p(x) \neq p(y)$.*

Corollary 4.3.17.1 (Weierstrass Approximation Theorem)**.** *The space of polynomials $\mathbb{R}[x]$ is dense in $C([a, b], \mathbb{R})$ under the sup norm. Equivalently, for any continuous function $f : [a, b] \to \mathbb{R}$, there exists a sequence of polynomials p_n such that $\|f - p_n\|_\infty \to 0$ as $n \to \infty$.*

Example 4.3.18 (Fall 2021 Day 1)**.** Suppose $f : [-1, 1] \to \mathbb{R}$ is a continuous function so that $\int_{-1}^1 x^{2n} f(x)dx = 0$ for all $n \geq 0$. Prove that f is odd.

Set $g(x) = f(x) + f(-x)$. Then g is an even function. In particular, $\int_{-1}^1 x^{2n+1} g(x)dx = 0$ trivially as g is even. On the other hand, the given condition on f implies also that $\int_{-1}^1 x^{2n} g(x)dx = 0$. Hence, by additivity, it follows that $\int_{-1}^1 p(x)g(x)dx = 0$ for all polynomials $p(x)$. Now, by Weierstrass' Approximation Theorem, we can take a sequence of polynomials $\{p_n\}$ such that $p_n \to g$ uniformly on $[-1, 1]$. Then, it follows that

$$\int_{-1}^1 g(x)^2 dx = \lim_{n\to\infty} \int_{-1}^1 p_n(x)g(x)dx = 0.$$

Hence, g is identically zero, and so f is odd.

There is a special characterization of compact subsets of the Banach space $(C(X), \|.\|_\infty)$ for X a compact Hausdorff topological space. Recall that in any metric space, a subset has compact closure (i.e. is *pre-compact*) iff for every sequence in the subset, there exists a subsequence that converges in the metric space (the limit is not necessarily in the subset).

Theorem 4.3.19 (Arzela-Ascoli)**.** *Suppose X is compact Hausdorff and the space of continuous functions $C(X)$ is equipped with the $\|.\|_\infty$ norm. A subset $B \subset C(X)$ is pre-compact iff*

- *B is uniformly bounded, i.e. $\sup_{f\in B} \|f\|_\infty < \infty$; and*
- *B is* equi-continuous*, i.e. for all $\epsilon > 0$ and for all $x \in X$, there exists an open neighbourhood U_x of x such that for any $y \in U_x$ and for any $f \in B$, we have $|f(x) - f(y)| < \epsilon$.*

If we take $X = [a, b]$, then Arzela-Ascoli states that for any $B \subset C([a, b])$ that is uniformly bounded and equicontinuous (i.e. for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in B$ for all $|x - y| < \delta$), every sequence of continuous functions $\{f_n\} \subset B$ has a subsequence $\{f_{n_k}\}$ that converges uniformly.

Example 4.3.20. We show that $T : (C([0, 1]), \|.\|_\infty) \to (C([0, 1]), \|.\|_\infty)$, $Tf(x) := \int_0^x f(t)dt$, is a compact operator.

Consider the open unit ball $B \subset C([0, 1])$ at the origin, i.e. $B = \{f \in C([0, 1]) : \|f\|_\infty \le 1\}$. Then we have

$$\|Tf\|_\infty = \sup_{x\in[0,1]} \left|\int_0^x f\right| \le \sup_{x\in[0,1]} x\|f\|_\infty \le 1$$

and so the set $T(B)$ is uniformly bounded in $C([0, 1])$. Now suppose $\epsilon > 0$ is arbitrary, and pick $\delta = \epsilon > 0$. Then, for all $x, y \in [0, 1]$ with $|x - y| < \delta$ and for all $f \in B$, we have (WLOG $x \le y$)

$$|Tf(y) - Tf(x)| = \left|\int_x^y f(t)dt\right| \le \int_x^y |f(t)|dt \le \|f\|_\infty \cdot |x - y| < 1 \cdot \delta = \epsilon.$$

Hence $T(B)$ is also equi-continuous, and so Arzela-Ascoli implies that $\overline{T(B)}$ is a compact subset of $C([0, 1])$. Therefore T is a compact operator.

4.4 PDEs and Fourier Analysis

We now consider functions integrable over $\mathbb{R}^n$. Denote $C^k(U)$ to be the space of all functions on U whose partial derivatives of order $\le k$ all exist and are continuous. Let $C^\infty(U)$ be the smooth functions on U. Let $C_c^\infty(U)$ be the space of all smooth functions with compact support contained inside U. A multi-index α is an ordered n-tuple $(\alpha_1, ..., \alpha_n)$ of positive integers, and we define $|\alpha| = \sum_j \alpha_j$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. We define $C_0(X)$ to be the space of continuous functions such that for any $\epsilon > 0$, the set $\{x \in \mathbb{R}^n : |f(x)| \ge \epsilon\}$ is compact. The *Schwartz space* $\mathcal{S}$ is the subspace of smooth functions that decay faster than any polynomial as $\|x\| \to \infty$:

$$\mathcal{S} = \left\{ f \in C^\infty : \sup_{x\in\mathbb{R}^n} \left((1 + \|x\|)^N |\partial^\alpha f(x)|\right) < \infty \ \forall N \in \mathbb{Z}_{\ge 0} \forall \text{ multi-indices } \alpha \right\}.$$

If $f \in C^\infty$, then $f \in \mathcal{S}$ iff $x^\beta \partial^\alpha f$ is bounded for all multi-indices α, β. In particular, the Schwartz space is closed under differentiation and under multiplication by polynomials.

Proposition 4.4.1. *Both C_c^∞ and $\mathcal{S}$ are dense in L^p $(1 \le p < \infty)$ and in C_0.*

We can consider functions on the n-dimensional torus T^n as simply multi-periodic functions on $\mathbb{R}^n$. We can normalize so that the period in every component is 1. Thus, for example, $f \in L^p(T^n)$ can be thought of as a function $f \in L^p(\mathbb{R}^n)$ such that $f(x + w) = f(x)$ for all $w \in \mathbb{Z}^n$. We can also think of T^n as the subset $[0, 1]^n$ of $\mathbb{R}^n$.

4.4.1 Convolution

Definition. Suppose f, g are measurable functions on $\mathbb{R}^n$. Their *convolution* is the measurable function $(f * g)(x) := \int_X f(x - y)g(y)dy$ (whenever this integral exists).

Proposition 4.4.2 (Young's Inequality)**.** *Suppose $1 \le p, q, r \le \infty$ with $p^{-1} + q^{-1} = r^{-1} + 1$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r$ and $\|f * g\|_r \le \|f\|_p \|g\|_q$.*

In particular, if p and q are conjugate exponents with $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g$ exists for all $x \in \mathbb{R}^n$, $f * g$ is bounded and uniformly continuous, and $\|f * g\|_\infty \le \|f\|_p \|g\|_q$. If $1 < p, q < \infty$, then $f * g \in C_0(\mathbb{R}^n)$.

Theorem 4.4.3. *$L^1(\mathbb{R}^n)$ is a Banach-algebra under convolution and the L^1-norm. Specifically, convolution of two L^1 functions is also L^1, convolution is a commutative and associative binary operation on L^1 which distributes over addition, so that L^1 is a $\mathbb{C}$-algebra under addition and convolution. Moreover, we have $\|f*g\|_1 \leq \|f\|_1\|g\|_1$.*

Proposition 4.4.4. *If $f \in L^1(\mathbb{R}^n), g \in C^k$, and $\partial^\alpha g$ is bounded on $\mathbb{R}^n$ for all $|\alpha| \leq k$, then $f * g \in C^k$ and $\partial^\alpha(f * g) = f * (\partial^\alpha g)$ for $|\alpha| \leq k$.*

Moreover, if $f, g \in \mathcal{S}$, then $f * g \in \mathcal{S}$.

The above results remain true if we replace $\mathbb{R}^n$ with T^n.

4.4.2 Fourier Series

For each $\kappa \in \mathbb{Z}^n$, define $E_\kappa(x) = \exp(2\pi i\kappa \cdot x)$. Then $\{E_\kappa : \kappa \in \mathbb{Z}^n\}$ is an orthonormal basis for the Hilbert space $L^2(T^n)$.

Definition. The *(semi-continuous) Fourier transform* of $f \in L^1(T^n)$ is the function $\hat{f} : \mathbb{Z}^n \to \mathbb{C}$ given by

$$\hat{f}(\kappa) := \langle f, E_\kappa \rangle = \int_{[0,1]^n} f(x)e^{-2\pi i\kappa\cdot x}dx.$$

The *Fourier series* of f is the (*a priori* formal series)

$$\sum_{\kappa\in\mathbb{Z}^n} \hat{f}(\kappa)e^{2\pi i\kappa\cdot x}.$$

We have various results on convergence:

1. The semi-continuous Fourier transform is a map from $L^1(T^n) \to \ell^\infty(\mathbb{Z}^n)$, and moreover $\|\hat{f}\|_\infty \leq \|f\|_1$.
2. The semi-continuous Fourier transform yields a unitary isomorphism from $L^2(T^n)$ to $\ell^2(\mathbb{Z}^n)$.
3. Due to the Riesz-Fisher Theorem, we know that $\hat{f}(\kappa)$ is the coefficient of E_κ. Hence, the Fourier series of $f \in L^2$ converges in the L^2-norm to f. Moreover, Parseval's Identity implies that
$$\int_{T^n} |f(x)|^2dx = \|f\|_2^2 = \|\hat{f}\|_2^2 = \sum_{\kappa\in\mathbb{Z}^n} |\hat{f}(\kappa)|^2.$$
4. (Hausdorff-Young Inequality) Suppose $1 \leq p \leq 2$ and q the conjugate exponent of p. If $f \in L^p(T^n)$ then $\hat{f} \in \ell^q(\mathbb{Z}^n)$ and $\|\hat{f}\|_q \leq \|f\|_p$.
5. If $f \in L^1(T^n)$ and $\hat{f} \in \ell^1(\mathbb{Z}^n)$, then $f \in L^2(T^n)$ and the Fourier series converges absolutely uniformly to a function g such that $g = f$ almost everywhere. The Fourier series also converges in the L^2-norm to f.

Now suppose that $n = 1$, i.e. we have a function on $S^1 = T^1$. Set $S_N(f)(x) = \sum_{n=-N}^N \hat{f}(n)e^{2\pi inx}$; clearly $S_N(f) \in C^\infty$. The number $\hat{f}(n)$ is called the n'th Fourier coefficient of f.

1. (*Riemann-Lebesgue Lemma*) If $f \in L^1(S^1)$, then $\hat{f}(n) \to 0$ as $|n| \to \infty$.
2. If f is twice continuously differentiable, then $\hat{f}(n) = O(|n|^{-2})$ as $|n| \to \infty$, and the Fourier series of f converges absolutely and uniformly to f.
3. If $f \in L^1(S^1)$ is differentiable at the point $t \in S^1$, then $S_N(f)(t) \to f(t)$ as $N \to \infty$.

Theorem 4.4.5. *If $f, g \in L^2(S^1)$, then the n'th Fourier coefficient of $f * g$ is $\hat{f}(n)\hat{g}(n)$.*

Definition. The k'th *Sobolev Space* $H^k(X)$ is the space consisting of those functions $f \in L^2(X)$ such that

$$\sum_n (1 + n^2 + n^4 + \cdots + n^{2k})|\hat{f}(n)|^2 < \infty.$$

Equivalently, it is the completion of the space of functions in $L^2(X)$ that have all derivatives up to k, and such that $f^{(i)} \in L^2$ for all $1 \leq i \leq k$. $H^k(X)$ equipped with the inner product $\langle u, v\rangle_{H^k} = \sum_{i=0}^k \left\langle u^{(i)}, v^{(i)}\right\rangle_{L^2}$ becomes a Hilbert space. In fact, $H^0(X) = L^2(X)$.

Example 4.4.6 (Fall 2019 Day 3)**.** Let $f \in H^1([0,1])$. Show that $\lim_{n\to\infty}\left(n\int_0^1 f(x)e^{-2\pi inx}dx\right) = 0$.

Since we only care about integrating f, we may suppose WLOG that $f(0) = f(1)$. Then $f \in H^1(S^1)$. It follows that both $\hat{f}$ and $\hat{f'}$ are well-defined, and by above we see that

$$\hat{f'}(n) = 2\pi in\hat{f}(n) = 2\pi in\int_0^1 f(x)e^{-2\pi inx}dx.$$

Since $\hat{f'}(n) \to 0$ as $n \to \infty$, the result follows.

Definition. Let $D_N(x) = \sum_{n=-N}^{N} e^{2\pi inx}$ be the N'th *Dirichlet kernel*; explicitly we have

$$D_N(x) = \frac{\sin(2N+1)\pi x}{\sin \pi x}.$$

Theorem 4.4.7. *If f is 1-periodic on $\mathbb{R}$ and of bounded variation in $[-\frac{1}{2}, \frac{1}{2}]$, then*

$$\lim_{N\to\infty}(S_N f)(x) = \frac{1}{2}\left(\lim_{y\to x^+} f(y)\right) + \frac{1}{2}\left(\lim_{y\to x^-} f(y)\right)$$

for all x. In particular, $S_N f(x) \to f(x)$ as $N \to \infty$ whenever f is continuous at x.

Theorem 4.4.8. *If $f, g \in L^1(S^1)$ and $f = g$ on some open interval I, then $S_m f - S_m g \to 0$ uniformly on compact subsets of I.*

Corollary 4.4.8.1. *Suppose $f \in L^1(S^1)$ and I an open interval with length ≤ 1.*

- *If f agrees on I with a function g such that $\hat{g} \in \ell^1(\mathbb{Z})$, then $S_n f \to f$ uniformly on compact subsets of I.*
- *If f is absolutely continuous on I and $f' \in L^p(I)$ for some $p > 1$, then $S_m f \to f$ uniformly on compact subsets of I.*

4.4.3 Continuous Fourier Transform

Definition. The *Fourier transform* of $f \in L^1(\mathbb{R}^n)$ is

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot x}dx$$

for all $\xi \in \mathbb{R}^n$. We have $\|\hat{f}\|_\infty \leq \|f\|_1$ and $\hat{f}$ is always continuous. Here, $\xi \cdot x$ is the usual inner product on $\mathbb{R}^n$.
We can use either $\mathcal{F}$ or $\hat{\bullet}$ to denote the Fourier transform.

We have the following properties if $f, g \in L^1(\mathbb{R}^n)$.

1. (*Riemann-Lebesgue Lemma*) $\mathcal{F}(L^1(\mathbb{R}^n)) \subseteq C_0(\mathbb{R}^n)$.
2. $\mathcal{F}(f * g) = \hat{f} \cdot \hat{g}$.
3. If $x^\alpha f \in L^1$ for $|\alpha| \leq k$, then $\hat{f} \in C^k$ and $\partial^\alpha(\hat{f}) = \mathcal{F}[(-2\pi i)^{|\alpha|}x^\alpha f]$.
4. If $f \in C^k$, $\partial^\alpha f \in L^1$ for all $|\alpha| \leq k$, and if $\partial^\alpha f \in C_0$ for $|\alpha| \leq k-1$, then $\mathcal{F}(\partial^\alpha f)(\xi) = (2\pi i)^{|\alpha|}\xi^\alpha \hat{f}(\xi)$ for all $\xi \in \mathbb{R}^n$.
5. $\mathcal{F}$ maps $\mathcal{S}$ to $\mathcal{S}$.
6. If $f, g \in L^1$, then $\int_{\mathbb{R}^n} \hat{f}g = \int_{\mathbb{R}^n} f\hat{g}$.
7. For any $y \in \mathbb{R}^n$, define $\tau_y(f)$ to be the (measurable) function $(\tau_y f)(x) = f(x - y)$. Clearly τ_y maps L^p to L^p, C^p to C^p, C_0 to C_0, and so on. Then, $\mathcal{F}(\tau_y f)(\xi) = e^{-2\pi i\xi\cdot y}\hat{f}(\xi)$ for any $y \in \mathbb{R}^n$. Also, $\tau_\eta(\hat{f}) = \hat{h}$ where $h(x) := e^{2\pi i\eta\cdot x}f(x)$, for any $\eta \in \mathbb{R}^n$.

Example 4.4.9 (Fall 2015)**.** Show that $f \in C^\infty \cap L^1(X)$ iff $\mathcal{F}(f)$ decays to 0 faster than one over any polynomial. In particular, this gives a proof of the fact that $\mathcal{F}$ preserves $\mathcal{S}$.

Definition. The *inverse Fourier transform* of $f \in L^1$, often denoted by $f^\vee$, is given by

$$f^\vee(x) := \int_{\mathbb{R}^n} f(\xi)e^{2\pi i\xi\cdot x}d\xi = \hat{f}(-x).$$

Theorem 4.4.10 (Fourier Inversion Theorem)**.** *If $f \in L^1$ and $\hat{f} \in L^1$, then f agrees almost everywhere with a continuous function f_0, and $(\hat{f})^\vee = \mathcal{F}(f^\vee) = f_0$.*

Corollary 4.4.10.1. *If $f \in L^1$ with $\hat{f} = 0$, then $f = 0$ almost everywhere.*

Corollary 4.4.10.2. *$\mathcal{F}$ is a linear isomorphism of $\mathcal{S}$ onto itself.*

Proposition 4.4.11. *If $f, g \in L^2(\mathbb{R}^n)$, then $(\hat{f}\hat{g})^\vee = f * g$.*

Theorem 4.4.12 (Plancherel Theorem)**.** *If $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$ and $\mathcal{F}|_{L^1\cap L^2}$ extends uniquely to a unitary isomorphism on L^2.*

More precisely, $\mathcal{F}$ is well-defined on the dense subspace $L^1 \cap L^2$ and satisfies $\left\langle \hat{f}, \hat{g} \right\rangle_2 = \langle f, g\rangle_2$. By continuity one can then extend $\mathcal{F}$ to L^2 uniquely. In particular, $\|\mathcal{F}(f)\|_2 = \|f\|_2$.

Theorem 4.4.13 (Hausdorff-Young Inequality)**.** *Suppose $1 \le p \le 2$ and q conjugate to p. If $f \in L^p(\mathbb{R}^n)$, then $\hat{f} \in L^q(\mathbb{R}^n)$ and $\|\hat{f}\|_q \le \|f\|_p$.*

Theorem 4.4.14. *Suppose $f \in L^1(\mathbb{R}^n)$. Then, the series $\sum_{\kappa\in\mathbb{Z}^n} \tau_\kappa f$ converges pointwise almost everywhere as well as in $L^1(T^n)$ to a function $Pf \in L^1(T^n)$ such that $\|Pf\|_1 \le \|f\|_1$. Moreover, for any $\kappa \in \mathbb{Z}^n$, the $\hat{P}f(\kappa)$ (semi-continuous Fourier transform) equals $\hat{f}(\kappa)$ (continuous Fourier transform).*

Theorem 4.4.15 (Poisson Summation Formula)**.** *If $f \in C(\mathbb{R}^n)$ satisfies $|f(x)| \le C(1+|x|)^{-n-\epsilon}$ and $|\hat{f}(\xi)| \le C(1+|\xi|)^{-n-\epsilon}$ for some $C, \epsilon > 0$, then*

$$\sum_{\kappa\in\mathbb{Z}^n} f(x+\kappa) = \sum_{\kappa\in\mathbb{Z}^n} \hat{f}(\kappa) e^{2\pi i\kappa\cdot x}$$

where both series converge absolutely and uniformly on T^n.

4.4.4 Solving PDEs

Suppose L is a differential operator

$$L = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha$$

where the $a_\alpha \in C^\infty$. If f is sufficiently well-behaved, say $f \in \mathcal{S}$, and if all of the a_α are constants, then

$$\mathcal{F}(Lf)(\xi) = \sum_{|\alpha|\le m} a_\alpha (2\pi i)^{|\alpha|} \xi^\alpha \hat{f}(\xi).$$

Some more precise results on this are given:

Proposition 4.4.16. *If $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$, then $\mathcal{F}(f') = (2\pi i k)\mathcal{F}(f)$.*

Using the Fourier inversion formula then allows us to solve the PDE.

Example 4.4.17 (Laplace Equation on the Half-Plane)**.** Consider a function f on $\mathbb{R}^n$. Suppose we want to find u on $\mathbb{R}^n \times [0,\infty)$ such that $u(x,0) = f(x)$ and $(\triangle + \partial_t^2)u = 0$ (where $\triangle = \sum_{i=1}^n \partial_i^2$ is the Laplacian on $\mathbb{R}^n$). By applying the Fourier transform on $\mathbb{R}^n$, the equation $(\triangle + \partial_t^2)u = 0$ becomes $(-4\pi^2|\xi|^2 + \partial_t^2)\hat{u} = 0$. This is a ODE in t for fixed ξ, with solution

$$\hat{u}(\xi, t) = c_1(\xi)e^{-2\pi t|\xi|} + c_2(\xi)e^{2\pi t|\xi|}.$$

The initial condition is $\hat{u}(\xi, 0) = \hat{f}(\xi)$, so that we want $c_1 + c_2 = \hat{f}$. If we take $c_1 = \hat{f}, c_2 = 0$, then $\hat{u}(\xi,t) = \hat{f}(\xi)e^{-2\pi t|\xi|}$. Taking inverse Fourier transform, we get that $u(x,t) = (f * P_t)(x)$ where $P_t = (e^{-2\pi t|\xi|})^\vee$.

The above calculation is formal since nothing has been justified by conditions on f. A precise result is as follows:

Proposition 4.4.18. *Suppose $f \in L^p(\mathbb{R}^n)$ for some $1 \le p \le \infty$. Then, the function $u = f * P_t$ satisfies $(\triangle + \partial_t^2)u = 0$ on $\mathbb{R}^n \times (0,\infty)$, and moreover $\lim_{t\to 0} u(x,t) = f(x)$ for almost all x, and in particular for every x at which f is continuous. Moreover, if $p < \infty$, then $\|u(\bullet, t) - f\|_p \to 0$ as $t \to 0$.*

Example 4.4.19 (Spring 2020 Day 2)**.** Let $g \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with Fourier transform

$$\hat{g}(k) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ik\cdot x} g(x) dx.$$

For $m > 0$, define $f : \mathbb{R}^3 \to \mathbb{C}$ by

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ik\cdot x} \frac{\hat{g}(k)}{|k|^2 + m^2} dx.$$

We show that f solves the PDE $-\Delta f + m^2 f = g$ in the distributional sense, i.e. for any text function $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have $\langle -\Delta f + m^2 f, \varphi \rangle = \langle g, \varphi \rangle$.

Note that f is the inverse Fourier transform to $\frac{\hat{g}(k)}{|k|^2+m^2}$. By Plancherel's Theorem, $\hat{g} \in L^2$ and moreover the Fourier transform is an isometry onto f. Since $\frac{1}{|k|^2+m^2}$ is in L^1 and is uniformly bounded on $\mathbb{R}^3$ for $m > 0$, we see that $\frac{\hat{g}(k)}{|k|^2+m^2} \in L^2$. Hence, $\hat{f} = \frac{\hat{g}(k)}{|k|^2+m^2}$ by the Fourier inversion formula. Notice that $\mathcal{F}(\Delta f) = -|k|^2 \hat{f}$. Now, the Fourier transform being an isometry on L^2 implies that

$$\begin{aligned} \langle -\Delta f + m^2 f, \varphi \rangle &= \langle \mathcal{F}(-\Delta f + m^2 f), \hat{\varphi} \rangle = \left\langle |k|^2 \hat{f} + m^2 \hat{f}, \hat{\varphi} \right\rangle \\ &= \langle \hat{g}, \hat{\varphi} \rangle = \langle g, \varphi \rangle . \end{aligned}$$

Therefore f solves $-\Delta f + m^2 f = g$ in the distributional sense.

4.5 Probability Theory

4.5.1 Definitions

Definition. A *probability measure* is a positive measure μ on $(X, \mathcal{M})$ such that $\mu(X) = 1$.

We have the following dictionary between measure theory and probability theory.

Measure Theory	Probability Theory
Measure space $(X, \mathcal{M}, \mu)$	Sample space $(\Omega, \mathcal{B}, P)$
Measurable set	Event
Measurable Real-valued function	Random variable X
Integral of f, $\int_X f d\mu$	Expectation or mean of X, denoted $\mathbb{E}(X)$
Convergence in measure	convergence in probability
Almost everywhere	almost surely
Borel probability measure on $\mathbb{R}$	Distribution
L^p	Having finite p'th moment
Characteristic Function	Indicator Function

Definition. The *variance* $\sigma^2(X)$ and the *standard deviation* $\sigma(X)$ is given by

$$\sigma^2(X) := \inf_{a\in\mathbb{R}} \mathbb{E}[(X-a)^2] \quad \text{and} \quad \sigma(X) = \sqrt{\sigma^2(X)}.$$

If $X \notin L^2$, then $\sigma^2(X) = \infty$. Otherwise,

$$\sigma^2(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}(X^2) - \mathbb{E}(X)^2.$$

Definition. Suppose $\phi : (\Omega, \mathcal{B}) \to (\Omega', \mathcal{B}')$ is a measurable map between two measurable spaces, and P is a probability measure on Ω. The *push-forward measure* $\phi_* P$ is a probability measure $\phi_* P$ on Ω' such that

$$(\phi_* P)(E) := P(\phi^{-1}(E)).$$

If $f : \Omega' \to \mathbb{R}$ is a measurable function, then $\int_{\Omega'} f d(\phi_* P) = \int_\Omega (f \circ \phi) dP$.

Definition. Suppose X is a random variable on the probability space $(\Omega, \mathcal{B}, P)$. The probability measure P_X on $\mathbb{R}$ induced by the push-forward measure of P by X is called the *distribution of* X, and the function $F_X(t) := P_X((-\infty, t]) = P(X \leq t)$ is called the *distribution function* of X.

A *distribution on* $\mathbb{R}$ is any Borel probability measure on $\mathbb{R}$.

The *density function* f_X is the Radon-Nikodym derivative of the probability measure P_X on $\mathbb{R}^n$ with respect to the Lebesgue measure. In other words, if X takes values in $\mathbb{R}^n$ and λ_n is the Lebesgue measure on $\mathbb{R}^n$, then f_X is the unique (almost everywhere) function such that $dP_X = f_X d\lambda_n$.

If $\{X_\alpha\}$ is a family of random variables such that $P_{X_\alpha} = P_{X_\beta}$ for all α, β, then the X_αs are *identically distributed.*

If $X_1, ..., X_n$ is a finite sequence of random variables, then the map $(X_1, ..., X_n) : \Omega \to \mathbb{R}^n$ induces the measure $P_{X_1,...,X_n}$ on $\mathbb{R}^n$, called the *joint distribution of* $X_1, ..., X_n$.

In terms of the distribution P_X of X, we have

$$\mathbb{E}(X) = \int_{\mathbb{R}} t dP_X(t), \quad \sigma^2(X) = \int_{\mathbb{R}} (t - E[X])^2 dP_X(t), \quad \text{and} \quad \mathbb{E}(X+Y) = \int_{\mathbb{R}^2} (t+s) dP_{X,Y}(t,s).$$

Definition. A collection $\{E_\alpha\}_{\alpha \in A}$ of events in Ω is *independent* if

$$P(E_{\alpha_1} \cap \cdots \cap E_{\alpha_k}) = \prod_{i=1}^{k} P(E_{\alpha_i})$$

for all possible finite subsets $\{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_k}\} \subset A$ of size k for all $k \in \mathbb{N}$.

A collection $\{X_\alpha\}_{\alpha \in A}$ of random variables in Ω is *independent* if for any collection $\{B_\alpha\}$ of Borel sets in $\mathbb{R}$, the collection of events $\{X_\alpha^{-1}(B_\alpha)\}_\alpha$ in Ω is independent. Equivalently, $\{X_\alpha\}$ is independent if for any finite subset $\{X_1, ..., X_n\} \subset \{X_\alpha\}$, the joint probability measure $P_{X_1,...,X_n}$ on $\mathbb{R}^n$ is the product of the individual probability measures P_{X_i} on $\mathbb{R}$.

We have some basic properties:

1. If $\{X_1, ..., X_n\}$ are independent random variables, then

$$P_{X_1+\cdots+X_n}(E) = \int_{\mathbb{R}^n} \chi_E(t_1 + \cdots + t_n) dP_{X_1} \cdots dP_{X_n}.$$

2. If $\{X_{in} : 1 \leq i \leq k_n, 1 \leq n \leq N\}$ are independent random variables, and if $f_n : \mathbb{R}^{k_n} \to \mathbb{R}$ are Borel-measurable functions for $1 \leq n \leq N$, then the N random variables $Y_n := f_n(X_{1n}, ..., X_{k_n n})$ are independent as well.

3. If $X_i \in L^1$ for $1 \leq i \leq n$ are n independent random variables, then $\prod_{j=1}^n X_j \in L^1$ and $\mathbb{E}(\prod_{i=1}^n X_i) = \prod_{i=1}^n \mathbb{E}(X_i)$. If moreover the X_i are in L^2, then

$$\sigma^2(X_1 + \cdots + X_n) = \sum_{i=1}^n \sigma^2(X_i).$$

4.5.2 Important Results

Lemma 4.5.1 (Markov's Inequality)**.** *For any non-negative random variable X and any $a > 0$, we have $P(X > a) \leq \frac{1}{a}\mathbb{E}(X)$.*

Proposition 4.5.2 (Jensen's Inequality)**.** *If φ is a convex real-valued function, and X a random variable in L^1, then $\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)]$.*

Proposition 4.5.3 (Chebyshev's Inequality)**.** *If X is a random variable with finite expectation $\mu = \mathbb{E}(X)$ and finite non-zero variance $\sigma^2 = \sigma^2(X)$, then for any $\alpha > 0$ we have $P(|X - \mu| \geq \alpha\sigma) \leq \frac{1}{\alpha^2}$.*

More generally, if X is a random variable that is extended P-integrable, and if g is a non-negative and non-decreasing real-valued measurable function on $\mathbb{R}$, then for any t with $g(t) \neq 0$ we have

$$P(X \geq t) \leq \frac{1}{g(t)} \mathbb{E}[g(X)].$$

Proposition 4.5.4 (Weak Law of Large Numbers)**.** *Suppose $\{X_j\}$ is an infinite sequence of independent L^2 random variables with expectations $\{\mu_j\}$ and variances $\{\sigma_j^2\}$. If $n^{-2}\sum_{j=1}^n \sigma_j^2 \to 0$ as $n \to \infty$, then $n^{-1}\sum_{j=1}^n (X_j - \mu_j) \to 0$ in probability as $n \to \infty$.*

Example 4.5.5 (Fall 2020 Day 3)**.** Suppose X_i are a sequence of random variables (NOT independent) such that $\mathbb{E}[X_i] =: \mu_i$ is finite and such that there exists a function $g : \{0\} \cup \mathbb{N} \to \mathbb{R}$ with $g(k) \to 0$ as $k \to \infty$ such that $\mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)] = g(|j - i|)$. Does the conclusion of the weak law of large numbers still hold, even though these random variables are not independent?

Set $S_n := \frac{1}{n}\sum_{i=1}^n (X_i - \mu_i)$. We want to check whether $S_n \to 0$ in probability as $n \to \infty$. It is clear that $\mathbb{E}[S_n] = 0$ by linearity of expectation. We calculate:

$$\begin{aligned}\sigma^2(S_n) = \mathbb{E}[S_n^2] &= \frac{1}{n^2}\sum_{i,j=1}^n \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)] = \frac{1}{n^2}\sum_{i,j=1}^n g(|j-i|) \\ &= \frac{1}{n^2}\sum_{d=0}^{n-1}(n-d)g(d).\end{aligned}$$

Now, let $\epsilon > 0$ be arbitrary, and let $N \in \mathbb{N}$ be such that $|g(n)| < \epsilon$ for all $n \geq N$. Let $M = \max\{|g(n)| : 1 \leq n < N\}$. Then, for $n \geq N$, we have

$$\begin{aligned}\sigma^2(S_n) &\leq \frac{1}{n^2}\left(\sum_{d=0}^{N-1}(n-d)|g(d)| + \sum_{d=N}^{n-1}(n-d)|g(d)|\right) \leq \frac{1}{n^2}\left(nM + \epsilon\sum_{d=N}^{n-1}(n-d)\right) \\ &= \frac{1}{n^2}\left(nNM + \frac{\epsilon}{2}(n-N)(n-N+1)\right).\end{aligned}$$

It follows that $\limsup_n \sigma^2(S_n) \leq \frac{\epsilon}{2} < \epsilon$ for any $\epsilon > 0$. It follows that $\lim_n \sigma^2(S_n) = 0$. Now, by Chebyshev's inequality, we have

$$0 \leq P\left(|S_n| > \alpha\right) \leq \frac{\sigma^2(S_n)}{\alpha}$$

for any $\alpha > 0$. Taking $n \to \infty$, it then follows that $P(|S_n| > \alpha) \to 0$ for all $\alpha > 0$. Hence $S_n \to 0$ in probability, i.e. the conclusion of the weak law of large numbers still holds.

Proposition 4.5.6 (Borel-Cantelli Lemma)**.** *Suppose $\{A_n\}$ is a sequence of events. Define* $\limsup_n A_n := \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty A_n$.

1. *If $\sum_{n=1}^\infty P(A_n) < \infty$, then $P(\limsup_n A_n) = 0$.*
2. *If the A_ns are independent and $\sum_{n=1}^\infty P(A_n) = \infty$, then $P(\limsup_n A_n) = 1$.*

Proof of Borel-Cantelli Lemma (1). Let $\epsilon > 0$ be arbitrary, and let $k \in \mathbb{N}$ such that $\sum_{n \geq k} P(A_n) < \epsilon$. Then,

$$P\left(\bigcap_{m=1}^\infty \bigcup_{n \geq m} A_n\right) \leq P\left(\bigcup_{n \geq k} A_n\right) \leq \sum_{n \geq k} P(A_n) < \epsilon.$$

As $\epsilon > 0$, it follows that $P(\limsup_n A_n) = 0$. ☐

Proof of Borel-Cantelli Lemma (2); also Spring 2018 Day 2. We first claim that the complements A_n^c are independent. Indeed, notice that for any finite subset $S \subset \mathbb{N}$, we have

$$\begin{aligned}P\left(\bigcap_{n \in S} A_n^c\right) &= 1 - P\left(\bigcup_{n \in S} A_n\right) = 1 - \sum_{T \subseteq S, T \neq \emptyset} (-1)^{|T|-1}\mathbb{P}\left(\bigcap_{n \in T} A_n\right) \\ &= 1 + \sum_{T \subseteq S, T \neq \emptyset} (-1)^{|T|} \prod_{n \in T} \mathbb{P}\left(A_n\right) = \prod_{n \in S}(1 - \mathbb{P}(A_n)) = \prod_{n \in S} P(A_n^c).\end{aligned}$$

The claim thus follows. Now, notice that

$$P\left(\limsup_{n \to \infty} A_n\right) = 1 - P\left(\bigcup_{k \geq 1}\bigcap_{n \geq k} A_n^c\right).$$

Since $\bigcap_{n \geq k} A_n^c$ is an increasing chain of events, it follows that

$$\begin{aligned}1 - P\left(\limsup_{n \to \infty} A_n\right) &= \lim_{k \to \infty} P\left(\bigcap_{n \geq k} A_n^c\right) = \lim_{k \to \infty}\lim_{N \to \infty}\prod_{n=k}^N (1 - P(A_n)) \\ &\leq \lim_{k \to \infty}\lim_{N \to \infty}\prod_{n=k}^N e^{-P(A_n)} = \lim_{k \to \infty}\lim_{N \to \infty}\exp\left(-\sum_{n=k}^N A_n\right) = 0\end{aligned}$$

where we use the standard inequality $1 - x < e^{-x}$, and we use the assumption that $\sum_{n \geq k} A_n = \infty$. ☐

Theorem 4.5.7 (Kolmogorov's Strong Law of Large Numbers)**.** *If $\{X_n\}$ is a sequence of independent L^2 random variables with means μ_n and variances σ_n^2 such that $\sum_{n=1}^{\infty} n^{-2}\sigma_n^2 < \infty$, then $n^{-1}\sum_{i=1}^{n}(X_j - \mu_j) \to 0$ (pointwise) almost surely as $n \to \infty$.*

Theorem 4.5.8 (Khinchine's Strong Law of Large Numbers)**.** *If $\{X_n\}$ is a sequence of independent and identically distributed L^1 random variables with means μ, then $n^{-1}\sum_{i=1}^{n} X_i \to \mu$ almost surely as $n \to \infty$.*

Theorem 4.5.9 (Portmanteau Theorem)**.** *Suppose P_n, P are probability measures on $\mathbb{R}$ with corresponding cumulative distribution functions F_n, F (resp.). Let $\mathbb{E}_n$ (resp $\mathbb{E}$) be the expectation with respect to P_n (resp. P). Then, the following are equivalent:*

1. $\lim_{n\to\infty} \mathbb{E}_n[f] = \mathbb{E}[f]$ *for all bounded continuous functions f on $\mathbb{R}$;*
2. $\lim_{n\to\infty} \mathbb{E}_n[f] = \mathbb{E}[f]$ *for all bounded Lipschitz continuous functions f on $\mathbb{R}$;*
3. $\limsup_{n\to\infty} P_n(C) \leq P(C)$ *for all closed subsets $C \subseteq \mathbb{R}$;*
4. $\liminf_{n\to\infty} P_n(U) \geq P(U)$ *for all open subsets $U \subseteq \mathbb{R}$;*
5. $\lim_{n\to\infty} F_n(x) = F(x)$ *for all $x \in \mathbb{R}$ such that F is continuous at x.*

If any of these hold, then we say that the probability measures P_n converge weakly *to the probability measure P.*

Definition. A sequence of random variables X_n on a probability space (Ω, P) is said to *converge weakly in distribution*, or *converge weakly in law*, to a random variable X on (Ω, P) if the sequence of push-forward measures $(X_n)_*(P)$ converges weakly to $X_*(P)$.

Example 4.5.10 (Fall 2018 Day 3)**.** Suppose W is Gumbel distributed, that is $P(W \leq x) = e^{-e^{-x}}$. Let X_i be independent and identically distributed exponential random variables with mean 1. Set $M_n = \max_{1\leq i\leq n} X_i$. Find sequences $\{a_n\}$ and $\{b_n\}$ such that $\frac{1}{a_n}(M_n - b_n)$ converges weakly in law to W, i.e. $\mathbb{E}[F(\frac{M_n - b_n}{a_n})] \to \mathbb{E}[F(W)]$ for all bounded continuous functions F.

Take $a_n = 1$ and $b_n = \log n$. Then, notice that $P\left(\frac{M_n - b_n}{a_n} \leq x\right) = 0$ for any $x \leq -b_n$, while for $x > -b_n$ we have

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right) = P(X_i \leq x + \log n, \forall 1 \leq i \leq n) = P(X_1 \leq x + \log n)^n = \left(1 - e^{-x-\log n}\right)^n = \left(1 - \frac{e^{-x}}{n}\right)^n.$$

Notice that $P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \left(1 - \frac{e^{-x}}{n}\right)^n \chi_{\{x \geq -\log n\}} \to e^{e^{-x}} = P(W \leq x)$ as $n \to \infty$. By the Portmanteau Theorem, it then follows that $\frac{1}{a_n}(M_n - b_n)$ converges weakly in law to W.

Alternatively, notice that the above convergence of cumulative distribution functions is uniform in x on compact subsets of $\mathbb{R}$, and so the density functions $f_n(x) = e^{-x}(1 - \frac{1}{n}e^{-x})^{n-1}\chi_{\{x\geq -\log n\}}$ of $\frac{M_n - b_n}{a_n}$ converge pointwise to the density function $f(x) = e^{-(x+e^{-x})}$ of W. Notice that $(1 - y) < e^{-y}$ for all $y \in (0, 1)$, and so $f_n(x) \leq e^{-x}(e^{-\frac{1}{n}e^{-x}})^n = e^{-x-e^{-x}} = f(x)$, where f is integrable. Hence $|f_n(x) - f(x)| \leq 2f(x)$, and so for any continuous bounded function F on $\mathbb{R}$ we have

$$\left|\mathbb{E}\left[F\left(\tfrac{M_n - b_n}{a_n}\right)\right] - \mathbb{E}[F(W)]\right| \leq \|F\|_\infty \int_{\mathbb{R}} |f_n(x) - f(x)| dx \to 0$$

by the dominated convergence theorem (noticing that $|f_n(x) - f(x)| \to 0$ pointwise).

Definition. Let $\mu \in \mathbb{R}$ and $\sigma > 0$. The *normal distribution with mean μ and variance σ* is the Borel probability measure on $\mathbb{R}$ given by

$$\nu_\mu^{\sigma^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{(t-\mu)^2/2\sigma} dt$$

where dt is the Lebesgue measure on $\mathbb{R}$. It has mean μ and variance σ^2. The distribution ν_0^1 is the *standard normal distribution.*

Theorem 4.5.11 (Central Limit Theorem)**.** *Suppose X_j is a sequence of independent identically distributed random variables with mean μ and variance σ^2. As $n \to \infty$, the sequence of random variables $Y_n := (\sigma\sqrt{n})^{-1}\sum_{i=1}^{n}(X_i - \mu)$ converges* vaguely *to ν_0^1, in the sense that we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f dP_{Y_n} = \int_{\mathbb{R}} f \nu_0^1$$

for any continuous function f on $\mathbb{R}$ such that $\{x \in \mathbb{R} : |f(x)| \geq \epsilon\}$ is compact in $\mathbb{R}$ for all $\epsilon > 0$.

Moreover, for all $a \in \mathbb{R}$, we have $\lim_{n\to\infty} F_{Y_n}(a) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-t^2/2}dt$, i.e. F_{Y_n} converges point-wisely to the function $a \mapsto \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-t^2/2}dt$.

4.5.3 Characteristic Functions

Definition. The *characteristic function* $\varphi_X : \mathbb{R} \to \mathbb{C}$ of a real-valued random variable X is $\varphi_X(t) := \mathbb{E}[e^{itX}]$.

More generally, if $X : \Omega \to \mathbb{R}^n$ is a random variable with values in $\mathbb{R}^n$, then for any $t \in \mathbb{R}^n$ the *characteristic function* $\varphi_X : \mathbb{R}^n \to \mathbb{C}$ is given by

$$\varphi_X(t) = \mathbb{E}[e^{i\langle t,X\rangle}]$$

where $\langle , \rangle$ is the standard inner product on $\mathbb{R}^n$.

Basic properties of the characteristic function:

1. φ_X is uniformly continuous on its entire domain, and $|\varphi_X(x)| \leq 1$.
2. Two random variables have the same characteristic function iff the two variables have the same probability distributions.
3. If a random variable $X \in L^k$ for all $1 \leq k \leq m$, then the characteristic function φ_X is m-times continuously differentiable on $\mathbb{R}$, and moreover $E[X^m] = i^{-m}\varphi_X^{(m)}(0)$.
4. If the characteristic function φ_X of X has a m-th derivative at 0, then $X \in L^k$ for all $1 \leq k \leq m - \delta$, where $\delta = 0$ if m is even and $\delta = 1$ if m is odd.
5. If $X_1, X_2, ..., X_k$ are independent random variables, then
$$\varphi_{a_1X_1+\cdots+a_kX_k}(t_1, ..., t_n) = \varphi_{X_1}(a_1t_1)\cdots\varphi_{X_k}(a_kt_k).$$
6. Suppose X is a real-valued random variable such that φ_X is L^1. Then F_X is absolutely continuous, and moreover (almost surely) the density function f_X satisfies
$$f_X(x) = F_X'(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\varphi_X(t)dt.$$
7. The characteristic function of some common distributions is below:
   - Bernoulli with parameter p: $1 - p + pe^{it}$.
   - Binomial with parameters n, p: $(1 - p + pe^{it})^n$.
   - Poisson with parameter λ: $e^{\lambda(e^{it}-1)}$.
   - Uniform on interval $[a, b]$: $\frac{e^{itb}-e^{ita}}{it(b-a)}$.
   - Normal $N(\mu, \sigma^2)$: $e^{it\mu-\frac{1}{2}\sigma^2t^2}$.
   - Exponential with mean λ: $\frac{1}{1-it\lambda}$.
   - Geometric (number of failures, probability of success p): $p\left(1 - e^{it}(1 - p)\right)^{-1}$.

Theorem 4.5.12. *Suppose $X : \Omega \to \mathbb{R}^m, Y : \Omega \to \mathbb{R}^n$ are random variables with characteristic functions φ_X, φ_Y. Consider the random variable $(X, Y) : \Omega \to \mathbb{R}^{m+n}$; it has characteristic function $\varphi_{X,Y} : \mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n} \to \mathbb{C}$. Then, X and Y are independent iff for all $s \in \mathbb{R}^m, t \in \mathbb{R}^n$, we have $\varphi_{X,Y}(s, t) = \varphi_X(s)\varphi_Y(t)$.*

Example 4.5.13 (Fall 2021 Day 3)**.** Suppose U and V are two random variables. We say that U and V are uncorrelated if $\mathrm{Cov}(U, V) = \mathbb{E}[UV] - \mathbb{E}[U]\mathbb{E}[V] = 0$. Prove or disprove that if U and V are uncorrelated then they are independent. Suppose X and Y are distributed by the bi-variate normal distribution with density

$$f(x, y) = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)}\right)$$

where $0 < \rho < 1$ is fixed. Let $U = X + aY$ and $V = X + bY$ with $a, b \neq 0$. Find a necessary condition for $\mathrm{Cov}(U, V) = 0$. Show also that this condition implies that U and V are independent.

It is not necessary for U and V to be independent if they are uncorrelated. Indeed, taking U to have density function $\frac{3}{2}x^2$ on $[-1,1]$ and taking $V = U^2$, a simple calculation shows that $\mathbb{E}[UV] = 0 = \mathbb{E}[U]\mathbb{E}[V]$ since Uf_U and $UVf_{U,V}$ are odd functions of U. It is obvious however that U and V are not independent.

Now, we compute the characteristic function of (X,Y), and of X and Y separately. Note first that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} e^{isx-\frac{1}{2\sigma^2}(x-\mu)^2}dx = e^{is\mu-\frac{1}{2}\sigma^2 s^2}.$$

Thus,

$$\begin{aligned}
\phi_{X,Y}(s,t) &= \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\int_{\mathbb{R}^2} e^{i(sx+ty)}e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}}dxdy = \frac{1}{2\pi}\frac{1}{\sqrt{1-\rho^2}}\int_{\mathbb{R}} e^{isx-\frac{1}{2}x^2}\int_{\mathbb{R}} e^{ity}e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}}dydx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{isx-\frac{1}{2}x^2}e^{it\rho x-\frac{1}{2}(1-\rho^2)t^2}dx = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(1-\rho^2)t^2}\int_{\mathbb{R}} e^{i(s+t\rho)x}e^{-\frac{1}{2}x^2}dx \\
&= e^{-\frac{1}{2}(1-\rho^2)t^2-\frac{1}{2}(s+t\rho)^2} = e^{-\frac{1}{2}(s^2+2\rho st+t^2)}.
\end{aligned}$$

Hence, by comparing derivatives, it follows that

$$\mathbb{E}[XY] = i^2\frac{\partial^2\varphi_{X,Y}}{\partial s\partial t}|_{(0,0)} = \rho$$

and similarly $\mathbb{E}[X] = 0 = \mathbb{E}[Y]$ and $\mathbb{E}[X^2] = 1 = \mathbb{E}[Y^2]$. Thus $\mathrm{Var}(X) = 1 = \mathrm{Var}(Y)$ and $\mathrm{Cov}(X,Y) = \rho$. Hence

$$\mathrm{Cov}(U,V) = \mathrm{Var}(X) + ab\mathrm{Var}(Y) + (a+b)\mathrm{Cov}(X,Y) = 1 + ab + (a+b)\rho.$$

Consequently, U and V are uncorrelated iff $1 + ab + (a+b)\rho = 0$.

Finally, we show that U and V are independent if $1 + ab + (a+b)\rho = 0$. Note that

$$\begin{aligned}
\varphi_{U,V}(\alpha,\beta) &= \mathbb{E}\left[e^{i\alpha(X+aY)+i\beta(X+bY)}\right] = \varphi_{X,Y}(\alpha+\beta, a\alpha+b\beta) \\
&= e^{-\frac{1}{2}\left((1+2\rho a+a^2)\alpha^2+(1+2\rho b+b^2)\beta^2+2(1+\rho a+\rho b+ab)\alpha\beta\right)} = e^{-\frac{1}{2}\left((1+2\rho a+a^2)\alpha^2+(1+2\rho b+b^2)\beta^2\right)}.
\end{aligned}$$

It follows that $\varphi_U(\alpha) = \mathbb{E}[e^{i\alpha U}] = \varphi_{U,V}(\alpha,0) = e^{-\frac{1}{2}(1+2\rho a+a^2)\alpha^2}$. Similarly $\varphi_V(\beta) = e^{-\frac{1}{2}(1+2\rho b+b^2)\beta^2}$. Thus we see that $\varphi_{U,V}(\alpha,\beta) = \varphi_U(\alpha)\varphi_V(\beta)$, and therefore U and V are independent iff they are uncorrelated.

4.5.4 Conditional Probability

Definition. Suppose E is an event in a probability space $(\Omega,\mathcal{B},P)$ such that $P(E) > 0$. We define a new probability measure on Ω, the *conditional probability* $P(\bullet|E)$ *with respect to* E, given by

$$P(\bullet|E) : \mathcal{B} \to \mathbb{R}, \quad F \mapsto P(F|E) = \frac{P(F\cap E)}{P(E)}.$$

The *conditional expectation* with respect to E is simply the expectation $\mathbb{E}[\bullet|E]$ with respect to the probability measure $P(\bullet|E)$.

We see that F is independent of E iff $P(F|E) = P(F)$.

Definition. Suppose $X : \Omega \to \mathbb{R}$ is an L^2 random variable. For any random vector $Y : \Omega \to \mathbb{R}^n$, the *conditional expectation* $\mathbb{E}[X|Y] : \mathbb{R}^n \to \mathbb{R}$ *of* X *given* Y is a measurable function such that

$$\min_{g:\mathbb{R}^n\to\mathbb{R}\text{ measurable}} \mathbb{E}[(X-(g\circ Y))^2] = \mathbb{E}[(X-\mathbb{E}[X|Y])^2].$$

Equivalently, it is a function satisfying $\mathbb{E}\big[(X-\mathbb{E}[X|Y])(f\circ Y)\big] = 0$ for all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbb{E}[(f\circ Y)^2] < \infty$.

While $y \mapsto \mathbb{E}[X|Y=y]$ need not be a uniquely defined function of y, it is unique up to a set of measure zero in $\mathbb{R}^n$ with respect to the push-forward measure Y^*P on $\mathbb{R}^n$.

Basic Properties:

1. $\mathbb{E}[\bullet|Y]$ is linear in the first entry, i.e. $\mathbb{E}[aX_1 + bX_2|Y] = a\mathbb{E}[X_1|Y] + b\mathbb{E}[X_2|Y]$.
2. If X and Y are independent, then $\mathbb{E}[X|Y]$ is the constant function $\mathbb{E}[X]$.
3. If f is any measurable function on $\mathbb{R}$, then $\mathbb{E}[f\circ X|X] = f$.
4. (Law of Total Expectation) $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.
5. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is measurable, then $\mathbb{E}\big[\mathbb{E}[X|Y]\big|f\circ Y\big] = \mathbb{E}[X|f\circ Y]$.

4.6 Miscellaneous Exercises

Example 4.6.1 (Fall 2020 Day 1)**.** Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence that converges to A. We show that $(1-x)\sum_{n=0}^{\infty} a_n x^n \to A$ as $x \to 1^-$.

Let $\epsilon > 0$ be arbitrary. Pick $N \in \mathbb{N}$ such that $|a_n - A| < \frac{\epsilon}{2}$ for all $n \geq N$. Now, the polynomial $(1-x)\sum_{n=0}^{N-1}(a_n - A)x^n$ vanishes at $x = 1$. By continuity, there exists some $\delta > 0$ (WLOG $\delta < 1$) such that for any $x \in [1-\delta, 1)$, we have

$$\left|(1-x)\sum_{n=0}^{N-1}(a_n - A)x^n\right| < \frac{\epsilon}{2}.$$

On the other hand, for $x \in [0, 1)$, we have the estimate

$$\left|(1-x)\sum_{n=N}^{\infty}(a_n - A)x^n\right| \leq \frac{\epsilon}{2}(1-x)\sum_{n=N}^{\infty} x^n = \frac{\epsilon}{2}x^N \leq \frac{\epsilon}{2}.$$

Combining these two estimates, it follows that for any $x \in (1-\delta, 1)$, we have

$$\left|(1-x)\sum_{n=0}^{\infty}(a_n - A)x^n\right| < \epsilon.$$

Hence $(1-x)\sum_{n=0}^{\infty}(a_n - A)x^n \to 0$ as $x \to 1^-$. Since for any $x \in (0, 1)$ we have

$$(1-x)\sum_{n=0}^{\infty} Ax^n = (1-x) \cdot \frac{A}{1-x} = A,$$

it follows that $(1-x)\sum_{n=0}^{\infty} Ax^n \to A$ as $x \to 1^-$, and therefore

$$\lim_{x \to 1^-}(1-x)\sum_{n=0}^{\infty} a_n x^n = A.$$

Example 4.6.2 (Fall 2019 Day 1)**.** Define *Dirichlet's function* $D : [0,1] \to \mathbb{R}$ and *Thomae's function* $T : [0,1] \to \mathbb{R}$ by

$$D = \chi_{\mathbb{Q}\cap[0,1]}, \qquad \text{and} \qquad T(x) = \begin{cases} 1/q & x = p/q \text{ where } p, q \in \mathbb{N}, \gcd(p,q) = 1, p < q, \\ 0 & x \in [0,1]\backslash\mathbb{Q}. \end{cases}$$

We show that D is discontinuous everywhere, T is continuous at irrational points and discontinuous at rational points, and T is nowhere differentiable.

Suppose $x \in [0,1]$. If $x \in \mathbb{Q}$, then by the density of the irrational numbers in $\mathbb{R}$ there exists a sequence $\{q_n\} \subset [0,1]\backslash\mathbb{Q}$ such that $q_n \to x$. However, $D(q_n) = 0$ while $D(x) = 1$, and so $D(q_n) \not\to D(x)$. Similarly, if $x \notin \mathbb{Q}$, then the density of rational numbers in $\mathbb{R}$ implies that there exists a sequence $\{q_n\} \subset [0,1] \cap \mathbb{Q}$ such that $q_n \to x$; however $D(q_n) = 1$ while $D(x) = 0$ and so $D(q_n) \not\to D(x)$. Hence D is discontinuous everywhere.

Suppose $x \in [0,1] \cap \mathbb{Q}$. Then $T(x) > 0$. However, there exists a sequence of irrational numbers $x_n \in [0,1]\backslash\mathbb{Q}$ such that $x_n \to x$. Since $T(x_n) = 0$ for all n, we have $T(x_n) \not\to T(x)$. Therefore T is discontinuous at every point in $[0,1] \cap \mathbb{Q}$.

Now suppose $x \in [0,1]\backslash\mathbb{Q}$. Let $\{x_n\} \to x$ be any sequence in $[0,1]$ converging to x. Let $\epsilon > 0$ be arbitrary, and pick $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. Now, the set $\{1\} \cup \{p/q : 0 \leq p < q \leq k-1\} = T^{-1}((\frac{1}{k}, \infty))$ is finite, and so there exists $N \in \mathbb{N}$ such that $x_n \notin T^{-1}((\frac{1}{k}, \infty))$ for all $n \geq N$. But then, we have $|T(x_n)| = T(x_n) \leq \frac{1}{k} < \epsilon$. Therefore T is continuous at x.

Since T is discontinuous at all rational points in $[0,1]$, it is obviously not differentiable at rational points as well. Now suppose T is differentiable at $x \in [0,1]\backslash\mathbb{Q}$. By taking a sequence of irrational numbers $x_n \to x$ in $[0,1]$, it follows that

$$T'(x) = \lim_{n\to\infty} \frac{T(x_n) - T(x)}{x_n - x} = 0$$

since $T(x_n) = 0 = T(x)$ for all $n \in \mathbb{N}$. Now, let x have binary expansion $\sum_{k\geq 0} a_k 2^{-k}$ where each $a_i \in \{0,1\}$. Set $p_n = \sum_{k=0}^{n} 2^{n-k}a_k$; then $0 \leq p_n \leq \sum_{k=0}^{n} 2^{n-k} = 2^n - 1 < 2^n$ and $\frac{p_n}{2^n} = \sum_{k=0}^{n} a_k 2^{-k} \to x$ as $n \to \infty$. Note that $x - p_n 2^{-n} > 0$ and

$$x - \frac{p_n}{2^n} = \sum_{k>n} \frac{a_k}{2^n} \leq \sum_{k>n} 2^{-k} = \frac{1}{2^{n+1}} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2^n}.$$

It follows that

$$|T'(x)| = \lim_{n\to\infty} \left| \frac{T(p_n 2^{-n}) - T(x)}{p_n 2^{-n} - x} \right| = \lim_{n\to\infty} \frac{2^{-n}}{x - p_n 2^{-n}} \geq \frac{2^{-n}}{2^{-n}} = 1$$

contradicting the fact that $T'(x) = 0$. Hence T is not differentiable on $[0,1]\backslash\mathbb{Q}$.

Chapter 5

Algebra

5.1 Groups and Group Actions

Throughout, suppose G a group, with identity e. If G is finite, then $|G|$ denotes the order.

Definition. If $A \subset G$, then the *centralizer* $C_G(A)$ of A in G is the subgroup $\{g \in G : ga = ag \ \forall a \in A\}$. The *centre* $Z(G)$ is the centralizer of G in G.

The *normalizer* of a subset $A \subseteq G$ is the subgroup $N_G(A) = \{g \in G : gAg^{-1} = A\}$. Clearly $Z(G) \leq C_G(A) \leq N_G(A)$.

The *commutator* subgroup $G' \leq G$ is the subgroup generated by all elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$. The commutator subgroup is always normal in G. Moreover G/H is an abelian group iff $G' \leq H$.

Definition. A *simple* group is a non-trivial group without any proper non-trivial normal subgroup.

A *composition series* is a sequence of subgroups $1 = N_0 \leq N_1 \leq \cdots \leq N_{k-1} \leq N_k = G$ of G such that N_{i-1} is normal in N_i and N_{i+1}/N_i is a simple group for all $1 \leq i \leq k$. If $1 = M_0 \leq \cdots \leq M_h = G$ is another composition series, then $h = k$ and there is a permutation $\pi \in S_k$ such that $M_i/M_{i-1} \cong N_{\pi(i)}/N_{\pi(i)-1}$ for all $1 \leq i \leq k$.

A group G is *solvable* if there is a sequence of subgroups $1 = G_0 \leq G_1 \leq \cdots \leq G_{k-1} \leq G_k = G$ of G such that G_{i-1} is normal in G_i and G_{i+1}/G_i is abelian for all $1 \leq i \leq k$. Equivalently, G is solvable if each composition factor N_i/N_{i-1} in the composition series of G is cyclic of prime order. If both N and G/N are solvable (where N normal in G), then G is solvable.

A group is a *p-group* if it is finite and has order a power of p (p prime). Subgroups that are p-groups are called p-subgroups. If $|G| = p^r m$ for some prime p with $p \nmid m$, then a subgroup of G order p^r is called a *Sylow p-subgroup*.

Definition. The *(external) direct product* $\prod_{i \in I} G_i$ of a collection of groups $\{G_i\}_{i \in I}$ is the Cartesian product of the sets with the operation defined component-wise.

Suppose now H and K are groups, and suppose $\varphi : K \to \mathrm{Aut}(H)$ is a given homomorphism (we denote the resulting group action of K on H by $\cdot$). The *(external) semi-direct product* $H \rtimes_\varphi K$ of H and K with respect to φ is the Cartesian product $H \times K$ equipped with the following operation:

$$(h_1, k_1)(h_2, k_2) = \big(h_1(k_1 \cdot h_2), k_1 k_2\big).$$

H and K can be embedded into $H \rtimes_\varphi K$ via $h \mapsto (h, 1)$ and $k \mapsto (1, k)$ respectively. Under this identification, H is a normal subgroup of $H \rtimes_\varphi K$ and K a subgroup of $H \rtimes_\varphi K$ with $H \cap K = 1$. Moreover, $khk^{-1} = k \cdot h = \varphi(k)(h)$ for all $h \in H, k \in K$. If φ is obvious, then we can omit φ and simply write $H \rtimes K$.

Definition. A group action G on X is *faithful* if the corresponding homomorphism $G \to \mathrm{Perm}(X)$ is injective, i.e. if $gb = b$ for all $b \in B$ then $g = e$.

The *kernel* of the action is the kernel of the homomorphism $G \to \mathrm{Perm}(X)$, i.e. it is $\{g \in G : gb = b \ \forall b \in X\}$.

For fixed $a \in X$, the *stabilizer subgroup* is $\mathrm{Stab}_G(a) = \{g \in G : ga = a\}$. The notation G_a is sometimes used as well.

For $a \in X$, the *orbit* Ga of a is the subset $\{ga : g \in G\}$ of X. A group action is transitive if $Ga = X$ for some (and thus all) $a \in X$. A group action is *doubly-transitive* if for some (and thus all) $a \in G$, the stabilizer subgroup G_a is transitive on $X \setminus \{a\}$.

- (*Lagrange's Theorem*) If $H \subseteq G$ is a subgroup, then G is the disjoint union of all left H-cosets, each of which have the same cardinality.

- Suppose $H, K \leq G$. Then $HK = \{hk : h \in H, k \in K\}$ is a subgroup iff $HK = KH$ as sets. In particular, this holds if either one of K or H is normal in G.
- If $|G| = p$, then $G \cong \mathbb{Z}/p\mathbb{Z}$.
- The *isomorphism theorems*:
  1. If $\varphi : G \to H$ is a homomorphism, then $\ker \varphi$ is normal in G and $G/\ker \varphi \cong \varphi(G)$.
  2. Suppose $A, B \leq G$ with $A \leq N_G(B)$. Then AB is a subgroup of G, B is normal in AB, $A \cap B$ is normal in A, and $AB/B \cong A/A \cap B$.
  3. Suppose H, K are normal subgroups of G with $H \leq K$. Then, K/H is normal in G/H and $(G/H)/(K/H) \cong G/K$.
  4. Suppose N is normal in G. Then there is an inclusion preserving lattice isomorphism between the lattice of subgroups of G/N and the lattice of subgroups of G containing N.
- (*Orbit-stabilizer theorem*) If G acts on X and $x \in X$, then there is a bijection between the left G_x-cosets of G and the orbit of x in X.
- (*Cauchy's Theorem*) If G is a finite group and $p \big| |G|$ is prime, then there is a subgroup of G of order p.
- If G is a finite group and p is the smallest prime dividing $|G|$, then any subgroup of index p is normal in G.
- For a subset $S \subseteq G$, the number of conjugates of S (i.e. number of sets T such that $T = gSg^{-1}$ for some $g \in G$) is the index of the normalizer $[G : N_G(S)]$. In particular, the size of the conjugacy class of $g \in G$ is the index $[G : C_G(g)]$ of the centralizer of g.
- (*Class Equation*) Suppose G is a finite group and $g_1, ..., g_r$ are representatives of the distinct conjugacy classes of G not contained in the centre $Z(G)$. Then
$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)].$$
- Every p-group has a non-trivial centre. Also, a p-group G with order p^r has subgroups of order p^s for all $0 \leq s \leq r$. In particular, if $|G| = p^2$ for p a prime then G is isomorphic to either $\mathbb{Z}/p^2\mathbb{Z}$ or $(\mathbb{Z}/p\mathbb{Z})^2$.
- Two elements of S_n are conjugates iff they have the same cycle type. Moreover, S_n is generated by a transposition and an n-cycle if n is prime.
- (*Sylow Theorems*) Suppose G is a group of order $p^r m$ where $p \nmid m$ and p is prime.
  1. Sylow p-subgroups of G exist.
  2. If P is a Sylow p-subgroup and Q a p-subgroup of G, then there exists $g \in G$ such that $Q \subseteq gPg^{-1}$. In particular, all Sylow p-subgroups are conjugate to each other. Equivalently, the conjugation action of G on the set of Sylow p-subgroups is transitive.
  3. If n_p denotes the number of Sylow p-subgroups of G, then $n_p = [G : N_G(P)]$ for any Sylow p-subgroup P (so in particular $n_p | m$), and $n_p \equiv 1 \pmod{p}$.

  In particular, if $n_P = 1$ then P is normal in G.
- If $|G| = pq$ with $p < q$, then either $G \cong \mathbb{Z}_{pq}$, or $p | (q-1)$ and $G \cong (\mathbb{Z}/q\mathbb{Z}) \rtimes_\varphi (\mathbb{Z}/p\mathbb{Z})$ where φ maps $\mathbb{Z}/p\mathbb{Z}$ onto some p-subgroup of $\mathrm{Aut}(\mathbb{Z}/q\mathbb{Z}) = (\mathbb{Z}/q\mathbb{Z})^*$.
- A_n is simple for all $n \geq 5$. In fact, A_5 is the unique non-abelian simple group with order < 100.
- (*Fundamental Theorem of Finitely Generated Abelian Groups*) If G is a finitely-generated abelian group, then $G \cong \mathbb{Z}^r \times (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_s\mathbb{Z})$ where $n_j \geq 2$ for all j, $r \geq 0$, and $n_{i+1} | n_i$ for $1 \leq i \leq s-1$. This expression is moreover unique.
- If $H, K \leq G$ such that H and K are normal in G and $H \cap K = 1$, then $G \cong H \times K$. The subgroup HK of G is the *internal direct product*.

Example 5.1.1 (Fall 2019 Day 2)**.** Let $\mathbb{F}_q$ be the finite field with q elements. Prove that the number of 3×3 nilpotent matrices over $\mathbb{F}_q$ is q^6.

Let N be an arbitrary 3×3 nilpotent matrix over $\mathbb{F}_q$. Then all of the eigenvalues of N is 0. By the Jordan decomposition theorem (noting that the characteristic polynomial of N splits over $\mathbb{F}_q$), there exists an element $X \in GL(3, \mathbb{F}_q)$ such that XNX^{-1} is exactly one of the following three matrices

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider the conjugation action of $GL(3, \mathbb{F}_q)$ on $M(3 \times 3, \mathbb{F}_q)$. Then the above implies that the set of all nilpotent elements in $M(3 \times 3, \mathbb{F}_q)$ is the disjoint union of the orbits of O, A, B under $GL(3, \mathbb{F}_q)$. It thus suffices to find the size of each of these orbits. For O, the orbit is obviously a singleton. For the other two, we use the orbit stabilizer theorem. Note first that $|GL(3, \mathbb{F}_q)| = (q^3 - 1)(q^3 - q)(q^3 - q^2)$ since we can choose the first column c_1 of an invertible matrix to be any non-zero vector in $\mathbb{F}_q^3$, we can choose the second column c_2 to be any vector in $\mathbb{F}_q^3$ not in $\mathbb{F}_q \cdot c_1$, and we can choose the third column to be any vector in $\mathbb{F}_q^3$ not in $\mathbb{F}_q \cdot c_1 \oplus \mathbb{F}_q \cdot c_2$.

- Let $X = (x_{ij})$ be any element of the stabilizer subgroup of A. Then

  $$\begin{pmatrix} x_{21} & x_{22} & x_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = AX = XA = \begin{pmatrix} 0 & x_{11} & 0 \\ 0 & x_{21} & 0 \\ 0 & x_{31} & 0 \end{pmatrix}$$

  implies that $x_{22} = x_{11}$ and $x_{21} = x_{23} = x_{31} = 0$. Then

  $$0 \neq \det X = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{11} & 0 \\ 0 & x_{32} & x_{33} \end{vmatrix} = x_{11}^2 x_{33}$$

  implies that $x_{11}, x_{33} \in \mathbb{F}_q^*$. Thus there are $q - 1$ choices each for $x_{11} = x_{22}$ and x_{33}, and q choices each for x_{12}, x_{13}, x_{32}. Thus the size of the stabilizer of A is $q^2(q-1)^3$.

- Let $Y = (y_{ij})$ be any element of the stabilizer subgroup of B. Then

  $$\begin{pmatrix} y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \\ 0 & 0 & 0 \end{pmatrix} = BY = YB = \begin{pmatrix} 0 & y_{11} & y_{12} \\ 0 & y_{21} & y_{22} \\ 0 & y_{31} & y_{32} \end{pmatrix}$$

  implies that $y_{11} = y_{22} = y_{33}$, $y_{12} = y_{23}$, and $y_{21} = y_{31} = y_{32} = 0$. Then

  $$0 \neq \det Y = \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ 0 & x_{11} & y_{12} \\ 0 & 0 & y_{11} \end{vmatrix} = y_{11}^3$$

  implies that $y_{11} \in \mathbb{F}_q^*$. Thus there are $q - 1$ choices for $y_{11} = y_{22} = y_{33}$, and q choices each for $y_{12} = y_{23}$ and y_{13}. Thus the size of the stabilizer of A is $q(q-1)^2$.

Therefore the number of nilpotent elements in $M(3 \times 3, \mathbb{F}_q)$ is

$$1 + \frac{(q^3-1)(q^3-q)(q^3-q^2)}{q^2(q-1)^3} + \frac{(q^3-1)(q^3-q)(q^3-q^2)}{q(q-1)^2} = 1 + (q^2+q+1)(q+1)q + (q^2+q+1)(q^3-q)q = q^6.$$

Example 5.1.2 (Fall 2020 Day 1)**.** We show that every finite group of order 72 is not simple.

Indeed, suppose G has order 72. Let n_3 be the number of Sylow 3-subgroups. Since $72 = 2^3 \cdot 3^2$, it follows from Sylow's third theorem that $n_3 | 2^3$ and $n_3 \equiv 1 \pmod 3$. Thus $n_3 = 1$ or $n_3 = 4$. If $n_3 = 1$, then it follows that the unique Sylow 3-subgroup of G is normal in G, so that G is not simple.

Suppose now that $n_3 = 4$. Let P_1, P_2, P_3, P_4 be the 4 Sylow 3-subgroups of G. Then, each P_i has order 9. Now, by Sylow's Theorems again, the conjugation action of G on the set $\{P_1, P_2, P_3, P_4\}$ is transitive, and induces a group homomorphism $\varphi : G \to S_4$. The transitivity of the group action implies in particular that $\ker \varphi \neq G$. Since $|G| = 72$ while $|S_4| = 24$, it follows that $\ker \varphi$ is non-trivial. Hence $\ker \varphi$ is a non-trivial proper normal subgroup of G, which implies that G is not simple.

Example 5.1.3 (Fall 2019 Day 1)**.** Prove that for any finite group G with n conjugacy classes, we have $|\mathrm{Hom}(\mathbb{Z}^2, G)| = n|G|$.

Since $\mathbb{Z}^2$ is freely generated by two commuting elements, it follows that there is a bijection

$$\mathrm{Hom}(\mathbb{Z}^2, G) \to \{(x, y) \in G \times G : xy = yx\}, \quad h \mapsto (h(1, 0), h(0, 1)).$$

Thus it suffices to count the number of pairs $(x, y) \in G \times G$ such that $xy = yx$. For any $x \in G$, note that $G_x := \{y \in G : yx = xy\}$ is the stabilizer subgroup of G for the conjugation action of G on itself. The orbit-stabilizer theorem implies that $|G_x| = |G|/|C(x)|$ where $C(x) := \{yxy^{-1} : y \in G\}$. However, $C(x)$ is the conjugacy class of x. Thus, we want to evaluate $|G| \sum_{x\in G} \frac{1}{|C(x)|}$. However, for each $z \in C(x)$, we have $|C(x)| = |C(z)|$, so that for a given conjugacy class C the sum $\sum_{z\in C} \frac{1}{C(z)}$ is simply 1. Hence, $|G| \sum_{x\in G} \frac{1}{|C(x)|} = n|G|$ as required.

Example 5.1.4 (Fall 2018 Day 3)**.** Suppose G is a group of order 78. Show that G contains a normal subgroup of index 6. Find an example of G that contains a non-normal subgroup of index 13.

By Sylow's Theorems, there exists a unique Sylow 13-subgroup, which must be a normal subgroup of index 6. Let C_n be the cyclic group of order n. Now consider $\varphi : S_3 : \mathrm{Aut}(C_{13})$ given by the composition of $\mathrm{sgn} : S_3 \to C_2$ and $C_2 \cong \{Id, (g \mapsto g^{-1})\} \subset \mathrm{Aut}(C_{13})$. This induces the semi-direct product $G := C_{13} \rtimes_\varphi S_3$. Using the presentation $S_3 = \langle r, s : r^3 = 1 = s^2, srs = r^2\rangle$, it follows that

$$G = \left\langle r, s, g : r^3 = s^2 = g^{13} = 1, srs = r^2, rg = gr, sgs = g^{-1} \right\rangle.$$

Notice that the copy of S_3 inside G is a subgroup of index 13, and since $gsg^{-1} = sg^{-2} \notin S_3$, it follows that this subgroup is not normal.

5.2 Rings and Modules

Throughout, assume R is a ring (not necessarily commutative, but always containing unity unless otherwise specified). Unity will always be denoted by 1 and zero will always be denoted by 0 (with subscripts if necessary). All modules will be considered as left-modules (unless otherwise specified); in particular, we assume ideals are left-ideals unless otherwise specified. Also, if we discuss ideals then we assume R has a unity. A homomorphism of rings $R \to S$ where both R and S have unities must map 1_R to 1_S. All subrings of rings containing unity must also contain unity. We assume integral domains must be commutative with identity. If we discuss prime/maximal ideals, then we also assume that the ring is commutative with unity. All modules over rings with unity are assumed unital, i.e. $1 \cdot m = m$ for all m in the module. R-algebras A must always be unital, A must always have unity, and the ring homomorphism $R \to A$ must map R into the center of A.

Definition. If I and J are ideals, then $I + J := \{x + y : x \in I, y \in J\}$ is an ideal. Also, $IJ = \langle xy : x \in I, y \in J\rangle$ is an ideal.

The *nil-radical* $N(R)$ of a commutative ring R is the ideal of nilpotent elements, i.e. elements x such that $x^n = 0$ for some n. The *radical* $rad(I)$ or $\sqrt{I}$ of an ideal I is the ideal of R of elements $x \in R$ such that $x^n \in I$ for some n. We have $I \subseteq \sqrt{I}$ and $N(R/I) = \sqrt{I}/I$.

Definition. Suppose M is an R-module.

- An R-module M is irreducible if $M \neq 0$ and there are no non-zero proper submodules of M. An R-module is irreducible iff it is isomorphic as R-modules to R/I where I is a maximal ideal of R.
- *Set of torsion elements* $Tor(M) := \{m \in M : rm = 0 \exists r \in R\setminus 0\}$. If R is an integral domain, then $Tor(M)$ is a submodule of M.
- If $N \subset M$ is a submodule, the *annihilator ideal of N in R* is the 2-sided ideal $Ann(N) = \{r \in R : rn = 0 \forall n \in N\}$.
- If $I \subset R$ is an ideal, the *annihilator submodule of I in M* is the submodule $\{m \in M : rm = 0 \forall r \in I\}$.

The annihilator submodule of $Ann(N)$ of M clearly contains N. Similarly, $Ann(N)$ contains I where N is the annihilator submodule of I in M.

An R-module M is *free on the subset* $A \subset M$ if every non-zero $x \in M$ can be written uniquely as $x = \sum_{i=1}^n r_i a_i$ where $r_i \in R \setminus \{0\}, a_i \in A \setminus \{0\}$, and $n \geq 1$. In this case, A is a *basis* or a *set of free generators* of M. If R is commutative then the *rank* of M is the cardinality of A. If $A = \{a_1, ..., a_n\}$, then $\mathrm{rank} M = n$ and $M = Ra_1 \oplus \cdots \oplus Ra_n \cong R^n$.

An R-module M is *Noetherian* iff all submodules of M are finitely generated iff for any increasing chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of R-submodules there exists $N \in \mathbb{N}$ such that $M_n = M_N$ for all $n \geq N$.

Definition. Suppose M is a right R-module and N a left R-module, where R has unity. The *tensor product* $M \otimes_R N$ is an abelian group generated by elements of the form $m \otimes n$ $(m \in M, n \in N)$ such that

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \quad m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \quad \text{and} \quad (mr) \otimes n = m \otimes (rn).$$

A map $\varphi : M \times N \to L$ of abelian groups is *R-balanced* or *middle-linear with respect to R* if φ is a bilinear map such that $\varphi(m, rn) = \varphi(mr, n)$. Then $\iota : M \times N \to M \otimes_R N, (m, n) \mapsto m \otimes n$ is R-balanced. The general tensor product above satisfied following universal rule: Every group homomorphism $\Phi : M \otimes_R N \to L$ (L abelian) is of the form $\varphi = \Phi \circ \iota$ where $\varphi : M \times N \to L$ is R-balanced (and vice versa).

If S is another ring with unity, then a right R-module M which is also a left S-module such that $s(mr) = (sm)r$ is called a (S, R)-bimodule. In particular, if R is commutative, then any left module M can be endowed with a (R, R)-bimodule structure given by $mr := rm$. This is the *standard bimodule structure on M*.

Now, if M is a (S, R)-bimodule and N a left R-module, then $M \otimes_R N$ is a left S-module via $s(m \otimes n) = (sm) \otimes n$. In particular, if R is commutative, then $M \otimes_R N$ is a left R-module where we endow M with the standard (R, R)-bimodule structure. The latter construction satisfies the following universal rule: Suppose $\iota : M \times N \to M \otimes_R N$ is the R-bilinear map $(m, n) \mapsto m \otimes n$. Then, every R-bilinear map $\varphi : M \times N \to L$ (L a left R-module) is of the form $\varphi = \Phi \circ \iota$ for some R-module homomorphism $\Phi : M \otimes_R N \to L$ (and vice versa).

Important results:

- *Isomorphism Theorems* for rings
  1. If $\varphi : R \to S$ is a homomorphism, then $\ker \varphi$ is a 2-sided ideal of R and $R/\ker \varphi \cong \varphi(R)$.
  2. If A a sub-ring of R and I an ideal of R. Then $A + I = \{a + x : a \in A, x \in I\}$ is a sub-ring of R, $A \cap I$ is an ideal of A, and $A + I/I \cong A/(A \cap I)$.
  3. If I and J are ideals with $I \subseteq J$, then J/I is an ideal of R/I and $(R/I)/(J/I) \cong R/J$.
  4. Fix an ideal I of R. The correspondence $A \leftrightarrow A/I$ is an inclusion preserving bijection between subrings of R containing I and subrings of R/I. Similarly for ideals.
- *Isomorphism theorems*: for R-modules
  1. If $\varphi : M \to N$ is an R-module homomorphism, then $\ker \varphi$ is an R-submodule in M and $M/\ker \varphi \cong \varphi(M)$.
  2. Suppose A, B are submodules of M. Then $A+B$ is an R-submodule of M and $(A+B)/B \cong A/A \cap B$.
  3. Suppose A, B are submodules of M with $A \subseteq B$. Then, B/A is an R-submodule in M/A and $(M/A)/(B/A) \cong M/B$.
  4. Suppose N is an R-submodule of M. Then there is an inclusion preserving lattice isomorphism between the lattice of R-submodules of M/N and the lattice of R-submodules of M containing N.
- Suppose R commutative. I is a prime ideal of R iff R/I is an integral domain; I is a maximal ideal of R iff R/I is a field. If R is commutative and I a prime ideal with no zero divisors, then R is an integral domain.
- Suppose $\varphi : R \to S$ is a homomorphism of commutative rings. If I is prime in S, then $\varphi^{-1}(I)$ is either R or is a prime ideal of R. If I is maximal in S and φ surjective, then $\varphi^{-1}(I)$ is a maximal ideal of R.
- If P is a prime ideal in R, and if I and J are ideals such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.
- (*Chinese Remainder Theorem* for rings) Suppose $I_1, ..., I_k$ are ideals in a commutative ring R. The map $R \to (R/I_1) \times \cdots \times (R/I_k)$, $r \mapsto (r + I_1, ..., r + I_k)$, is a ring homomorphism with kernel $I_1 \cap \cdots \cap I_k$. If moreover $I_i + I_j = R$ for all $i \neq j$, then $I_1 \cap \cdots \cap I_k = I_1 \cdots I_k$ and the above homomorphism is surjective.
- All groups are $\mathbb{Z}$-modules. All vector spaces are modules over fields. Action of a polynomial ring $F[x]$ (F a field) on a module V are in bijection with vector spaces V over F along with a fixed linear operator $T : V \to V$. Moreover, $(a_n x^n + \cdots + a_1 x + a_0) \cdot v = a_n T^n v + \cdots + a_1 T v + a_0$, where $T^n := T \circ T \circ \cdots$. Moreover, $F[x]$-submodules are precisely the T-invariant sub-spaces.
- (Universal Property of Free Modules) Suppose A is an arbitrary set and R a ring. Then there exists a unique (up to R-module isomorphisms) free R-module $F(A)$ with $A \subset F(A)$ as basis such that for any R-module M and any set map $\varphi : A \to M$ there exists a unique R-module homomorphism $\Phi : \Phi(A) \to M$ such that $\Phi(a) = f(a)$ for all $a \in A$.

- (*Chinese Remainder Theorem* for R-modules) Suppose $I_1, ..., I_k$ are ideals in a commutative ring R, and suppose M is an R-module. The map $M \to (M/I_1M) \times \cdots \times (M/I_kM)$, $m \mapsto (m + I_1M, ..., m + I_kM)$, is an R-module homomorphism with kernel $I_1M \cap \cdots \cap I_kM$. If moreover $I_i + I_j = R$ for all $i \neq j$, then $I_1M \cap \cdots \cap I_kM = (I_1 \cdots I_k)M$ and the above homomorphism is surjective.

From now on, we assume R is always an integral domain, and is commutative with identity.

Definition. R is a *Euclidean domain (ED)* if it is an integral domain and there exists a map $v : R \to \mathbb{Z}^+ \cup \{0\}$ with $v(0) = 0$ such that for any $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ with $a = qb + r$ and $v(r) < v(b)$.

R is a *Principal Ideal Domain (PID)* if it is an integral domain in which all ideals are principal.

R is a *Unique Factorization Domain (UFD)* if it is an integral domain such that for any $a \in R$, there exist distinct irreducible elements $p_1, ..., p_r \in R$ and integers $n_1, ..., n_r$ such that $a = \prod_{i=1}^r p_i^{n_i}$, and moreover such a factorization is unique (up to multiplication by units).

R is *Noetherian* iff every ideal of R is finitely generated iff for any ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$, there exists $N \in \mathbb{N}$ such that $I_n = I_N$ for all $n \geq N$.

- Prime elements of R are always irreducible. If R is a UFD, then all irreducible elements are prime.
- A principal ideal generated by a prime element is always prime.
- Suppose R is an integral domain but not a field. If for every non-zero non-unit $u \in R$ there exists $x \in R$ such that $u \nmid x$, and such that for all units $z \in R^*$ we have $u \nmid (x - z)$, then R is not a Euclidean domain.
- We have Fields $\implies ED \implies PID \implies UFD$. The ring $\mathbb{Z}[\frac{1+\sqrt{19}}{2}]$ is a PID but not a ED. We also have PID $\implies$ Noetherian ring.
- Prime ideals $\implies$ maximal ideal in a PID.
- $R[x]$ is a PID iff R is a field; in this case $R[x]$ is in fact an ED. $R[x]$ is a UFD iff $R[x]$ is a UFD. The latter implies that $\mathbb{Z}[x]$ is a UFD but not a PID.
- Suppose R an integral domain. Then, R is a PID iff every prime ideal of R is principal iff it has a *Dedekind-Hasse norm*, i.e. there exists a map $N : R \to \mathbb{Z}_{\geq 0}$ with $N(a) = 0$ iff $a = 0$ such that, for any non-zero $a, b \in R$, either $a \in \langle b \rangle$ or there exists $s, t \in R$ such that $0 < N(sa + tb) < N(b)$.
- R is a PID iff R is a UFD and if for any $a, b \in R$, there exists $d \in R$ such that $\langle a, b \rangle = \langle d \rangle$.
- R is a UFD iff every irreducible element of R is prime and for any ascending chain of principal ideals $\langle a_0 \rangle \subseteq \langle a_1 \rangle \subseteq \cdots$, there exists $N \in \mathbb{N}$ such that $a_m = a_N$ for all $m \geq N$. In particular, if every irreducible element is prime in a Noetherian domain, then it is a UFD.
- (*Hilbert Basis Theorem*) If R is Noetherian then so is $R[x]$.
- (*Gauss' Lemma*) Suppose R is a UFD with field of fractions F, and let $p \in R[x]$. If p is reducible in $F[x]$ then it is reducible in $R[x]$. In particular, if $p \in R[x]$ and the greatest common divisor of the coefficients of p is 1, then p is irreducible in $R[x]$ iff it is irreducible in $F[x]$.
- Suppose I is a proper ideal of R, and let $p \in R[x]$ be a non-constant monic polynomial. If the image of p in $(R/I)[x]$ is irreducible in $(R/I)[x]$, then p is irreducible in $R[x]$. The converse is false; for instance, $x^4 + 1 \in \mathbb{Z}[x]$ is irreducible, but modulo p is reducible for all primes p.
- (*Eisenstein's Criterion*) Suppose I is a prime ideal of R, and let $p = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in R[x]$. If $a_0, a_1, ..., a_{n-1} \in I$ but $a_0 \notin I^2$, then f is irreducible in $R[x]$.
- Suppose R is an integral domain. Consider two polynomials $f, g \in R[x]$ of degree d and e respectively. Then the map $P_{e-1} \times P_{d-1} \to P_{d+e-1}, (a, b) \mapsto (f, g)$ is a linear map on isomorphic free modules over R, where P_k is the space of all polynomials in $R[x]$ with degree $\leq k$. The determinant of this map is thus well-defined, and is called the *resultant* $Res(f, g)$ of f and g. Now, notice that this map has image $\langle f, g \rangle \cap P_{d+e-1}$, and so is surjective iff f and g are co-prime in $R[x]$ iff f and g share a common root (if R is some field in which f and g split). Hence, the *resultant* of f and g is non-zero iff f and g are co-prime.

By taking bases and computing, we see that

$$Res\left(\sum_{n=0}^{d} a_{d-n}x^n, \sum_{n=0}^{e} b_{d-n}x^n\right) = \begin{vmatrix} a_0 & 0 & 0 & \cdots & 0 & b_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 & b_1 & b_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & b_2 & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_d & a_{d-1} & a_{d-2} & \cdots & \vdots & 0 & b_e & b_{e-1} & \cdots & b_0 \\ 0 & a_d & a_{d-1} & \cdots & \vdots & 0 & 0 & b_e & \cdots & \vdots \\ 0 & 0 & a_d & \cdots & \vdots & 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & a_{d-1} & \vdots & \vdots & \vdots & \ddots & b_{e-1} \\ 0 & 0 & 0 & \cdots & a_d & 0 & 0 & 0 & \cdots & b_e \end{vmatrix}$$

where the determinant is of a $(d+e) \times (d+e)$ matrix. Notice that the resultant is homogeneous of degree e in the coefficients of f, and is homogeneous of degree d in the coefficients of g. Now, recall that a polynomial f has a multiple roots iff f and f' are coprime (assuming the characteristic is 'nice'). Also recall that, for R a field, the *discriminant* $Disc(f)$ of a polynomial f is given by $Disc(f) = \prod_{i<j}(\alpha_i - \alpha_j)^2$ where the roots of f (with multiplicities) are $\alpha_1, ..., \alpha_d$. These two are related via

$$a_0 Disc(f) = (-1)^{d(d-1)/2} Res(f, f')$$

where a_0 is the leading coefficient of f. The discriminant is thus homogeneous of degree $2\deg f - 2$ in the coefficients of f (the resultant $Res(f, f')$ is homogeneous of degree $2\deg f - 1$ in the coefficients of f, but we then divide by a_0 thus reducing the degree by 1). It is also known that the discriminant is an irreducible polynomial in the coefficients of f.

If we instead consider homogeneous polynomials $f = \sum_n a_n x^n y^{d-n}$ and $g = \sum_n b_n x^n y^{e-n}$ in two variables, we have $Res_x(f(x,1), g(x,1)) = Res_y(f(1,y), g(1,y))$, and so we can define the homogeneous resultant $Res(f,g)$ similarly. The same homogeneity properties hold. Similarly, the homogeneous discriminant $Disc_h$ can be defined by $Disc(f(x,1)) = Disc(f(1,y))$ (here, if $f(x,1)$ or $f(1,y)$ have degree strictly less than the total degree of f, then we compute the determinant as if they are polynomials of degree n, i.e. we compute as if the coefficients of f are indeterminate).

- Suppose M is a free R-module with rank m, where R is a PID. Then, any sub-module N of M is also free with rank $n \leq m$. Moreover, there exists a basis $y_1, ..., y_m$ of M so that $a_1y_1, ..., a_ny_n$ where $a_1, ..., a_n \in R$ are non-zero elements such that $a_i | a_{i+1}$ for $1 \leq i \leq n-1$.

- (*Fundamental Theorem of Finitely Generated Modules over PID; Invariant Factor Form*) Suppose R is a PID and M a finitely generated R-module. Then, there exists a unique $r \geq 0$, unique $m \in \mathbb{N}$, and non-zero non-unit elements $a_1, ..., a_m \in R$ (unique up to units) satisfying $a_i | a_{i+1}$ for $1 \leq i \leq n-1$ such that
$$M \cong R^r \times (R/\langle a_1 \rangle) \times (R/\langle a_2 \rangle) \times \cdots \times (R/\langle a_m \rangle).$$
Moreover, $Tor(M) \cong (R/\langle a_1 \rangle) \times (R/\langle a_2 \rangle) \times \cdots \times (R/\langle a_m \rangle)$ and $Ann(M) = \langle a_m \rangle$, and M is torsion free iff it is free. (Torsion free finitely generated modules are not free over general integral domains.)

  The r above is called the *free rank* of M, and the elements $a_1, ..., a_m \in R$ (defined up to multiplication by units in R) are called *the invariant factors of* M.

- (*Fundamental Theorem of Finitely Generated Modules over PID; Elementary Divisor Form*) Suppose R is a PID and M a finitely generated R-module. Then, there exists a unique $r \geq 0$, unique $t \in \mathbb{N}$, primes $p_1, ..., p_t \in R$ (unique up to units; not necessarily distinct), and unique integers $\alpha_i \geq 0$ for $1 \leq i \leq t$ (up to permutation) such that
$$M \cong R^r \times (R/\langle p_1^{\alpha_1} \rangle) \times (R/\langle p_1^{\alpha_2} \rangle) \times \cdots \times (R/\langle p_t^{\alpha_t} \rangle).$$
Moreover, if $Ann(M) = \langle a \rangle$, then $a = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ is the prime factorization of $a \in R$.

  The elements $p_i^{\alpha_i} \in R$ are called *the elementary divisors of* M.

- If V is a finite dimensional vector space over F, and if $T : V \to V$ is some linear operator, then we have the action of $F[t]$ on V via T. Since $F[x]$ is a PID and V is finitely generated, the above two results hold.

1. (*Rational Canonical Form*) Using the invariant factor decomposition, we have

$$V \cong F[t]/\langle a_1\rangle \times \cdots \times F[t]/\langle a_m\rangle$$

where $a_1, ..., a_m \in F[t]$ are monic (so that they are uniquely defined), and $a_1|a_2|\cdots|a_m$. If $1, \bar{t}, ..., \bar{t}^{d_i-1}$ is a basis for the F-vector space $F[t]/\langle a_i\rangle$ ($d_i := \deg a_i$), then T acts on $F[t]/\langle a_i\rangle$ via $\bar{t}^j \mapsto \bar{t}^{j+1}$ for $j < d_i - 1$, and $\bar{t}^{d_i-1} \mapsto -b_0^{(i)} - \cdots - b_{d_i-1}^{(i)}\bar{t}^{d_i-1}$ where $a_i(t) = t^{d_i} + b_{d_i-1}^{(i)}t^{d_i-1} + \cdots + b_0^{(i)}$. Writing T in this basis for V (i ranges from 1 to m), we obtain the matrix

$$\begin{bmatrix} C(a_1) & & & \\ & C(a_2) & & \\ & & \ddots & \\ & & & C(a_m) \end{bmatrix}, \quad \text{where} \quad C(b_0+b_1t+\cdots+b_{d-1}t^{d-1}+t^d) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 0 & 1 & \cdots & 0 & -b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -b_{d-1} \end{pmatrix},$$

which if we stipulate $a_1|a_2|\cdots|a_m$, is uniquely defined. This matrix is called the *rational canonical form of* T. The matrix $C(p)$ attached to a polynomial p is called the *companion matrix* of p; it is has characteristic polynomial equal to p. We also have the following facts:

(a) The minimal polynomial of T is a_m.

(b) The characteristic polynomial $\det(tI - T)$ of T is $a_1(t)\cdots a_m(t)$.

(c) The characteristic polynomial and the minimal polynomial share the same roots (ignoring multiplicity).

2. (*Jordan Canonical Form*) The elementary factor decomposition implies that

$$V \cong F[t]/\langle p_1^{\alpha_1}\rangle \times \cdots \times F[t]/\langle p_m^{\alpha_m}\rangle.$$

If we suppose that all eigenvalues of T belong to F, then each $p_i = t - \lambda_i$ where $\lambda_1, ..., \lambda_m$ are (not necessarily distinct) eigenvalues of T. The matrix of T in the basis $(t-\lambda_i)^j$ $1 \le j \le \alpha_i$, $1 \le i \le m$, then yields

$$[T] = \begin{bmatrix} J_{\alpha_1}(\lambda_1) & & & \\ & J_{\alpha_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{\alpha_m}(\lambda_m) \end{bmatrix}, \quad \text{where} \quad J_\alpha(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \in M_{\alpha\times\alpha}(F).$$

Such a decomposition is called the *Jordan decomposition* (each block is a *Jordan block*) or the *Jordan canonical form*. This decomposition is unique up to permutation of the Jordan blocks along the diagonal.

The number of times a given λ occurs in $\{\lambda_1, ..., \lambda_m\}$ is the geometric multiplicity of the eigenvalue λ. The sum $\sum_{\lambda_i=\lambda}\alpha_i$ is the algebraic multiplicity of the eigenvalue λ.

Example 5.2.1 (Spring 2020 Day 1)**.** The ring $\mathbb{Z}[\sqrt{p}]$ for p a prime congruent to 1 modulo 4 is not a UFD. This is because we have $2 \cdot \frac{p-1}{2} = (\sqrt{p}+1)(\sqrt{p}-1)$. Since the field norm $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}$ of 2 is 4, and no elements of $\mathbb{Z}[\sqrt{p}]$ have field norm 2, it follows that 2 is irreducible. However, it is easy to see that 2 does not divide $\sqrt{p}\pm 1$ in the ring $\mathbb{Z}[\sqrt{p}]$. Hence $\mathbb{Z}[\sqrt{p}]$ is not a UFD.

Alternatively, $\mathbb{Z}[\sqrt{p}]$ is not integrally closed in its quotient field since the polynomial $x^2+x+\frac{1-p}{4} \in (\mathbb{Z}[\sqrt{p}])[x]$ has roots $\frac{1}{2}(1 \pm \sqrt{p}) \notin \mathbb{Z}[\sqrt{p}]$. Since all UFDs are integrally closed, it follows that $\mathbb{Z}[\sqrt{p}]$ is not a UFD.

Example 5.2.2 (Spring 2020 Day 3)**.** Suppose V is an n dimensional vector space over an arbitrary field K, and suppose $T_1, ..., T_n : V \to V$ are pair-wise commuting nilpotent operators on V. Claim: $T_1\cdots T_n = 0$.

Indeed, since T_i is nilpotent, we have $T_i^{k_i} = 0$ for some $k_i \in \mathbb{N}$. Picking any non-zero vector $w \in V$, we have $T_i(T_i^{k_i-1}w) = 0$ and so $\ker T_i$ is non-empty. By the rank-nullity theorem, we have $\text{rank} T_i \le n-1$, so that the dimension of $T_i(V)$ is at least one less than the dimension of V. In particular, $T_n(V)$ has dimension at most $n-1$. Since T_{n-1} commutes with T_n, it follows that $T_n(V)$ is T_{n-1}-invariant. The restriction $T_{n-1}|_{T_n(V)}$ is obviously also nilpotent, and so the same reason as above implies that $T_{n-1}(T_n(V)) = T_{n-1}T_n(V)$ has dimension at most $n-2$. Continuing inductively, and noting that T_i commutes with $T_{i+1}\cdots T_n$ for all $1 \le i \le n-1$, it follows that $T_iT_{i+1}\cdots T_n(V)$ has dimension at most $i-1$. In particular, $T_1\cdots T_n(V)$ has dimension at most 0, and so $T_1\cdots T_n(V)=0$. Therefore $T_1\cdots T_n=0$.

The condition that these linear operators are pair-wise commuting is necessary, since for instance we may take $V=\mathbb{R}^2$, $K=\mathbb{R}$, T_1 and T_2 the left multiplication by the matrices $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$. Clearly $T_1^2=0=T_2^2$, so that T_1,T_2 are nilpotent. We see that T_1T_2 is left multiplication by $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ (in particular is non-zero), while T_2T_1 is left multiplication by $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$.

Example 5.2.3 (Fall 2020 Day 2)**.** Suppose R is a commutative ring with unity in which every proper ideal is prime. Show that R is a field.

First, notice that the zero ideal is proper and thus prime. Hence R is an integral domain. Now let $x\in R$ be non-zero, and consider the ideal (x^2). If $(x^2)=R$, then $x^2y=1$ for some $y\in R$, and so x is invertible with inverse xy. Otherwise, (x^2) is a proper ideal and thus prime, and since $x\cdot x\in(x^2)$ it follows that $x\in(x^2)$. Thus $x=x^2y$ for some $y\in R$. As R is an integral domain, it follows that $xy=1$, and once again x is invertible. Therefore R is a field.

Example 5.2.4 (Fall 2021 Day 3)**.** Suppose R is a commutative ring with unit, I an ideal of R, and M a finitely generated R-module. If $IM=M$, prove there exists $r\in R$ such that $r-1\in I$ and $rM=0$.

Let $m_1,...,m_k$ be generators for M. Since Let $r_{ij}\in I$ be such that $m_i=\sum_j r_{ij}m_j$ for all $1\leq i\leq k$. Setting $m=(m_i)_{1\leq i\leq k}$ and $R=(r_{ij})_{1\leq i,j\leq n}$, we have $Rm=m$, and so $(I-R)m=0$. By multiplying throughout by the matrix $\operatorname{adj}(I-R)$, it follows that $\det(I-R)Im=0$. Setting $r=\det(I-R)$, it follows that $rM=0$. Also since all entries of R are in I, we have $\det(I-R)\equiv\det I=1\,(\operatorname{mod} I)$. Thus $r\in R$ satisfies $r-1\in I$ and $rM=0$.

5.3 Fields and Galois Theory

We denote the group of automorphisms of L over K (i.e. fixing K pointwise) by $Gal(L/K)$ even if L is not a Galois extension of K. We also set $Aut(K):=Gal(K/P)$ where P is the prime sub-field of K. If $f\in K[x]$ is separable, then we denote by $Gal(f/K)$ the Galois group of the splitting field of f over K.

If $H\leq Gal(L/K)$ is a subgroup, then the fixed field of H (elements $\alpha\in L$ such that $\sigma\alpha=\alpha$ for all $\sigma\in H$) is denoted by L^H or $Fix(H)$.

Definition. Types of Fields:

- *Algebraically Closed:* K is algebraically closed if every polynomial in $K[x]$ has a root in K.
- *Perfect:* K is perfect if every irreducible polynomial in $K[x]$ is separable over K. K is perfect iff char $K=0$, or char $K=p$ and for every $\alpha\in K$ there exists $\beta\in K$ with $\beta^p=\alpha$.

Definition. Suppose L/K is a field extension. The following are various types of extensions:

- *Algebraic:* An element $\alpha\in L$ is algebraic over K if there exists a polynomial $f\in K[x]$ such that $f(\alpha)=0$. The extension L/K is algebraic if all elements of L are algebraic over K.
- *Transcendental:* If L is not algebraic over K.
- *Simple:* If $L=K(\alpha)$ for some $\alpha\in L$.
- *Splitting Field for $f\in K[x]$:* if L is the smallest field extension of K in which f factors into linear factors. Splitting fields exist and are unique (up to non-unique non-canonical isomorphism). The splitting field of $f\in K[x]$ has degree at most $(\deg f)!$, and must be divisible by $\deg f$.
- *Normal:* L is a normal field extension of K if every irreducible polynomial $f\in K[x]$ that has a root in L splits completely in L. If L is a finite normal extension of K, then it must be the splitting field of some polynomial over K.
- *Separable:* A polynomial $f\in K[x]$ is separable if f has no multiple roots iff $\gcd(f,f')=1$. Otherwise, f is inseparable. The field L is separable over K if every element of L is the root of a separable polynomial over K iff the minimal polynomial in $K[x]$ of each element of L is separable. Any finite extension over a perfect field is separable.
- *Galois:* L is Galois over K if $|Gal(L/K)|=[L:K]$.
- *Abelian:* L is an abelian extension of K if L/K is Galois and $Gal(L/K)$ is an abelian group.

- *Root Extensions:* L is a root extension of K if there exists a chain of field extensions $K = K_0 \subset K_1 \subset \cdots \subset K_s = K$ where for each $i = 1, ..., s$, there exists $\alpha_i \in K_{i-1}$ such that $K_i = K_{i-1}(\sqrt[n]{\alpha_i})$. Any element of a root extension of K is said to be *expressible by radicals.* A polynomial $f \in K[x]$ is *solved by radicals* if each root of f is expressible by radicals.

- *Purely Inseparable:* an algebraic extension L/K is purely inseparable if the minimal polynomial over K of every element of L has only one distinct root in L. This is only possible if char $K = p > 0$. L/K is purely inseparable iff every separable element of L over K is contained in K iff for each $\alpha \in E$ there exists $n \in \mathbb{Z}_{\geq 0}$ with $\alpha^{p^n} \in F$.

Definition. Suppose K/F is any finite extension, and suppose $\alpha \in K$. Then multiplication by α induces an F-linear operator of the F-vector space K (which is invertible iff $\alpha \neq 0$). The determinant of this linear operator is called the *norm* $N_{K/F}(\alpha)$ *of* α *from* K *to* F, and the trace of this linear operator is called the *trace* $Tr_{K/F}(\alpha)$ *of* α *from* K *to* F. Clearly $N_{K/F} : K^* \to F^*$ is a multiplicative group homomorphism, and $Tr_{K/F} : K \to F$ is an additive group homomorphism.

- If $\varphi : F \to F'$ are homomorphisms of fields, then either φ is identically zero or it is injective.

- Let K_1, K_2 be two extensions of K, and let K_1K_2 be their compositum (i.e. smallest field extension of K containing both K_1, K_2). If $K_1 = K(\alpha_1, ..., \alpha_m)$ and $K_2 = K(\beta_1, ..., \beta_n)$ then $K_1K_2 = K(\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n)$. Also, $[K_1K_2 : K] \leq [K_1 : K] \cdot [K_2 : K]$, with equality iff an F-basis for one of the fields remains linearly independent over the other field. For instance, if $[K_1 : K]$ and $[K_2 : K]$ are co-prime integers, then $[K_1K_2 : K] = [K_1 : K][K_2 : K]$.

- An element $\alpha \in \mathbb{R}$ is obtainable from F by compass and straight-edge constructions iff $[F(\alpha) : F] = 2^k$ for some $k \geq 0$.

- Suppose $\varphi : F \to F'$ is an isomorphism, and suppose $f \in F[x]$. Let $\bar{f} = \varphi(f) \in F'[x]$. Let E, E' be the splitting fields of f and $\bar{f}$ respectively. Then, there exists a field isomorphism $\Phi : E \to E'$ such that $\Phi(F) = F'$ and $\Phi|_F = \varphi$.

- If L/K and K/F are both algebraic, then L/F is also algebraic.

- Suppose char $K = p$ and $f \in K[x]$ is irreducible. Then, there exists a unique integer $k \geq 0$ and a unique irreducible separable polynomial $f_{sep} \in K[x]$ such that $f(x) = f_{sep}(x^{p^k})$. The degree of f_{sep} is called the *separable degree* of f, and the integer p^k is called the *inseparable* degree of p.

- (*Primitive Element Theorem*) A finite separable extension must be simple.

- A finite extension K/F is simple iff there are only finitely many subfields of K containing F.

- If $\alpha \in L$ is algebraic over K, then for any $\sigma \in Gal(L/K)$ the element $\sigma\alpha \in L$ is also a root of the minimal polynomial of α over K. Thus $Gal(L/K)$ permutes the roots of irreducible polynomials (however, it is not necessary that every permutation corresponds to an automorphism of L over K).

- Suppose $L = K(\alpha)$ is a finite simple extension and $p \in K[x]$ is the minimal polynomial of α. Let R_p denote the set of roots of p in L. Then the map $Gal(L/K) \to R_p, \sigma \mapsto \sigma(\alpha)$ is a bijection.

- If an irreducible polynomial splits in L, then $Gal(L/K)$ acts transitively on the roots of f. If L is the splitting field of f, then $Gal(L/K)$ acts freely and transitively on the roots of f.

- If $F \subseteq K \subseteq L$ is any tower of field extensions, then $Gal(L/K) \leq Gal(L/F)$. If $H_1 \leq H_2 \leq Aut(K)$ are any subgroups, then $Fix(H_2) \subseteq Fix(H_1)$.

- If E is the splitting field over F of some polynomial $f \in F[x]$, then $|Gal(E/F)| \leq [E : F]$ with equality if f is separable over F. More generally, $|Gal(K/F)| \leq [K : F]$ for any finite field extension K/F.

- (*Artin's Theorem*) Suppose $G \leq Aut(K)$ is any finite subgroup, and let $F = K^G$ is the fixed field of G. Then, $[K : F] = |G|$ and $Gal(K/F) = G$.

- More generally, if K/F is *any* finite field extension, then $|Gal(K/F)| \leq [K : F]$ with equality iff F is the fixed field of $Gal(K/F)$.

- (*Characterization of Galois Extensions*) Suppose K/F is any finite field extension. Then the following statements are equivalent.

- K/F is Galois, i.e. $|Gal(K/F)| = [K : F]$.
- F is the fixed field of $Gal(K/F)$.
- K/F is normal and separable.
- L is the splitting field over K of some separable polynomial in $K[x]$.

- (*Fundamental Theorem of (Finite) Galois Theory*) Suppose K/F is a Galois extension, and set $G = Gal(K/F)$. Then, there is an inclusion reversing lattice isomorphism between fields E with $F \subset E \subset K$ and subgroups H of G, given by the correspondences $E \mapsto Gal(K/E)$ and $H \mapsto K^H$. Moreover, for a fixed subfield $F \subset E \subset K$ with the corresponding subgroup $H \leq G$, we have
  - K/E is Galois with Galois group $Gal(K/E) = H$;
  - $[E : F] = [G : H]$;
  - The isomorphisms of E (into a fixed algebraic closure of F containing K) which fix F are in one-to-one correspondence (as sets only) with the set of left cosets of H in G; and
  - E/F is a Galois extension iff H is normal in G. Furthermore, if this holds, then $Gal(E/F) \cong G/H$.

  If E_1 and E_2 are subfields of K containing F with corresponding subgroups $H_1, H_2 \leq G$, then
  - the subfield $E_1 \cap E_2$ of K (containing F) is the subgroup of G generated by H_1 and H_2; and
  - the compositum E_1E_2 in K corresponds to $H_1 \cap H_2 \leq G$.

- Suppose K/F is an arbitrary finite extension, and suppose L is a Galois extension of F containing K with Galois group G. Let $H = Gal(L/K) \leq G$. Then, $N_{K/F}(\alpha) = \prod_{\sigma \in G/H} \sigma(\alpha)$, where the product is over a set of H-coset representatives of in G. Similarly, $Tr_{K/F}(\alpha) = \sum_{\sigma \in G/H} \sigma(\alpha)$. In particular, $N_{K/F} = \prod_{\sigma \in Gal(K/F)} \sigma\alpha$ and $Tr_{K/F} = \sum_{\sigma \in Gal(K/F)} \sigma\alpha$ if K/F is Galois.

  If $m = x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0 \in F[x]$ is the minimal polynomial of $\alpha \in K$, then d divides $n := [K : F]$, and we have the formulae $N_{K/F}(\alpha) = (-1)^n a_0^{n/d}$ and $Tr_{K/F}(\alpha) = -\frac{n}{d} a_{d-1}$. In particular, $N_{K/F}(a\alpha) = a^n N_{K/F}(\alpha)$ and $Tr_{K/F}(a\alpha) = aTr_{K/F}(\alpha)$ for any $a \in F$.

  In fact, we have $\prod_\sigma (x - \sigma\alpha) = (m(x))^{n/d}$ where the product is over a set of H-coset representatives of in G.

- (*Hilbert's Theorem 90*) Suppose K is a Galois extension of F with cyclic Galois group of order n generated by σ. If $\alpha \in K$ satisfies $N_{K/F}(\alpha) = 1$, then $\alpha = \beta/\sigma\beta$ for some non-zero $\beta \in K$. If $\alpha \in K$ satisfies $Tr_{K/F}(\alpha) = 0$, then $\alpha = \beta - \sigma\beta$ for some $\beta \in K$.

- Fix a prime p, and let $\mathbb{F}_q$ denote *the* finite field with $q = p^n$ elements $n = [\mathbb{F}_q : \mathbb{F}]$ (finite fields of given order are unique up to non-unique isomorphism). The extension $\mathbb{F}_q/\mathbb{F}_p$ ($\mathbb{F}_p = (\mathbb{Z}/p\mathbb{Z})$) is a Galois extension with cyclic Galois group canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$, given by the splitting field of $x^{p^n} - x \in \mathbb{F}_p[x]$. The generator of the Galois group is the *Frobenius automorphism* $\sigma_p : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}, \alpha \mapsto \alpha^p$. The subfields of $\mathbb{F}_{p^n}$ are precisely $\mathbb{F}_{p^d}$ for $d|n$. The polynomials $x^{p^n} - x$ are precisely the product (in $\mathbb{F}_p[x]$) of all distinct irreducible polynomials in $\mathbb{F}_p[x]$ with degree d as d runs through divisors of n. The number of irreducible polynomials of degree n over $\mathbb{F}_p$ is $\frac{1}{n}\sum_{d|n} \mu(d)p^{n/d}$ where μ is the Möbius μ-function.

- The cyclotomic extensions $\mathbb{Q}(\zeta_n)$ (ζ_n a primitive n'th root of unity) are Galois extensions of degree n, given by the splitting field of the n'th cyclotomic polynomial $\Phi_n(x) = \prod_{d \in (\mathbb{Z}/n\mathbb{Z})^*}(x - \zeta_n^d)$. The n'th cyclotomic polynomial is a monic irreducible polynomial in $\mathbb{Z}[x]$, and $x^n - 1 = \prod_{d|n} \Phi_d(x)$. We have $\deg \Phi_n = \varphi(n)$ (Euler's totient function). The Galois group $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$ via $a \,(\mathrm{mod}\, n) \mapsto \sigma_a$, where σ_a is the unique automorphism of $\mathbb{Q}(\zeta_n)$ satisfying $\sigma_a(\zeta_n) := \zeta_n^a$.

- Suppose K/F is Galois and F'/F is any extension. Then KF' is a Galois extension of F', with Galois group $Gal(KF'/F') = Gal(K/K \cap F')$ (which in turn is isomorphic to a subgroup of $Gal(K/F)$). In particular,
$$[KF' : F] = \frac{[K : F][F' : F]}{[K \cap F' : F]}.$$

- Suppose K_1, K_2 are two Galois extensions of F. Then the intersection $K_1 \cap K_2$ is Galois over F. Also, the compositum K_1K_2 is Galois with Galois group isomorphic to the subgroup $H = \{(\sigma, \tau) : \sigma|_{K_1 \cap K_2} = \tau|_{K_1 \cap K_2}\}$ of the direct product $Gal(K_1/F) \times Gal(K_2/F)$.

If K is Galois over F and $Gal(K/F)$ is the direct product of two groups G_1 and G_2, then K is the compositum of two Galois extensions K_1 and K_2 of F with $K_1 \cap K_2 = F$ and $Gal(K_i/F) = G_i$ for $i = 1, 2$.

In particular, if E/F is any finite separable extension, then there exists a unique (up to non-unique isomorphism) Galois extension K over F such that $E \subset K$ and is minimal in the sense that in any fixed algebraic closure of K, every Galois extension of F containing E contains K. Such a field extension K of F containing E is the *Galois closure of E over F*.

- (*Kronecker-Weber Theorem*) Suppose $K/\mathbb{Q}$ is a finite extension. Then $K/\mathbb{Q}$ is abelian iff K is contained in some cyclotomic extension of $\mathbb{Q}$. Moreover, for every finite abelian group G there exists a subfield K of a cyclotomic field with $Gal(K/\mathbb{Q}) = G$.

- (*Fundamental Theorem of Symmetric Functions*) Any symmetric rational function $f(x_1, ..., x_n)$ (i.e.

$$f(x_{\sigma 1}, ..., x_{\sigma n}) = f(x_1, ..., x_n)$$

for all $\sigma \in S_n$) is a rational function in the elementary symmetric functions $s_1, ..., s_n$, where

$$s_k := \sum_{I \subset \{1,...,n\}, |I|=k} \prod_{i \in I} x_i.$$

- Suppose $f \in F[x]$, and let $D(f)$ be the *discriminant* of f given by $D(f) = \prod_{i<j}(\alpha_i - \alpha_j)^2$ where $\alpha_1, ..., \alpha_n$ are the roots of f in a splitting field of f over F. The discriminant is always an element of F and is a polynomial in the coefficients in f; also, $D(f) \neq 0$ iff f is separable. Since $Gal(f/F)$ acts as permutations on the roots of f, we can embed $Gal(f/F)$ into S_n ($n = \deg f$). We have the following two facts about $Gal(f/F)$ as a subgroup of S_n:

  1. The Galois group of separable $f \in F[x]$ (when considered as a subgroup of S_n) is a subgroup of A_n iff $x^2 - D(f) \in F[x]$ splits.
  2. Suppose $F = \mathbb{Q}$ and $f \in \mathbb{Z}[x]$. For any prime $p \nmid D(f)$, the Galois group of f over $\mathbb{Q}$ contains an element with cycle decomposition $(n_1, ..., n_k)$ where $n_1, ..., n_k$ are the degrees of the irreducible factors of f reduced modulo p.

- Suppose char $K = 0$. A polynomial $f \in K[x]$ can be solved by radicals iff $Gal(f/K)$ is a solvable group, i.e. there exists a chain of subgroups $1 = G_s \leq G_{s-1} \leq \cdots \leq G_0 = Gal(f/K)$ where G_i/G_{i+1} is cyclic, $i = 0, ..., s-1$. Moreover, these subgroups correspond to the chain $K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_s$ where K_s is the splitting field of f over K, there exists an element $\alpha_i \in K_i$ such that $K_i = K_{i-1}(\sqrt[n_i]{\alpha_i})$ for $1 \leq i \leq s$ with $n_i = |G_i/G_{i+1}|$, and $K_i = K_s^{G_i}$.

- Now suppose E/F is transcendental. A subset $\{a_1, ..., a_n\} \subset E$ is *algebraically independent* over F if for all $f \in F[x_1, ..., x_n]$, we have $f(a_1, ..., a_n) \neq 0$. An arbitrary subset S of E is algebraically independent if every finite subset of S is algebraically independent. A *transcendence base* for E/F is a maximal algebraically independent subset (over F) of E. Every transcendental extension E/F has a transcendence base, and any two transcendence bases have the same cardinality (called the *transcendence degree of* E/F).

- The compositum of two separable (resp purely inseparable) extension is also separable (resp purely inseparable). Thus, for any algebraic extension E/F, there is a unique field E_{sep} such that $F \subseteq E_{sep} \subseteq E$ such that E_{sep} is separable over F and E is purely inseparable over E_{sep}. The field E_{sep} is called the *maximal separable subextension* of E/F. E_{sep} is the set of elements of E which are separable over F.

Example 5.3.1 (Fall 2020 Day 3)**.** Fix a prime p.

1. Let F be any field with characteristic p, and consider the polynomial $P(X) = X^p - X - c \in F[X]$ where $c \in F$ is such that $P(X)$ does not have a root in F. Prove that P is irreducible, and that $F[X]/\langle P \rangle$ is Galois over F with Galois group $\mathbb{Z}/p\mathbb{Z}$.

2. Let $Q \in \mathbb{Z}[X]$ be a monic polynomial of degree p with exactly $p - 2$ real roots such that $Q(X) \equiv X^p - X - c \pmod p$ for some non-zero $c \in \mathbb{Z}/p\mathbb{Z}$. Show that the Galois group of Q is S_p.

Note first that P is separable since it isn't a polynomial in X^p (or equivalently, $P' = -1$ so that $\gcd(P, P') = 1$). Let K be a splitting field of P over F, and let $\alpha \in K$ be a root of P. Then, for any $a \in \mathbb{F}_p$ (the finite field with p-elements), we see that $P(\alpha + a) = (\alpha + a)^p - (\alpha + a) - c = \alpha^p - \alpha - c + a^p - a = 0$ (since $a^p = a$ for all

$a \in \mathbb{F}_p$). Hence, all roots of P are of the form $\alpha + a$ for $a \in \mathbb{F}_p$. In particular, we see that $K = F(\alpha)$ is a Galois extension. Now, let $\sigma \in Gal(K/F)$ be non-trivial; such a Galois group element exists since P does not have a root in F and so $K \neq F$. Since $K = F(\alpha)$, we must have $\sigma(\alpha) \neq \alpha$, and since σ permutes the roots of P, it follows that $\sigma(\alpha) = \alpha + a$ for some $a \in \mathbb{F}_p$. By taking a power of σ corresponding to an inverse k of a in $\mathbb{Z}$ modulo p, and noting that $\sigma^k(\alpha) = \alpha + ka = 1$, we may suppose WLOG that $a = 1$. Then we have p different elements of $Gal(K/F)$ corresponding to $\sigma^0 = Id, \sigma, \sigma^2, ..., \sigma^p$, and moreover the action of $\langle\sigma\rangle$ and thus $Gal(K/F)$ on the roots of P is transitive. This implies that P is irreducible, and that $F[X]/\langle P\rangle \cong F(\alpha) = K$ is Galois over F. That the Galois group is $\mathbb{Z}/p\mathbb{Z}$ follows from the fact that $\deg P = p$ so that $|Gal(F(\alpha)/F)| = [F(\alpha) : F] = p$.

Now, notice that Q is irreducible in $\mathbb{Z}[X]$ since its reduction modulo p is irreducible by part (a); here, we take $F = \mathbb{F}_p$ and notice that $a^p - a = 0$ for all $a \in \mathbb{F}_p$ and so any non-zero $c \in \mathbb{Z}/p\mathbb{Z}$ satisfies the requirements of part (a). By Gauss' Lemma, noting that Q is monic, it follows that Q is irreducible in $\mathbb{Q}[X]$. Since Q has exactly $p-2$ real roots, it has a complex conjugate pair of roots, and so complex conjugation belongs to the Galois group of Q. Moreover, the permutation action induced by complex conjugation on the roots of Q simply swaps the two complex roots. Hence, identifying the Galois group of Q as a subgroup of S_p, the Galois group contains a transposition of S_p. Also, as $[\mathbb{Q}[X]/\langle Q\rangle : \mathbb{Q}] = p$, it follows that p divides the degree of the splitting field of Q over $\mathbb{Q}$, and thus p divides the order of the Galois group of Q. By Cauchy's Theorem, it follows that the Galois group of Q contains an element of order p. Since all elements of S_p of order p are necessarily cyclic, it follows that the Galois group of Q (regarded as a subgroup of S_p) contains a transposition and a p-cycle. As p is prime, the Galois group must then be all of S_p.

Example 5.3.2 (Fall 2019 Day 3)**.** Suppose p and q are distinct primes. Set $f_p(x) := \frac{x^p-1}{x-1}$. Prove that f_p is irreducible modulo q iff q is a primitive residue of p (i.e. q generated $(\mathbb{Z}/p\mathbb{Z})^*$). Show also that f_7 is the product of two irreducible cubic factors modulo 23.

We consider $f_p \in \mathbb{F}_q[x]$. Note first that f_p is separable since $(x^p - 1)' = px^{p-1}$ is separable as p and q are distinct. Let K be the splitting field of f over $\mathbb{F}_q$; it is a Galois extension of $\mathbb{F}_q$. Let $\sigma \in Gal(K, \mathbb{F}_q)$ be the Frobenius automorphism $\sigma(x) = x^q$. Then σ generates $Gal(K, \mathbb{F}_q)$. Now, if ζ is a root of f_p, then we see that ζ^k is a root of f_p for all $k \in \mathbb{Z}, p \nmid k$; moreover these are all the roots of f_p. In particular, as $q \in (\mathbb{Z}/p\mathbb{Z})^*$, $\sigma^k(\zeta) = \zeta^{q^k}$ is a root of f_p for all $k \in \mathbb{N}$. Thus q is a primitive residue of p iff the orbit of a root ζ of f_p under $Gal(K/\mathbb{F}_q)$ contains all roots of f_p iff f_p is irreducible.

To factor f_7, notice that $23^3 \equiv 2^3 \equiv 1 \pmod 7$ while $23^i \not{e}quiv 1 \pmod 7$ for $i = 1, 2$, and so the orbits of a root of f_7 under $Gal(K/\mathbb{F}_{23})$ have 3 roots each. Hence, f_7 has two irreducible factors of degree 3 each. To compute these factors, let ζ be a fixed root of f_7. Then $f_7(x) = \prod_{i=1}^{6}(x - \zeta^i)$.

While the question doesn't ask us to factorize it, we still find the factorization. The two orbits are $\{\zeta, \zeta^2, \zeta^4\}, \{\zeta^3, \zeta^5, \zeta^6\}$. Let $a = \zeta + \zeta^2 + \zeta^4 \in \mathbb{F}_q$ and $b = \zeta^3 + \zeta^5 + \zeta^6$. Since $\zeta \cdot \zeta^2 \cdot \zeta^4 = 1$ and $\zeta \cdot \zeta^2 + \zeta \cdot \zeta^4 + \zeta^2 \cdot \zeta^4 = b$, and similarly for the other orbit, it follows that

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = f_7(x) = (x^3 - ax^2 + bx - 1)(x^3 - bx^2 + ax - 1).$$

Comparing the coefficients of x and x^2, it follows that $a + b = -1$ and $ab + a + b = 1$, and hence a, b are the roots of $x^2 + x + 2$ in $\mathbb{F}_{23}$. The discriminant is $1 - 4 \cdot 2 = -7 = 16$, and so the roots are $\frac{1}{2}(-1 \pm 4) = -10, 9$. Therefore

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = (x^3 + 10x^2 + 9x - 1)(x^3 - 9x^2 - 10x - 1).$$

Example 5.3.3 (Fall 2021 Day 2)**.** Prove that the Galois group G of $f = x^4 + 1$ over $\mathbb{Q}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Is f irreducible modulo p for some prime p?

It is clear that f is irreducible. Let ζ be one of the roots of f in $\mathbb{C}$. Then notice that $\zeta^3, \zeta^5, \zeta^7$ are all roots of f, and so the splitting field of f is $\mathbb{Q}[\zeta]$ which is a degree 4 Galois extension of $\mathbb{Q}$. Hence G is a group of order 4. Since G acts transitively on roots of f, there exist $\sigma, \tau \in G$ such that $\sigma(\zeta) = \zeta^3$ and $\tau(\zeta) = \zeta^7$. Then notice that $\sigma^2(\zeta) = \zeta = \tau^2(\zeta)$, and so σ, τ are distinct elements of order 2 in G. Hence $G \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Now suppose p is an odd prime, and let $\mathbb{F}_q$ be the splitting field of f, where $q = p^k$, $k = [\mathbb{F}_q : \mathbb{F}_p]$. Let $\sigma(\alpha) = \alpha^p$ be the Frobenius automorphism over $\mathbb{F}_p$. Then, $Gal(\mathbb{F}_q/\mathbb{F}_p)$ is a cyclic group generated by σ, and σ has order k. Now, if ζ is any root of f, then $\sigma^2(\zeta) = \zeta^{p^2} = \zeta$ since $p^2 \equiv 1 \pmod 8$ for all odd numbers p. Since $\mathbb{F}_q$ is generated by the roots of f, and since σ^2 fixes all of the roots of f pointwise, it follows that σ has order 2 and so $k = 2$. However, if f were irreducible, then there would a subfield of $\mathbb{F}_q$ isomorphic to $\mathbb{F}_p[x]/\langle f\rangle$, the latter of which is a degree 4 extension of $\mathbb{F}_p$. This contradicts the fact that $\mathbb{F}_q$ is a degree 2 extension of $\mathbb{F}_p$. Therefore f is reducible modulo every prime $p > 2$.

5.4 Representation Theory

Throughout, we suppose all groups are finite groups (with '1' denoting the identity), and all vector spaces are finite dimensional vector spaces over $\mathbb{C}$, unless otherwise specified. $GL(V)$ denotes the general linear group of

V. Throughout, $\bar{\bullet}$ denotes complex conjugation in $\mathbb{C}$.

We denote by $\mathbb{C}[G]$ the $\mathbb{C}$ vector space for which the elements of G form a basis; it is the space of formal linear combinations $\sum_{g\in G} c_g g$ where $c_g \in \mathbb{C}$. Its dimension is clearly $|G|$.

5.4.1 Definitions

Definition. A *(finite linear) representation* of a finite group G is a pair (V, φ) where V is a finite finite dimensional vector space over $\mathbb{C}$ and $\varphi : G \to GL(V)$ is a group homomorphism. The image of $g \in G$ is denoted by $\varphi_g : V \to V$. V is the *representation space* of G. The dimension of V is the *degree* of the representation.

A representation is *faithful* if ρ is injective, i.e. $\rho_g = I_V$ iff $g = 1$.

Two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ are *similar* or *isomorphic* if there exists a linear isomorphism $T : V \to V'$ such that $T \circ \rho_g = \rho'_g \circ T$ for all $g \in G$. We write $(V, \rho) \cong (V', \rho')$

Suppose $\rho : G \to GL(V)$ is a representation, and $W \subset V$ a subspace. W is *stable or invariant under the action of* G if $\rho_g(W) \subseteq W$ for all $g \in G$. The restriction $\rho^W : G \to GL(W)$ sending g to $\rho_g|_W$ is also a representation, and so W *is a sub-representation of* (V, ρ).

The *trivial* or *unit* representation of G is the representation $G \to \mathbb{C}^*, g \mapsto 1$. The degree is clearly 1.

The *regular representation* is the representation $G \to GL(\mathbb{C}[G])$ where $g \in G$ is sent to the linear map induced on $\mathbb{C}[G]$ by left-multiplication of g. The degree of this representation is clearly $|G|$. The subspace W spanned by $\sum_{g\in G} g \in \mathbb{C}[G]$ is a sub-representation of the regular representation isomorphic to the unit representation.

Suppose G acts on a finite set X. The *permutation representation* associated to X is the representation $G \to GL(V)$ where V is a vector space of all linear combinations $\sum_{x\in X} c_x x$ for $c_x \in \mathbb{C}$, and the action of G on X is linearly extended to all of V (i.e. the image of $g \in G$ under the permutation representation sends $x \in X$ to $g \cdot x$). The degree of this representation is the size of X.

The representation theory of compact groups (i.e. Hausdorff second-countable topological groups that are compact) is analogous to the theory of representations of finite groups. Every compact group G has a unique left and right-invariant Borel measure μ (i.e. $\int_G f(g)d\mu(g) = \int_G f(agb)dmu(g)$ for all $a, b \in G$) such that $\mu(G) = 1$; this measure is called the *Haar measure of* G. For example, any finite group G equipped with the discrete topology has the Haar measure $\mu(S) = |S|/|G|$ for all subsets $S \subset G$. We let $L^2(G)$ denote the space of complex valued square integrable functions on G; if G is finite, then clearly $L^2(G) = \text{Map}(G, \mathbb{C})$.

A finite representation of a compact group G is a finite-dimensional $\mathbb{C}$-vector space V and a continuous homomorphism $\rho : G \to GL(V)$ (where $GL(V)$ is equipped with the topology of $GL(n, \mathbb{C})$). If H is instead a Hilbert space, then one can define a continuous homomorphism from $G \to O(H)$; however such a representation is always a direct sum of finite dimensional Hilbert space representations. We thus always assume that representations of compact groups are finite. From now on, we denote the Haar measure of G as dg. Thus, if G is finite, then

$$\int_G f(g)dg := \frac{1}{|G|}\sum_{g\in G} f(g).$$

The unit representation still works for G compact. The regular representation of G is the Hilbert space $L^2(G, \mathbb{C})$ with group action given by $\rho_g : L^2(G, \mathbb{C}) \to L^2(G, \mathbb{C})$ where $\rho_g(f)$ is the L^2 function $h \mapsto f(g^{-1}h)$ for all $h \in G$. The regular representation for an infinite compact group is not finite dimensional; however, as $L^2(G, \mathbb{C})$ is separable it does have countable dimension.

Lemma-Definition. Suppose $\rho : G \to GL(V)$ is a representation of the compact group G, and $W \subset V$ is a subspace stable under the action of G. Then there exists a subspace W' of V such that $V = W \oplus W'$ (as vector spaces) and W' is stable under G. In this case, the action of G on W and W' completely determines the action of G on V. In such a case, we say that the representation (ρ, V) is the *direct sum* of the representations (ρ^W, W) and $(\rho^{W'}, W')$, written $V = W \oplus W'$.

Example 5.4.1 (Fall 2021 Day 1)**.** We prove the above lemma. Let notation be as above.

Pick any subspace W_0 such that $V = W \oplus W_0$ as vector spaces; this is possible by simply extending any basis of W to a basis for V. Let $p_0 : V \to W$ be the projection with respect to this direct sum, i.e. p_0 is the unique map satisfying $p_0|_W = I_W$ and $p_0|_{W_0} = 0$. Consider the map

$$p = \frac{1}{|G|}\sum_{g\in G} \rho_g \circ p_0 \circ \rho_g^{-1} : V \to V.$$

Notice first that $\rho_g \circ p \circ \rho_g^{-1} = p$ and that $Im(p) = W$ since W is a sub-representation of V and since p_0 maps V to W. Also, as W is a sub-representation of V, we have $\rho_g^{-1} w = \rho_{g^{-1}} w \in W$ for all $w \in W$, and so

$p_0 \circ \rho_g^{-1}(w) = \rho_g^{-1} w$. It follows that $\rho_g \circ p_0 \circ \rho_g^{-1}|_W = I_W$ for all $g \in G$, and so $p|_W = I_W$. Let $W' = \ker p$. The identity $\rho_g \circ p = p \circ \rho_g$ implies that $\ker p$ is ρ_g invariant for all $g \in G$. Hence W' is a sub-representation of V. Since $W = Im(p)$ with $p|_W = I_W$, and since $W' = \ker p$, it then follows that $V = W \circ W'$.

Definition. A representation (V, ρ) of a compact group G is *irreducible* or *simple* if $V \neq 0$ and there are no non-zero proper subspaces of V invariant under G, i.e. iff V is not the direct sum of two non-zero representations. Every representation is a direct sum of irreducible representations. It is known that for compact G, any irreducible representation is necessarily finite.

Definition. Consider two representations (W_i, ρ^i) of G. Let $V = W_1 \otimes_{\mathbb{C}} W_2$ be the tensor product of these two vector spaces. The representation $\rho : G \to GL(V)$ given by $\rho_g : V \to V, w_1 \otimes w_2 \mapsto \rho_g^1(w_1) \otimes \rho_g^2(w_2)$ is called the *tensor product of the representations* (W_1, ρ^1) *and* (W_2, ρ^2). We write $\rho = \rho^1 \otimes \rho^2$.

Suppose (V, ρ) is a representation of G. Then the linear involution $\theta : V \otimes V \to V \otimes V$ on the vector space $V \otimes V$ given by $\theta(x \otimes y) = y \otimes x$ decomposes $V \otimes V$ into $\ker(\theta - I_V)$ and $\ker(\theta + I_V)$, both of which are invariant under G. The *symmetric square* $Sym^2(V)$ and *alternating square* $Alt^2(V)$ are the sub-representations $\ker(\theta - I_V)$ and $\ker(\theta + I_V)$ respectively. $Sym^2(V)$ is spanned by elements of the form $x \otimes y + y \otimes x$ while $Alt^2(V)$ is spanned by elements of the form $x \otimes y - y \otimes x$. The degree of $Sym^2(V)$ and $Alt^2(V)$ are $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$ respectively.

Definition. Suppose (V, ρ) is a representation of G. Let V' be the dual space of V, and let $\langle , \rangle : V \times V' \to \mathbb{C}$ denote the natural pairing. The unique representation $\rho' : G \to GL(V')$ given by $\langle \rho_g v, \rho'_g f \rangle = \langle v, f \rangle$ for all $v \in V, f \in V', g \in G$ is called the *dual* or *contragredient* representation of ρ.

5.4.2 Character Theory

Definition. Suppose (V, ρ) is a finite representation of the compact group G. The map $\chi_\rho : G \to \mathbb{C}$ given by $\chi_\rho(g) = \text{Trace}(\rho_g)$ is called the *character* of the representation G.

The character is said to be *irreducible* if it is the character of an irreducible representation.

Clearly, characters cannot be defined for infinite dimensional V.

Basic properties of characters (here, we denote $\chi = \chi_\rho$):

1. If ρ is a degree 1 representation, then $\chi_\rho = \rho$.
2. $\chi(1)$ is the degree of the corresponding representation.
3. $\chi(g^{-1}) = \overline{\chi(g)}$ and $\chi(ghg^{-1}) = \chi(h)$ for all $g, h \in G$. In particular, $\chi : G \to \mathbb{C}$ induces a set map from the set of conjugacy classes of G to $\mathbb{C}$.
4. $\chi_{\rho^1 \oplus \rho^2} = \chi_{\rho^1} + \chi_{\rho^2}$ and $\chi_{\rho^1 \otimes \rho^2} = \chi_{\rho^1} \chi_{\rho^2}$.
5. If χ_s^2 and χ_a^2 denotes the character of the symmetric square and alternating square of a representation ρ respectively, and if χ is the character of ρ, then $\chi_s^2(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right)$, $\chi_a^2(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$, and $\chi(g)^2 = \chi_s^2(g) + \chi_a^2(g)$ for all $g \in G$.
6. The character of the dual representation to ρ is $\overline{\chi_\rho}$.
7. The number of distinct irreducible characters of a finite group is finite. If G is compact, then the set of all irreducible characters (up to isomorphism) is countable.
8. For any character χ of a finite group G, the numbers $\chi(g)$ are algebraic integers for all $g \in G$.

Theorem 5.4.2 (Schur's Lemma)**.** *Suppose $\rho^i : G \to GL(V_i)$ are two irreducible representations of the compact group G. Suppose $f : V_1 \to V_2$ is a linear map such that $\rho_g^2 \circ f = f \circ \rho_g^1$ for all $g \in G$. Then, either $f = 0$, or $(V_1, \rho^1) \cong (V_2, \rho^2)$ and f is a scalar multiple of the identity under the identification $V_1 = V_2$, $\rho^1 = \rho^2$.*

Proof of Schur's Lemma; see also Fall 2018 Day 1. The identity $\rho_g^2 \circ f = f \circ \rho_g^1$ implies that $\ker f$ and $f(V_1)$ are both G-invariant under ρ^1 and ρ^2 respectively. Since ρ^1 is irreducible, either $\ker f = V_1$ (and thus $f \equiv 0$), or f is injective. Similarly, irreducibility of ρ^2 implies that $f(V_1) = 0$ (i.e. $f \equiv 0$) or $f(V_1) = V_2$. Thus f is either the zero map, or (V_1, ρ^1) and (V_2, ρ^2) are isomorphic.

Now suppose $f : V \to V$ is an automorphism such that $\rho_g \circ f = f \circ \rho_g$ for all $g \in G$ (where $\rho_g : G \to GL(V)$). Let λ be an eigenvalue of f; since we are over $\mathbb{C}$, one such must exist. Then, for any $v \in V$ such that $f(v) = \lambda v$, we have

$$f(\rho_g v) = \rho_g(f(v)) = \lambda \rho_g v$$

and so $\rho_g v$ is also a λ-eigenvector. Hence, the eigenspace corresponding to the eigenvalue λ is G-invariant. Since this eigenspace is non-zero and since the representation is irreducible, it follows that the λ-eigenspace is all of V, i.e. $f = \lambda I_V$. □

Corollary 5.4.2.1. *If (V_i, ρ^i) $(i = 1, 2)$ are non-isomorphic irreducible representations of the compact group G, then for any linear mapping $T : V_1 \to V_2$ the linear map*

$$\int_G (\rho_g^2)^{-1} \circ T \circ \rho_g^1 dg : V_1 \to V_2$$

is the zero map.

Corollary 5.4.2.2. *If (V, ρ) is an irreducible representation of the compact group G, and if $T : V \to V$ is any linear map, then we have the following equality of linear maps on V:*

$$\int_G \rho_{g^{-1}} \circ T \circ \rho_g dg = \left(\frac{\text{Trace}(T)}{\dim V} \right) I_V.$$

For G compact, we have the usual L^2-inner product on $L^2(G, \mathbb{C})$ given by

$$(\phi|\psi) = \int_G \phi(g)\overline{\psi(g)} d\mu(g).$$

In particular, if G is finite, then we have the inner product $(|)$ on the $\mathbb{C}$-space $L^2(G, \mathbb{C}) = \text{Map}(G, \mathbb{C})$ of all complex-valued set functions on G, given by $(\phi|\psi) = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}$. We also have the pairing $\langle , \rangle$ on $\text{Map}(G, \mathbb{C})$ given by $\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\psi(g^{-1})$. For any function $\phi \in \text{Map}(G, \mathbb{C})$, define the map $\hat{\phi} \in \text{Map}(G, \mathbb{C})$ given by $\hat{\phi}(g) = \overline{\phi(g^{-1})}$. We have $\hat{\chi} = \chi$ for all characters χ of G. Clearly $(\phi|\psi) = \left\langle \phi, \hat{\psi} \right\rangle$.

A *class function on G* is a map $f \in L^2(G, \mathbb{C})$ such that $f(ghg^{-1}) = f(h)$ for all $g, h \in G$, i.e. f depends only on the conjugacy class of G. The set $\mathcal{H} = \mathcal{H}_G$ of class functions on G is a closed subspace of $L^2(G, \mathbb{C})$, so in particular is a Hilbert space.

Lemma 5.4.3. *Suppose $f \in \mathcal{H}_G$, and suppose (V, ρ) is an irreducible representation of degree n with character χ. Then, as linear maps on V, we have the equality*

$$\int_G f(g)\rho_g dg = \frac{(f|\bar{\chi})}{n} I_V.$$

Theorem 5.4.4. *The set of irreducible characters of the compact group G forms an orthonormal basis for $\mathcal{H}_G$ equipped with the inner product $(|)$.*

We let $\mathcal{X}$ denote the set of all irreducible characters of G, so that $\mathcal{X}$ is an orthonormal basis for the Hilbert space $\mathcal{H}_G$.

Theorem 5.4.5. *Suppose V is a representation of the compact group G with character χ such that V decomposes into a direct sum $V = W_1 \oplus \cdots \oplus W_k$ of irreducible representations W_i. For any irreducible representation W with character ψ, the number of W_i isomorphic to W is (ψ, χ) (which is equal to $\langle \psi, \chi \rangle$ if G is finite). Moreover, the number of such W_i isomorphic to W does not depend on the chosen decomposition.*

In particular, two representations with the same character are isomorphic.

Corollary 5.4.5.1. *For each $\chi \in \mathcal{X}$, let W_χ be the corresponding representation space. Then, any representation V of G with character ψ is isomorphic to the direct sum $\bigoplus_{\chi \in \mathcal{X}} m_\chi W_\chi$ where $m_\chi \in \mathbb{Z}_{\geq 0}$ satisfy $m_\chi = (\psi|\chi)$ (which is non-zero for only finitely many $\chi \in \mathcal{X}$), $\psi = \sum_{\chi \in \mathcal{X}} m_\chi \chi$, and $(\phi|\phi) = \sum_{\chi \in \mathcal{X}} m_\chi^2$.*

In particular, $(\psi|\psi)$ is a positive integer for all representations V with character ψ, and V is irreducible iff $(\psi|\psi) = 1$.

Corollary 5.4.5.2. *The number of distinct irreducible characters of a finite group G up to isomorphism (which is equal to $\dim H_G$) is equal to the number of conjugacy classes of G.*

Proposition 5.4.6. *For each $g \in G$ (G a finite group), let $c(g)$ be the number of elements in the conjugacy class of g. Suppose $\chi_1, ..., \chi_h$ are the set of distinct irreducible characters of G. Then*

$$\sum_{i=1}^{h} |\chi_i(g)|^2 = \frac{|G|}{c(g)}, \quad \text{and} \quad \sum_{i=1}^{h} \chi_i(g_1)\overline{\chi_i(g_2)} = 0$$

where g_1 and g_2 are in distinct conjugacy classes of G.

Example 5.4.7 (Spring 2020 Day 2)**.** Suppose G is finite and $g \in G$. Then we show that the following statements are equivalent:

1. $g \in Z[G]$ (the centre of G);
2. For any irreducible representation (V, ρ) of G, the image $\rho(g)$ is a scalar multiple of the identity; and
3. For any irreducible representation (V, ρ) of G with character χ, we have $|\chi(g)| = \dim V$.

($1 \implies 2$) Fix the map $T = \rho_g : V \to V$. Then, for all $x \in G$, we have $\rho_x \circ T = \rho_{xg} = \rho_{gx} = T \circ \rho_x$ since g is in the centre of G. By Schur's Lemma, it follows that $T = \rho_g$ is a scalar multiple of the identity.

($2 \implies 3$) Suppose $\rho_g = \lambda I_V$. Since G is finite, g has finite order in G, and so there exists $k \in \mathbb{N}$ such that $\lambda^k I_V = (\rho_g)^k = \rho_e = I_V$. Thus λ is a root of unity, and in particular has absolute value 1. Hence

$$|\chi(g)| = |\text{Trace}(\rho_g)| = |\text{Trace}(\lambda I_V)| = |\lambda \cdot \dim V| = \dim V.$$

($3 \implies 1$) Let $\chi_1, ..., \chi_h$ be all the irreducible representations of G, with degrees $n_1, ..., n_h$ respectively. Then, we know that $\sum_{i=1}^h |\chi_i(g)|^2 = |G|/c(g)$, where $c(g)$ denotes the size of the conjugacy class of g. However, by assumption $|\chi_i(g)| = n_i$, and so we have $|G|/c(g) = \sum_{i=1}^h n_i^2 = |G|$. Hence $c(g) = 1$, which implies that $xgx^{-1} = g$ for all $x \in G$. Equivalently, $g \in Z(G)$.

Proposition 5.4.8. *Let $\chi_1, ..., \chi_h$ be characters of the finite group G corresponding to representations $W_1, ..., W_h$ with corresponding degrees $n_i = \dim W_i = \chi_i(1)$.*

- *The character r_G of the regular representation $\mathbb{C}[G]$ of G is given by $\rho_G(1) = |G|$ and $\rho_G(g) = 0$ for all $g \in G \setminus \{1\}$.*
- *Every irreducible representation W_i is contained in $\mathbb{C}[G]$ with multiplicity n_i (this is in fact true for compact G as well, replacing $\mathbb{C}[G]$ with $L^2(G, \mathbb{C})$).*
- *We have $\sum_{i=1}^h n_i^2 = |G|$, $n_i | [G : Z(G)]$ for all i ($Z(G)$ the centre), and $\sum_{i=1}^h n_i \chi_i(g) = 0$ for all $g \in G, g \neq 1$.*

Corollary 5.4.8.1. *A compact group G is abelian iff all irreducible representations of G have degree 1. More generally, each irreducible representation of G has degree at most $[G : Z(G)]$ (where $Z(G)$ is the centre of G).*

Theorem 5.4.9 (Canonical Decomposition of a Representation)**.** *Let $\mathcal{X}$ be the set of irreducible characters of the compact group G. For each $\chi \in \mathcal{X}$, let the corresponding representation space be W_χ with degree $n_\chi = \dim W_\chi = \chi(1)$. For any finite representation (V, ρ) with character ψ and degree n, there exists a* unique *decomposition $V = \bigoplus_{\chi \in \mathcal{X}} V_\chi$ into sub-representations such that each V_χ is a direct sum of $(\psi|\chi)$ copies of W_χ (here, $V_\chi \neq 0$ for finitely many $\chi \in \mathcal{X}$ since V is finite dimensional). This decomposition is called the* canonical decomposition of the representation (V, ρ).

This decomposition of V_χ into multiples copies of W_χ may not be unique; in fact, there is a 1-1 correspondence between decompositions $V_\chi = W_\chi \oplus \cdots \oplus W_\chi$ and choices of bases of the vector space of linear mappings $T : W_\chi \to V$ such that $\rho_g \circ T = T \circ \rho_g$ for all $g \in G$ (the image is necessarily in V_χ, and T is necessarily injective unless it is zero). This correspondence takes any basis $\{T_1, ..., T_t\}$ of such linear maps $W_\chi \to V$ to the decomposition $V_\chi = T_1(W_\chi) \oplus \cdots \oplus T_t(W_\chi)$.

Furthermore, the projection $p_\chi : V \to V_\chi$ onto the component corresponding to $\chi \in \mathcal{X}$ ($\ker p_i = \bigoplus_{\phi \in \mathcal{X}, \phi \neq \chi} V_\phi$ and $p_\chi|_{V_\chi} = I_{V_\chi}$), is given by

$$p_\chi(v) = n_\chi \int_G \overline{\chi(g)} \rho_g(v) dg \quad \forall v \in V.$$

Proposition 5.4.10. *Suppose ρ is an irreducible representation of G of degree n with character χ, and let G have centre $Z(G)$. Then, $\rho_g = \frac{\chi(g)}{|G|} \cdot I_V$ and $|\chi(g)| = n$. We have $n^2 \leq [G : Z(G)]$. If ρ is faithful, then $Z(G)$ must be a cyclic group.*

Theorem 5.4.11 (Peter-Weyl Theorem (if G compact))**.** *Suppose R_G is the regular representation of a compact group G ($R_G = \mathbb{C}[G]$ if G finite, $R_G = L^2(G, \mathbb{C})$ if G infinite). On R_G we have a natural $\mathbb{C}$-algebra structure as follows: if G is finite, it is simply given by $\left(\sum_g a_g g\right)\left(\sum_g b_g g\right) = \sum_{g,h} a_g b_h gh$. If G is compact, then for $f_1, f_2 \in L^2(G, \mathbb{C})$ we have the convolution $f_1 * f_2(h) = \int_G f_1(g) f_2(g^{-1}h) dg$. Any action of G on V for V a representation can be extended to an action of the $\mathbb{C}$-algebra R_G on V by linearity if G is finite, and by*

$$f \cdot v := \int_G f(g)(g \cdot v) dg \in V.$$

Thus for any representation V of a compact group G, we get a corresponding $\mathbb{C}$-algebra homomorphism $R_G \to End(V)$.

For each $\chi \in \mathcal{X}$ (set of irreducible characters of G), let $\rho_\chi : G \to GL(W_\chi)$ be the corresponding representation with degree n_χ. Extend the group homomorphism $\rho_\chi : G \to GL(W_\chi)$ to a $\mathbb{C}$-algebra homomorphism $\tilde{\rho}_\chi : R_G \to End(W_\chi) \cong M(n_\chi, \mathbb{C})$ (space of $n_\chi \times n_\chi$ matrices) as above. Then, the family of maps $(\rho_\chi)_{\chi\in\mathcal{X}} : R_G \to \prod_{\chi\in\mathcal{X}} M(n_\chi, \mathbb{C})$ is an isomorphism. This map is the Fourier transform *of an element of R_G (this terminology is especially used if G is compact infinite).*

Proposition 5.4.12 (Fourier Inversion Formula)**.** *For G finite, the inverse isomorphism $\prod_{\chi\in\mathcal{X}} M(n_\chi, \mathbb{C}) \to [\sim] R_G$ is*

$$(u_\chi)_{\chi\in\mathcal{X}} \mapsto \frac{1}{|G|}\sum_{g\in G}\left(\sum_{\chi\in\mathcal{X}} n_\chi \mathrm{Trace}_{W_\chi}(\rho_\chi(g^{-1}))u_\chi\right)\cdot g.$$

Proposition 5.4.13 (Plancherel Formula)**.** *Suppose G finite, and let $u, v \in \mathbb{C}[G]$ be written as $u = \sum_g u(g)g$ and $v = \sum_g v(g)g$. Set $\langle u, v\rangle = |G|\sum_{g\in G} u(g^{-1})v(g)$. Then,*

$$\langle u, v\rangle = \sum_{\chi\in\mathcal{X}} n_\chi \mathrm{Trace}_{W_\chi}(\tilde{\rho}_\chi(uv)).$$

5.4.3 Products Of Groups

Definition. Suppose G_1, G_2 are two compact groups, and suppose we have representations (V_1, ρ^1) and (V_2, ρ^2) on G_1 and G_2 respectively. The representation $G_1 \times G_2 \to GL(V_1 \otimes V_2), (g_1, g_2) \mapsto \rho^1(g_1) \otimes \rho^2(g_2)$ is called the *tensor product* of ρ^1 and ρ^2, and is denoted by $\rho^1 \otimes \rho^2$.

The Haar measure on $G_1 \times G_2$ is simply the product measure of the two Haar measures on G_1 and G_2.
Properties of tensor products of representations:

1. If $G_1 = G_2 = G$, then the tensor product of two representations ρ^1, ρ^2 gives a representation $\rho^1 \otimes \rho^2$ of $G \times G$. When restricted to the diagonal, this tensor product $\rho^1 \otimes \rho^2$ yields the representation $\rho^1 \otimes \rho^2$ of G earlier called the tensor product of ρ^1 and ρ^2.

2. If ρ^1 and ρ^2 are irreducible, then $\rho^1 \otimes \rho^2$ is irreducible. Conversely, every irreducible representation of $G_1 \times G_2$ is of the form $\rho^1 \otimes \rho^2$ for irreducible representations ρ^1, ρ^2 of G_1, G_2 respectively.

3. The character χ of $\rho^1 \otimes \rho^2$ is $\chi(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$.

Example 5.4.14 (Spring 2018 Day 1)**.** Suppose $\rho_m : G_m \to GL(V_m)$ are n representations, and set $G = G_1 \times \cdots \times G_n$, $V = V_1 \otimes \cdots \otimes V_n$, and $\rho = \rho_1 \otimes \cdots \otimes \rho_n$. We prove that ρ is irreducible iff ρ_m is irreducible for all m.

For a representation π of a group H with character ψ, recall that

$$n_\pi := \frac{1}{|H|}\sum_{h\in H} |\psi(h)|^2$$

is always an integer, and π is irreducible iff $n_\pi = 1$. Let χ_m be the character of ρ_m, and χ the character of ρ. Then, we know that $\chi(g_1, ..., g_n) = \prod_{m=1}^n \chi_m(g_m)$. Hence, we see that

$$\begin{aligned} n_\rho = \frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 &= \frac{1}{|G_1|\cdots|G_n|}\sum_{g_1\in G_1,\ldots,g_n\in G_n} |\chi_1(g_1)\cdots\chi_n(g_n)|^2 \\ &= \prod_{m=1}^n \frac{1}{|G_m|}\sum_{g_m\in G_m}|\chi_m(g_m)|^2 = n_{\rho_1}\cdots n_{\rho_n}. \end{aligned}$$

Since all these numbers are integers, it follows that $n_\rho = 1$ iff $n_{\rho_m} = 1$ for all m. Hence (V, ρ) is irreducible iff (V_m, ρ_m) are irreducible for all m.

5.4.4 Induced Representations

Definition. Suppose (V, ρ) is a representation of a compact group G, and suppose $H \subset G$ is a closed subgroup of finite index (if G is finite, then H is any subgroup). Let (W, θ) be a representation of H such that $W \subset V$. The representation ρ of G is *induced* by the representation θ of H if

- V is equal to the direct sum $\sum_{i=1}^{m} \rho_{g_i} W$ where $\{g_1 = 1, ..., g_m\}$ is a set of left H-coset representatives of G; and
- for any $g \in H$, we have $\rho_g|_W = \theta_g$.

Properties of Induced Representations (throughout, G is a compact group and H a subgroup of finite index in G):

1. Notice that the subspace $\rho_g W$ of V depends only on the coset gH, and that ρ_g permutes the various $\rho_{g_i} W$ among themselves.
2. The regular representation of G is induced by the regular representation of H.
3. The permutation representation associated to the left-action of G on the set of left H-cosets G/H is induced by the unit representation of H (here $W = \mathbb{C} \cdot (1H)$).
4. If ρ_i is induced by θ_i for $i = 1, 2$, then $\rho_1 \oplus \rho_2$ is induced by $\theta_1 \oplus \theta_2$.
5. Suppose R is a set of coset representatives of G modulo H. If (V, ρ) is induced by (W, θ), and if W_1 is a H-stable subspace of W, then the subspace $V_1 = \sum_{g \in R} \rho_g W_1$ is G-stable, and the representation V_1 of G is induced by the representation W_1 of H.
6. If ρ is induced by θ, ρ' is a representation of G, and if ρ'_H is the restriction of ρ' to H, then the representation $\rho \otimes \rho'$ of G is induced by the representation $\theta \otimes \rho'_H$ of H.
7. Suppose (V, ρ) is induced by (W, θ). Let (V', ρ') be any other representation of G, and suppose $f : W \to V'$ is a linear map such that $f \circ \theta_h = \rho'_h \circ f : W \to V'$ for all $h \in H$. Then, there exists a unique linear map $F : V \to V'$ such that $F|_W = f$ and $F \circ \rho_g = \rho'_g \circ F$ for all $g \in G$.
8. Let (W, θ) be any representation of H, where $H \leq G$. Then, there exists a unique representation (V, ρ) of G such that ρ is induced by θ.
9. Suppose (V, ρ) is induced by (W, θ), and let χ_ρ and χ_θ be the characters of ρ and θ. Then,

   $$\chi_\rho(g) = \sum_{r \in R, r^{-1} g r \in H} \chi_\theta(r^{-1} g r),$$

   where R is any system of H-coset representatives. If G is a finite group and $C(g)$ denotes the conjugacy class of g in G, then

   $$\chi_\rho(g) = \frac{1}{|H|} \sum_{x \in G, x^{-1} g x \in H} \chi_\theta(x^{-1} g x) = \frac{1}{|H|} \sum_{x \in H \cap C(g)} \chi_\theta(x).$$

10. (*Frobenius Reciprocity Formula*) Suppose (V, ρ) is induced by (W, θ), with characters χ_ρ and χ_θ. Let $(|)_G$ (resp. $(|)_H$) denote the inner product on $\mathcal{H}_G$ (resp. $\mathcal{H}_H$). Then, for any class function $f \in \mathcal{H}_G$ on G, we have $(f|_H|\chi_\theta)_H = (f|\chi_\rho)_G$, where $f|_H$ is the restriction of f to H.
11. Suppose G is the direct product of subgroups H and K. Let ρ be a representation on G induced by the representation θ on H. Let r_K denote the regular representation of K. Then $\rho \cong \theta \otimes r_K$.

Remark 5.4.15. If H is a closed subgroup of the compact group G of infinite index, then the representation of G induced by (W, θ) is defined to be the Hilbert subspace of $L^2(G, W)$ of functions $f : G \to W$ such that $f(hg) = \theta_h(f(g))$ for all $g \in G$ for each $h \in H$. The action of G on this subspace is $\rho_g(f)(x) = f(xg)$ for $g, x \in G$.

5.4.5 Examples

We now find *all* irreducible characters of some examples of groups. A *character table* for a finite group G is a table whose rows correspond to every irreducible character G and the columns are elements of G, and the entry in the row χ and column $g \in G$ is the value $\chi(g)$.

Example 5.4.16. Consider the cyclic group $\mathbb{Z}/n\mathbb{Z}$. One checks that the set of all irreducible characters are $\chi_k : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}, [m] \mapsto e^{2\pi i m k/n}$ for $k = 0, ..., n-1$. Here, χ_0 is the trivial representation.

Example 5.4.17. Consider the compact group $S^1 \subset \mathbb{C}^*$. The Haar measure on S^1 is $\frac{1}{2\pi}d\alpha$, where $\alpha : S^1 \to [0, 2\pi]$ gives the (principal) argument. All irreducible representations of S^1 are of the form $\chi_n : S^1 \to \mathbb{C}$, $\chi_n(z) = z^n$, for all $n \in \mathbb{Z}$. The orthogonality relations are simply the obvious formulae

$$(\chi_m|\chi_n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha}e^{im\alpha}d\alpha = \delta_{m,n}.$$

Example 5.4.18. Consider the dihedral group $D_n = \langle r, s | r^n = 1, s^2 = 1, srs = r^{-1}\rangle = (\mathbb{Z}/n\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$, where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ acts by conjugation on $(\mathbb{Z}/n\mathbb{Z})$ via $m \mapsto -m$.

- If n is even, then there are 4 irreducible representations of degree 1 whose characters $\psi_0, ..., \psi_3$ are given by $\psi_i(s) = (-1)^i$ and $\psi_i(r) = (-1)^{\lfloor i/2 \rfloor}$. For each $h = 1, 2, ..., (n/2)-1$, we have the irreducible representation ρ^h induced by the representation of $\langle r\rangle \cong \mathbb{Z}/n\mathbb{Z}$ with character χ_h (or χ_{n-h}; the induced representations are isomorphic). Explicitly, we have $\rho^h(r) = \begin{pmatrix} \exp(2\pi ih/n) & 0 \\ 0 & \exp(-2\pi ih/n)\end{pmatrix}$ and $\rho^h(s) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$. The corresponding characters $\chi_h := \chi_{\rho_h}$ are given by $\chi_h(r^k) = 2\cos\frac{2\pi hk}{n}$ and $\chi_h(sr^k) = 0$. All of these representations are mutually non-isomorphic, since their characters are distinct. Since $4 \cdot 1^2 + (\frac{n}{2} - 1) \cdot 2^2 = 2n = |D_n|$, it follows that these are the only possible irreducible characters. In particular, this implies that D_n for n even has $\frac{n}{2} + 3$ conjugacy classes.
- If n is even, then there are 2 irreducible representations of degree 1 whose characters ψ_0, ψ_1 are given by $\psi_0(r) = 1 = \psi_1(r)$ and $\psi_0(s) = 1, \psi_1(s) = -1$. We have the degree 2 irreducible representations ρ^h $(1 \le h \le \frac{n-1}{2})$ given above, with characters χ_h given by $\chi_h(r^k) = 2\cos\frac{2\pi hk}{n}$ and $\chi_h(sr^k) = 0$. Again, since all of these characters are distinct and $2 \cdot 1^2 + \frac{n-1}{2} \cdot 2^2 = 2n$, it follows that these are precisely all the distinct irreducible characters. In particular, D_n for n odd has $\frac{n+3}{2}$ conjugacy classes.

Example 5.4.19. Consider the group $D_\infty = S^1 \rtimes (\mathbb{Z}/2\mathbb{Z})$ where the conjugation action of $\mathbb{Z}/2\mathbb{Z}$ on S^1 consists of the identity map on S^1 and the complex conjugation map on S^1. We have $D_\infty = \langle r, s | r \in S^1, s^2 = 1, srs = r^{-1}\rangle$. The Haar measure is $\frac{1}{4\pi}d\alpha$, i.e. for any f we have

$$\int_G f(g)dg = \frac{1}{4\pi}\int_0^{2\pi} f(e^{i\alpha})d\alpha + \frac{1}{4\pi}\int_0^{2\pi} f(s \cdot e^{i\alpha})d\alpha.$$

There are two irreducible representations of degree 1, with characters ψ_0, ψ_1 given by $\psi_i(s^\epsilon \cdot z) = (-1)^{i\epsilon}$ ($\epsilon \in \{0, 1\}$ and $z \in S^1$). For each $n \in \mathbb{N}$, there is an irreducible representation ρ_n of D_∞ given by $\rho^h(z) = \begin{pmatrix} z^h & 0 \\ 0 & z^{-h}\end{pmatrix}$ for $z \in S^1$ and $\rho^h(s) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$. It can be shown that these are all the irreducible representations of D_∞.

Example 5.4.20. Consider the alternating group $A_4 = (\mathbb{Z}/2\mathbb{Z})^2 \rtimes (\mathbb{Z}/3\mathbb{Z})$ where $(\mathbb{Z}/2\mathbb{Z})^2 \cong \{e, (12)(34), (13)(24), (14)(23)\}$ (normal in A_4) and $(\mathbb{Z}/3\mathbb{Z}) \cong \langle(123)\rangle$. One checks readily that there are four conjugacy classes namely $\{e\}$, $C_1 = \{(12)(34), (13)(24), (14)(23)\}$, $C_2 = \{(123), (134) = (123)(12)(34), (243) = (123)(13)(24), (142) = (123)(14)(23)\}$ and $C_3 = \{(132), (234) = (123)^2(12)(34), (124) = (123)^2(13)(24), (143) = (123)^2(14)(23)\}$. Hence, there are four irreducible characters $\chi_0, \chi_1, \chi_2, \chi_3$ with degrees $n_0 = 1, n_1, n_2, n_3$ (here χ_0 is the unit representation). Then $1 + n_1^2 + n_2^2 + n_3^2 = 12$ and $n_i|12$ implies that either $(n_1, n_2, n_3) = (1, 1, 3)$ or $(2, 2, 2)$. However, note that $\chi_j : A_4 \to A_4/(\mathbb{Z}/2\mathbb{Z})^2 \cong (\mathbb{Z}/3\mathbb{Z}) \to \mathbb{C}$ $(j = 1, 2)$ which maps C_k $(k = 1, 2, 3)$ to $e^{2\pi ij(k-1)/3}$ are all distinct irreducible characters. Thus the remaining character χ_3 has degree 3. By using $\sum_{i=0}^2 \chi_i(g) + 3\chi_3(g) = 0$ for all $g \neq 1$, we get the following character table for A_4:

	e	(12)(34)	(123)	(132)
χ_0	1	1	1	1
χ_1	1	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$
χ_2	1	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$
χ_3	3	-1	0	0.

Example 5.4.21. Consider S_4. Let $H = \{e, (12)(34), (13)(24), (14)(23)\} \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $K = \{\sigma \in S_4 : \sigma(4) = 4\} \cong S_3 \cong D_3$; then H is normal in S_4 and $S_4 \cong H \rtimes K$. Thus all representations of K can be extended to S_4 by declaring them to be invariant under action by H. This yields three representations of S_4 with degrees 1, 1, and 2. Also, all irreducible representations of A_4 give rise to representations of S_4 since A_4 is normal in S_4 and $S_4 \cong A_4 \rtimes (\mathbb{Z}/2\mathbb{Z})$. This along with the formulae from the character theory of finite groups yields the following character table for S_4:

	e	(ab)	$(ab)(cd)$	(abc)	$(abcd)$
χ_0	1	1	1	1	1
sgn	1	-1	1	1	-1
θ	2	0	2	-1	0
ψ	3	1	-1	0	-1
$\text{sgn} \cdot \psi$	3	-1	-1	0	1